LATTICE-VALUED BOREL MEASURES

S. S. KHURANA

Abstract. A Riesz representation type theorem is proved for measures on locally compact spaces, taking values in some ordered vector spaces.

In a series of papers ([4], [5], [6]), J. M. Maitland Wright has established, among other things, some Riesz representation type theorems for positive linear mappings from $C(X)$ to $E$, $X$ being a compact Hausdorff space and $E$ a complete (or $\sigma$-complete) vector-lattice. In this paper we prove these results (Theorem 4) by using the properties of order convergence in vector lattices.

We shall use the notations of ([2], [3]). For a compact Hausdorff space $X$, we denote by $C(X)$ the vector space of all continuous real-valued functions on $X$ with sup norm, by $L(X)$ and $M(X)$ the dual and bidual of $C(X)$, respectively, and by $\beta(X)$ and $\beta_1(X)$ the sets of all bounded Borel and Baire measurable real-valued functions on $X$, respectively. In the natural order $C(X)$ is a vector lattice and $\beta(X)$ and $\beta_1(X)$ are boundedly $\sigma$-complete lattices. Also $L(X)$ and $M(X)$ are boundedly complete vector lattices and $C(X)$ is a sublattice of $M(X)$. Let $S(X)$ be the subspace of $M(X)$ generated by those elements of $M(X)$ which are suprema of bounded subsets of $C(X)$.

Let $E$ be a vector lattice (always assumed to be over the field of real numbers). Order convergence, order closure ($\sigma$-closure), order continuity ($\sigma$-continuity) in vector lattices are taken in the usual sense ([1], [2], [3]). If $A$ is a subset of $E$, let $A_1$ be the set of order limits, in $E$, of sequences in $A$, $A_2$ be the set of order limits of sequences in $A \cup A_1$ (= $A_1$), and so on. Continuing this process transfinetly, if necessary, and taking the union of all these subsets, we get the order $\sigma$-closure of the set $A$. A vector subspace $B$ of $E$ we shall call monotone order closed ($\sigma$-closed), if for any net (sequence) $\{x_\alpha\}$, such that $x_\alpha \uparrow x$ in $E$, $x \in B$ ($x_\alpha \uparrow x$ means $\{x_\alpha\}$ is increasing and its sup is $x$). Now if $A$ is a vector sublattice of a boundedly $\sigma$-complete vector lattice $E$, $E_1$ a monotone order $\sigma$-closed vector subspace of $E$, and $E_1 \supset A$, then $E_1 \supset A_1$ ($A_1$ as defined above); since $A_1$ is also a vector sublattice of $E$, $E_1 \supset A_2$, and so continuing this (transfinetly if necessary) we get $E_1 \supset$ order $\sigma$-closure of $A$. This result will be needed later. Monotone order continuity ($\sigma$-continuity) can be defined between ordered...
vector spaces in a similar way. For any real-valued function on a topological space \( Z \), \( \sup f = \text{closure of } \{ z \in Z : f(z) \neq 0 \} \) in \( Z \).

The order \( \sigma \)-closure of \( S(X) \) (\( C(X) \)), in \( M(X) \), will be denoted by \( \beta(X) \) (\( \beta_1(X) \)). We denote by \( B(X) \) the set of all bounded real-valued functions on \( X \) with the natural point-wise order. \( \beta_1(X) \) is the order \( \sigma \)-closure of \( C(X) \) in \( B(X) \), and \( \beta(X) \) is the order \( \sigma \)-closure of the vector space generated by bounded lower semicontinuous functions on \( X \). If \( X \) is Stonian (\( \sigma \)-Stonian), \( C(X) \) is a boundedly complete (\( \sigma \)-complete) vector lattice, and in this case \( H = \{ f \in B(X) : \exists g \in C(X) \text{ such that } f = g \text{ except on a meagre subset of } X \} \supseteq \beta(X) \cap \beta_1(X) \); this gives a mapping \( \psi : \beta(X) \to C(X) \) (\( \psi_1 : \beta_1(X) \to C(X) \)). We prove first the following simple lemmas.

**Lemma 1.** There exists a 1-1, onto, linear, both way positive, mapping \( \varphi : \beta(X) \to \beta(X) \) (\( \varphi_1 : \beta_1(X) \to \beta_1(X) \)), such that

(i) \( \varphi(f) = f(\varphi_1(f) = f) \), \( \forall f \in C(X) \);

(ii) \( \varphi, \varphi^{-1}, \varphi_1, \varphi_1^{-1} \) are all order \( \sigma \)-continuous;

(iii) for any increasing net \( \{ f_a \} \) in \( C(X) \), \( \varphi(\sup f_a) = \sup \varphi(f_a) \), and \( \varphi^{-1}(\sup f_a) = \sup \varphi^{-1}(f_a) \).

**Proof.** On \( B(X) \), the space of all bounded, real-valued functions on \( X \), we take the topology of point-wise convergence. Since the identity map \( i : (C(X), \| \cdot \|) \to B(X) \) is a weakly compact linear operator, its second adjoint \( i^{**} : (M(X), \sigma(M(X), L(X))) \to B(X) \) is continuous, and so is order continuous, since order convergence in \( M(X) \) implies \( \sigma(M(X), L(X)) \)-convergence. This means that for an increasing net \( \{ f_a \} \) in \( C(X) \), \( \varphi(\sup f_a) = \sup \{ \varphi(f_a) \} \) in \( B(X) \). This proves that \( i^{**} \cap \beta(X) \supseteq \beta_1(X) \) and is order \( \sigma \)-closed, and so \( i^{**} \cap \beta_1(X) \supseteq \beta(X) \); similar results hold for \( \beta_1(X) \). Let \( \varphi = i^{**} \mid \beta_1(X) \) (\( \varphi_1 = i^{**} \mid \beta_1(X) \)). Then \( \varphi : \beta(X) \to \beta(X) \) (\( \varphi_1 : \beta_1(X) \to \beta_1(X) \)). If \( f \in \beta(X) \) and \( f \geq 0 \), then there exists a net \( \{ f_a \} \subset C(X) \) such that \( f_a \to f \) (i.e., order converges to \( f \) in \( M(X) \)) \[2\]. This means \( f_a \to \varphi(f) \) which means that \( \varphi(f) \geq 0 \). Now suppose that for some \( f \in \beta(X) \) \( \varphi(f) = 0 \). Take \( \{ f_a \} \subset C(X) \) such that \( f_a \to f \) and so \( f_a \to \varphi(f) \) which means that \( \varphi(f) = 0 \). This means \( f_a \to 0 \) \( \forall x \in X \), and so \( \langle f, \epsilon_x \rangle = 0 \) \( \forall \) point measure \( \epsilon_x \in L(x) \), which proves that \( f = 0 \) \([2\], p. 83\), and thus \( \varphi \) is 1-1. To prove that \( \varphi^{-1} \) is positive, take \( f \in \beta(X) \), such that \( \varphi(f) \geq 0 \). There exists a net \( \{ f_a \} \subset C(X) \) such that \( f_a \to f \) in \( M \), which means that \( f_a \to \varphi^+ \) and \( f_a \to \varphi^- \). This gives that \( \lim f_a \) (\( x \)) = 0, \( \forall x \in X \), and so \( \langle f^-, \epsilon_x \rangle = 0 \) for any point measure \( \epsilon_x \) in \( L(X) \) which means \( f^- = 0 \) \([2\], p. 83\). This proves \( \varphi^{-1} \) is positive. To prove that \( \varphi \) is onto, take a lower semi-continuous function \( f \) in \( B(X) \). Then there exists an increasing net \( \{ f_a \} \) in \( C(X) \) such that \( f_a \to f \). Taking
g = \sup\{f_n^-\} in M(X), we get f = \varphi(g). Also if an increasing sequence \(h_n \uparrow h\) in \(B(X)\), by positivity of \(\varphi^{-1}\), \(g_n = \varphi^{-1}(h_n)\) is increasing in \(M(X)\); and so \(\varphi(g) = h\), where \(g = \sup\{g_n\} in M(X)\). This proves \(\varphi\) is onto. The order \(\sigma\)-continuity and other properties of \(\varphi^{-1}\) are easily verified. Similar arguments prove the corresponding results for \(\varphi_1\). This completes the proof.

**Lemma 2.** If \(X\) is Stonian (\(\sigma\)-Stonian), the mapping \(\psi : \beta(X) \to C(X)\)
\((\psi_1 : \beta_1(X) \to C(X))\) is a positive order \(\sigma\)-continuous linear mapping. Also if \(\{f_n\}\) is an increasing net in \(C(X)\) such that \(\sup f_n = f\) in \(\beta(X)\), then \(\psi(f) = \sup \psi(f_n), if X\) is Stonian.

**Proof.** The linearity and positivity of \(\psi\) are obvious. Also if \(\{f_n\}\)
is an increasing net in \(C(X)\), then pointwise \(\sup\{f_n\} = f\) and \(\sup\{f_n\} = h^*\) in \(C(X)\) are equal except on a meagre subset of \(X\) [4], and so \(\psi(f) = h = \sup \psi(f_n). If \{h_n\}\) is an increasing sequence in \(\beta(X)\) such that \(h_n = f_n \in C(X)\) on \(X\setminus A_n, A_n\) being meagre for every \(n, and h_n \uparrow h\) in \(\beta(X)\), then \(f = \sup f_n^-\) in \(C(X)\), and \(g = \) pointwise \(\sup\{f_n\}\) are equal on \(X\setminus A\) being a meagre subset of \(X\). This proves \(\psi(h_n) \uparrow \psi(h)\), and so \(\psi\) is order \(\sigma\)-continuous. The corresponding results for \(\psi_1\) can be proved in a similar way.

**Lemma 3.** Let \(X\) and \(S\) be compact Hausdorff spaces with \(S\) also a Stonian (\(\sigma\)-Stonian) space, and \(\mu : C(X) \to C(S)\) a positive linear mapping. Then \(\mu\) can be uniquely extended to a positive linear mapping \(\bar{\mu} : \beta(X) \to C(S)\)
\((\bar{\mu} : \beta_1(X) \to C(S))\), satisfying the following conditions.

(i) \(\bar{\mu}\) is order \(\sigma\)-continuous;
(ii) for any increasing net \(\{f_a\} \subset C(X), with sup f_a = f in \beta(X), \bar{\mu}(f) = sup \bar{\mu}(f_a), in case X\) is Stonian.

**Proof.** Assume first that \(S\) is Stonian. The second adjoint of \(\mu : C(X) \to C(S)\), \(\mu'' : M(X) \to M(S)\), is an order-continuous positive linear mapping ([3], p. 525), and so \(\mu''^{-1} \circ Bo(S) \subseteq Bo(X)\). Using Lemmas 2 and 3 we get a mapping \(\bar{\mu} : \beta(X) \to C(S)\), satisfying the conditions of the lemma. If \(\nu\) is another extension satisfying the conditions of the theorem, then \(\bar{\mu}\) and \(\nu\) are equal on the subspace generated by l.s.c. bounded functions on \(X\), and so by order \(\sigma\)-continuity, they are equal on \(\beta(X)\). The \(\sigma\)-Stonian case can be dealt with in a similar way.

Let \(Y\) be a locally compact Hausdorff space, \(\beta'(Y) (\beta_1'(Y))\) all bounded Borel (Baire) measurable functions with compact supports, \(B'(Y)\) all bounded real-valued functions on \(Y\) with compact supports, and \(K(Y)\) all continuous real-valued functions on \(Y\) with compact supports. For any open (open \(F_\sigma\)) relatively compact subset \(V \subset Y\), let
$\beta'(Y, V) = \{ f \in \beta'(Y) : f \equiv 0 \text{ on } Y \setminus V \}$ \( \beta_1'(Y, V) = \{ f \in \beta_1'(Y), f \equiv 0 \text{ on } Y \setminus V \} \). If \( K(Y, V) = \{ f \in K(Y), \sup f \subset V \} \) and \( S'(Y, V) \) is the subspace of \( B'(Y) \) generated by \( \{ f \in B'(Y) : \exists \text{ an increasing net } \{ f_\alpha \} \subset K(Y, V), \text{ with } \sup f_\alpha = f \} \), then \( \beta'(Y, V) = \text{order } \sigma \)-closure of \( S'(Y, V) \) \( \beta_1'(Y, V) = \text{order } \sigma \)-closure of \( K(Y, V) \). Also \( \beta'(Y) = \bigcup \{ \beta'(Y, V) : V \text{ open relatively compact in } Y \} \) \( \beta_1'(Y, V) = \bigcup \{ \beta_1'(Y, V) : V \text{ open relatively compact in } Y \} \).

**Theorem 4.** Let \( E \) be a boundedly monotone complete \((\sigma\text{-complete})\) ordered vector and \( \mu : K(Y) \to E \) a positive linear map. Then \( \mu \) can be uniquely extended to \( \tilde{\mu} : \beta'(Y) \to E \ (\tilde{\mu} : \beta_1'(Y) \to E) \) with the properties that (i) \( \tilde{\mu} \) is monotone \( \sigma \)-continuous, (ii) for any increasing net \( \{ f_\alpha \} \) in \( K(Y) \) with \( \sup f_\alpha = f \in \beta'(Y) \), \( \tilde{\mu}(f) = \sup \mu(f_\alpha) \), in \( \beta_1'(Y) \), \( \tilde{\mu}(f) = \sup \mu(f_\alpha) \), in \( \beta_1'(Y) \).

**Proof.** Let \( V \) be an open relatively compact subset of \( Y \). Take \( \{ g_\alpha \} (\alpha \in I) \), an increasing net in \( K(Y) \), with \( \sup g_\alpha \subset V, 0 \leq g_\alpha \leq 1, \forall \alpha \), and \( \sup \{ g_\alpha \} = x_V \). Also take \( g \in K(Y), 0 \leq g \leq 1, \) and \( g = 1 \) on \( V \). Assuming \( E \) to be boundedly monotone complete, let \( e = \sup \{ \mu(g_\alpha) : \alpha \in I \} \) (note \( \mu(g_\alpha) \) is increasing and \( \mu(g_\alpha) \leq \mu(g), \forall \alpha \)). For any \( f \in K(Y) \), with \( \sup f \subset V \), and \( f \equiv x_V \) \( (= \sup g_\alpha) \), we first prove that \( \mu(f) \leq e \). Let \( C = \sup f \subset V, n \) any positive integer and \( V_\alpha = \{ x \in V : f(x) < g_\alpha(x) + 1/n \} \). Using the facts that \( \{ V_\alpha \} \) is increasing and \( \bigcup V_\alpha \subset C \), a compact set, we get \( V_{\alpha(n)} \subset C, \) for some \( \alpha(n) \in I \). Thus \( f < g_{\alpha(n)} + (1/n)g \) and so \( \mu(f) \leq e + (1/n) \mu(g), \forall n \), which gives \( \mu(f) \leq e, \) since \( \inf \{ (1/n)\mu(g) : n, \text{ a positive integer} \} = 0 \), (note \( \mu(g) \geq 0 \)).

Let \( E_0 = \{ p \in E : -\lambda e \leq p \leq \lambda e, \text{ for some real } \lambda > 0 \} \). Then \( E_0 \) is a boundedly monotone complete, directed, integrally closed, ordered vector subspace of \( E \) ([1], p. 290; to prove the integral closedness of \( E_0 \), we need the boundedly monotone \( \sigma \)-completeness of \( E \). Thus the completion by non-void cuts of \( E_0 \), say \( E_1 \), will be a boundedly complete vector lattice ([1], Theorem 9, p. 357). Let \( E_2 = \{ p \in E_1 : -\lambda e \leq p \leq \lambda e, \text{ for some } \lambda > 0 \} \). This means \( E_2 \) is a boundedly complete vector lattice with a strong unit \( e \) and so there exists a compact Hausdorff Stonian space \( S \), such that \( E_2 \) and \( C(S) \) are vector lattice isomorphic (i.e., there exists a 1-1, onto, both-way positive linear map from \( E_2 \) to \( C(S) \) which preserves arbitrary suprema and infima). Let \( V' = V \cup \{ x_0 \} \) be the Alexandroff one point compactification of the locally compact space \( V \) (if \( V \) is compact, we take \( V' = V \), and \( A \) the subspace of \( C(V') \) generated by constant functions and \( K(Y, V) \). Any element of \( A \) can be uniquely written in the form \( \lambda + f \), where \( \lambda \in \mathbb{R} \).
Define a linear mapping $f^\land: A \to C(S)$ as $(X + f)^\land = Xe + f(x)$. We first prove that $f^\land$ is positive. Suppose first that $X > 0$ and $X + f \geq 0$ on $V'$. This gives $1 + (1/X)f \geq 0$, and so $-((1/X)f) \leq \sup g_x$. From what is proved above it follows that $-((1/X)f) \leq e$, and so $Xe + f \geq 0$. If $X = 0$, there is nothing to prove. If $X < 0$ and $V$ is not compact, take $x \in V \setminus \sup f$. Then $f(x) = 0$ and $X + f(x) < 0$, a contradiction. If $X < 0$ and $V$ is compact, then $x_V \in K(Y, V)$, $x_V \geq g_x$, $\forall \alpha$, and $x_V \leq \sup g_x$. So from what is proved above it follows that $\mu(x_V) = e$. Now $X + f \geq 0$ on $V = V'$ implies that $x_V + f \geq 0$, and so $\mu(x_V + f) \geq 0$, which gives $Xe + f \geq 0$. This proves $f^\land$ is positive.

Also considering $A$ as a subspace of $C(V')$, with sup norm topology, $f^\land$ is also continuous and as such has a unique extension $\mu_V: C(V') \to C(S)$, since, by the Stone-Weierstrass approximation theorem, $A$ is dense in $C(V')$. It is easy to verify that this extension is also a positive linear operator. By Lemma 3, $\mu_V$ can be uniquely extended to $\bar{\mu}_V: \beta(V') \to E_1$, which is order $\sigma$-continuous, and if $\{f_a\}$ is an increasing net in $C(V')$ with $\sup f_a = f \in \beta(V')$, then $\bar{\mu}_V(f) = \sup \mu_V(f_a)$. It immediately follows that $\bar{\mu}_V(x_{(x)}) = 0$, i.e., $\bar{\mu}_V(f) = \bar{\mu}_V(f)_{i}$ if $f \in \beta(V')$ $(i = 1, 2)$ and $f_{1|V} = f_{2|V}$. We define $\bar{\mu}_V: \beta'(Y, V) \to E_1$ as: for any $f \in \beta'(Y, V)$, $\bar{\mu}_V(f) = \bar{\mu}_V(f')$, where $f = f'$ on $V$, and $f'(x_0) = 0$: this mapping is positive, linear and order $\sigma$-continuous and has the property that for any increasing net $\{f_a\} \subset K(Y, V)$ with $\sup f_a = f \in \beta(Y, V)$, $\bar{\mu}_V(f) = \sup \mu_V(f_a)$. Now $\bar{\mu}_V^{-1}(E_0) \supset K(Y, V)$, and so, by bounded monotone completeness at $E_0$, $\bar{\mu}_V^{-1}(E_0) \supset S(Y, V)$.

Since $\bar{\mu}_V^{-1}(E_0)$ is a boundedly monotone order $\sigma$-closed (since $\bar{\mu}_V$ is order $\sigma$-continuous) subspace of $B'(Y, V)$, and $S(Y, V)$ is a vector sub-lattice of $B'(Y, V)$, $\bar{\mu}_V^{-1}(E_0) \supset S(Y, V) = \beta'(Y, V)$. Thus $\bar{\mu}_V: \beta'(Y, V) \to E (E \supset E_0)$ is a positive, linear, and monotone order $\sigma$-continuous map, and for any increasing net $\{f_a\} \subset K(Y, V)$ with $\sup f_a = f \in \beta'(Y, V)$, $\bar{\mu}_V(f) = \sup \mu_V(f_a)$. Now define $\bar{\mu}: \beta'(Y) \to E$ as: For any $f \in \beta'(Y)$, $f \in \beta'(Y, V)$ for some open relatively compact subset $V$ of $Y$. We define $\bar{\mu}(f) = \bar{\mu}_V(f)$. To see that this mapping is well-defined, let $f \in \beta'(Y, V_i)$ $(i = 1, 2)$; this means $f \in \beta'(Y, V_1 \cap V_2)$. Since $\bar{\mu}_V = \bar{\mu}_V$ on $K(Y, V_1 \cap V_2)$, they are equal on $S'(Y, V_1 \cap V_2)$ and so are equal on $\beta'(Y, V_1 \cap V_2) = \beta'(Y, V_1) \cap \beta'(Y, V_2)$ (using $\sigma$-continuity of these measures). This proves $\bar{\mu}$ is well-defined. Also it is easily verified that $\mu$ is linear, positive, monotone order $\sigma$-continuous, and for any increasing net $\{f_a\}$ in $K(Y)$ with $\sup f_a = f \in \beta'(Y)$, $\bar{\mu}(f) = \sup \mu(f_a)$. Uniqueness of $\bar{\mu}$ is easily verified. Also the case when $E$ is boundedly monotone $\sigma$-complete can be proved in a similar way. This completes the proof.
Remark. For compact $Y$, this result is proved in [6] by an entirely different method.

REFERENCES