A TRANSFORMATION FORMULA FOR PRODUCTS ARISING IN PARTITION THEORY¹
M. V. SUBBARAO AND V. V. SUBRAHMANYASASTRI²

Abstract. We obtain a transformation formula involving Euler products. The formula can be utilized to obtain a large variety of partition-theoretic identities.

1. A transformation formula. Let \( f(a, x) \) be the product given by

\[
(1.1) \quad f(a, x) = \prod_{n=1}^{\infty} (1 - \alpha(n)x^n)^{g(n)/n},
\]

where \( \alpha(n), g(n) \) are totally multiplicative functions of \( n \) (that is, \( \alpha(mn) = \alpha(m)\alpha(n), g(mn) = g(m)g(n) \) for all positive integers \( m \) and \( n \)). Then we shall prove in this note that

\[
(1.2) \quad \prod_{\omega = 0}^{k-1} f(a, \omega^k x) = \prod_{d|k} \prod_{\delta(k,d)} f(a^{(k/d)\omega(d^k)}, x^k)^{g/(d^k)\mu(\delta)},
\]

\( \omega \) being a primitive \( k \)-th root of unity.

This result is a generalization of the identity proved earlier in [3]:

\[
(1.3) \quad \prod_{\omega = 0}^{k-1} \phi(\omega^k x) = \prod_{d|k} \phi(x^{kd})^{\sigma(kd\omega(d))},
\]

where

\[
(1.4) \quad \phi(x) = \prod_{n=1}^{\infty} (1 - x^n),
\]

and \( \sigma(n) \) denotes the sum of the positive divisors of \( n \). This is an important tool in deriving partition-theoretic identities such as the celebrated Ramanujan identity.

Received by the editors on June 17, 1974, and in revised form on November 4, 1974.

¹This research was supported in part by National Research Council Grant #3103.
²On leave from Sri Venkateswara University, India.

Key words and phrases: Euler and Jacobi identities, primitive roots of unity.
\[
\sum_{n=0}^{\infty} p(5n + 4)x^n = 5\{\phi(x^5)\}^5/\{\phi(x)\}^6,
\]

\(p(n)\) denoting as usual the number of unrestricted partitions of \(n\).

The result (1.3) is easy when \(k\) is a prime and was noted by Kolberg [1], while the proof of (1.3) for general values or \(k\) was given by Subrahmanyasastri [3] by using multiplicative induction of \(k\).

Many partition functions have generating functions of the form (1.1). For example,

(1) When \(g(n) = n^2, a = 1, f(a, x)^{-1}\) generates the plane partitions, for which an asymptotic formula was obtained by Wright [5].

(2) When \(g(n) = n \min(k, n), a = 1, f(a, x)^{-1}\) generates \(p^{(k)}(n)\), the number of \(k\)-rowed partitions of \(n\). In this case \(g(n)\) is not a totally multiplicative function. However, the \(f(a, x)\) in this case can be related to the function for which \(g(n) = n, and n\) is a totally multiplicative function. Whenever the generating function is related to an \(f(a, x)\) with a totally multiplicative \(g(n)\), the formula (1.2) will be useful.

(3) When

\[
g(n) = \begin{cases} 
n^2, & \text{if } n = 2^\alpha, \alpha \geq 0 \\
0, & \text{otherwise,}
\end{cases}
\]

and \(a = 1\), we have a simple and interesting case. Here \(g(n)\) is totally multiplicative and \(f(a, x)^{-1}\) generates \(P(n)\), the number of partitions of \(n\) into powers of 2 (including 1), with each summand occurring at most in as many different colors as the magnitude of the summand, with repetitions allowed. That is, \(n\) has representations of the form

\[
n = \sum_{\alpha=0}^{\infty} \sum_{j=1}^{2^\alpha} a_{\alpha j}(2^\alpha)_{j}, \quad (a_{\alpha j} \geq 0),
\]

\(a_{\alpha j}\) denoting the multiplicity of the summand \(2^\alpha\) in the color \(j\). The notion of partitions with summands occurring in different colors goes back to MacMahon [2]. We can also interpret \(P(n)\) as the number of weighted partitions into summands \(2^\alpha (\alpha \geq 0)\), where the weight of the summand \(2^\alpha\) (of multiplicity \(a_\alpha\)) in a partition of

\[
n = \sum_{\alpha=i_1} a_\alpha 2^\alpha
\]

is to be taken as (the binomial coefficient)
In other words,

\[ P(n) = \sum \left( \frac{2^{i_1} + a_{i_1} - 1}{a_{i_1}} \right) \left( \frac{2^{i_2} + a_{i_2} - 1}{a_{i_2}} \right) \cdots \left( \frac{2^{i_r} + a_{i_r} - 1}{a_{i_r}} \right), \]

the summation being over all those non-negative integers \( i_r \) and \( a_{i_r} \) for which \( n = 2^{i_1} + a_{i_1} 2^{i_2} + \cdots + a_{i_r} 2^{i_r} + \cdots \).

To illustrate the applications of (1.2) we shall derive the following simple partition-theoretic identities for \( p(n), p^{(3)}(n) \) and \( P(n) \).

(A) In the case \( g(n) = n, a = 1, k = 4 \), we derive

\[
\sum_0^\infty p(4n)x^{2n} = \frac{1}{2} \frac{\phi(x^2)}{\phi^3(x)} A_1(x) + \frac{1}{2} \frac{\phi^3(x)\phi^4(x^4)}{\phi^8(x^2)} A_2(x),
\]

(1.5)

\[
\sum_0^\infty p(4n + 1)x^{2n} = \frac{1}{2} \frac{\phi(x^2)\phi(x^4)}{\phi^3(x)} A_3(x) + \frac{1}{2} \frac{\phi^3(x)\phi^4(x^4)\phi(x^{24})}{\phi^8(x^2)} A_4(x),
\]

(1.6)

\[
\sum_0^\infty p(4n + 2)x^{2n} = \frac{1}{2} \frac{\phi(x^2)\phi(x^4)}{\phi^3(x)} A_1(x) - \frac{1}{2} \frac{\phi^3(x)\phi^4(x^4)\phi(x^{24})}{\phi^8(x^2)} A_2(x),
\]

(1.7)

and

\[
\sum_0^\infty p(4n + 3)x^{2n + 1} = \frac{1}{2} \frac{\phi(x^2)\phi(x^4)}{\phi^3(x)} A_3(x) - \frac{1}{2} \frac{\phi^3(x)\phi^4(x^4)\phi(x^{24})}{\phi^8(x^2)} A_4(x),
\]

(1.8)

where

\[ A_1(x) = \prod_{m=1}^\infty \frac{1}{(1 + x^{2^{m-1}})(1 + x^{2^{m-1}})} \]
\[ -x \prod_{m=1}^{\infty} \left( 1 + x^{2 \cdot 4m - 19} \right) \left( 1 + x^{2 \cdot 4m - 5} \right), \]

\[ A_2(x) = \prod_{m=1}^{\infty} \left( 1 - x^{2 \cdot 4m - 13} \right) \left( 1 - x^{2 \cdot 4m - 11} \right) \]

\[ + x \prod_{m=1}^{\infty} \left( 1 - x^{2 \cdot 4m - 19} \right) \left( 1 - x^{2 \cdot 4m - 5} \right), \]

\[ A_3(x) = \prod_{m=1}^{\infty} \left( 1 + x^{2 \cdot 4m - 17} \right) \left( 1 + x^{2 \cdot 4m - 7} \right) \]

\[ - x^2 \prod_{m=1}^{\infty} \left( 1 + x^{2 \cdot 4m - 23} \right) \left( 1 + x^{2 \cdot 4m - 1} \right), \]

and

\[ A_4(x) = \prod_{m=1}^{\infty} \left( 1 - x^{2 \cdot 4m - 17} \right) \left( 1 - x^{2 \cdot 4m - 7} \right) \]

\[ - x^2 \prod_{m=1}^{\infty} \left( 1 - x^{2 \cdot 4m - 23} \right) \left( 1 - x^{2 \cdot 4m - 1} \right). \]

(B) In the case \( g(n) = n \min (3, n), a = 1 \), we derive

\[
\sum_{n=0}^{\infty} p^{(3)}(3n)x^{3n} = \frac{\phi^3(x^3)\phi^6(x)}{\phi^{12}(x^3)} (1 + 2x^3) + \frac{6x\phi^6(x^3)\phi^3(x)}{\phi^{12}(x^3)} (1 + 2x^3) + \frac{9x^2\phi^6(x^3)}{\phi^{12}(x^3)} (1 - 2x + 2x^3 - x^4),
\]

\[
(1.9)
\]

\[
\sum_{n=0}^{\infty} p^{(3)}(3n + 1)x^{3n+1} = \frac{-x\phi^3(x^3)\phi^6(x)}{\phi^{12}(x^3)} (2 + x^3) + \frac{3x\phi^6(x^3)\phi^3(x)}{\phi^{12}(x^3)} (1 - 4x + 2x^3 - 2x^4) + \frac{9x^2\phi^6(x^3)}{\phi^{12}(x^3)} (1 - 2x + 2x^3 - x^4),
\]

\[
(1.10)
\]
\[
\sum_{n=0}^{\infty} p^{(3)}(3n + 2)x^{3n+2} = \frac{-3x^{2}\phi^0(x^9)\phi^3(x)}{\phi^{12}(x^3)}(2 + x^3)
\]

\[
+ \frac{9x^2\phi^8(x^9)}{\phi^{12}(x^3)}(1 - 2x + 2x^3 - x^4).
\]

Incidentally, we note from (1.11) that

\[
(1.12) \quad p^{(3)}(3n + 2) \equiv 0 \text{ (mod 3)}.
\]

(C) In the case

\[
g(n) = \begin{cases} 
n^2, & \text{if } n = 2^a, a \geq 0, \\
0, & \text{otherwise} \end{cases}, \quad a = 1,
\]

we derive

\[
(1.13) \quad \left( \sum_{0}^{\infty} P(4n)x^{4n} \right)x = \left( \sum_{0}^{\infty} P(4n + 1)x^{4n+1} \right)
\]

\[
= \frac{x(1 + 3x)}{f(x)(1 + x^2)^3(1 + x)}
\]

and

\[
(1.14) \quad \left( \sum_{0}^{\infty} P(4n + 2)x^{4n+2} \right)x = \left( \sum_{0}^{\infty} P(4n + 3)x^{4n+3} \right)
\]

\[
= \frac{x^3(3 + x^4)}{f(x)(1 + x^2)^3(1 + x)}. \]

2. Proof of the formula (1.2). We require the following

\textbf{Lemma 2.1.} Let \(A\) be any set of positive integers and \(F(k, n)\) any arithmetic function with values in the complex number field. Then for every positive integer \(k\)

\[
(2.1) \quad \prod_{n \in A \atop \gcd(n, k) = 1} F(k, n) = \prod_{d \mid k} \prod_{m \in A \atop \gcd(md, k) = 1} \{F(k, md)\}^{\mu(d)},
\]

where \(\mu(d)\) is the Mőbius function.

This is easily proved using the Mőbius inversion formula by setting \(L(k, n) = \log F(k, n)\) (the principal value), \(\sum_{n \in A \atop \gcd(n, \kappa) = d} L(k, n) = G(k/d)\), \(\sum_{md \in A} L(k, nd) = H(k/d)\) and noting that \(\sum_{d \mid k} G(k/d) = H(k)\).
Proof of (1.2). Left side of (1.2) =

\[
\prod_{d|k} \prod_{n=1}^{\infty} \prod_{r=0}^{k-1} (1 - a^{\alpha(n)} \omega^{rn} x^n)^{g(n)/n} = \prod_{d|k} \prod_{n=1}^{\infty} F(k_1, n_1)
\]

with \(k = k_1d\), \(n = n_1d\) and

\[
F(k_1, n_1) = \prod_{r=0}^{k_1-1} (1 - a^{\alpha(n_1)} \omega^{r \cdot n_1 \cdot d} \cdot x^{n_1 \cdot d})^{g(n_1 \cdot d)/n_1 \cdot d}
\]

where \(\omega_1 = \omega^d\), a primitive \(k_1\)-th root of unity. \(\omega_2 = \omega_1^{n_1}\) is also a primitive \(k_1\)-th root of unity, and as \(r\) runs through a complete residue system mod \(k_1\) once, it runs through a complete residue system (mod \(k_1\)) \(d\) times. Hence

\[
F(k_1, n_1) = \prod_{r=0}^{k_1-1} (1 - a^{\alpha(n_1)} \omega_2^{r \cdot n_1 \cdot d} \cdot x^{n_1 \cdot d})^{g(n_1 \cdot d)/n_1 \cdot d}
\]

so that by Lemma 2.1

\[
\prod_{n=1}^{\infty} F(k_1, n_1) = \prod_{\delta|k_1} \prod_{m=1}^{\infty} (1 - a^{\alpha(m \cdot d)} \cdot x^{k_1 \cdot n_1 \cdot d})^{g(m \cdot d)/m \cdot d}.
\]

Substituting this in (2.2) and using the fact that \(\alpha(n)\) and \(g(n)\) are totally multiplicative, (1.2) follows.

Corollary. In the case \(a = 1\), (1.2) takes the form

\[
(2.3) \quad \prod_{r=0}^{k-1} f(\omega^r x) = \prod_{\delta|k} \prod_{\delta|k} \prod_{\delta|k} (f(x^{k \cdot \delta}))(h(k \cdot \delta, \delta, \delta | \delta), \delta)
\]

where

\[
(2.4) \quad f(x) = \prod_{n=1}^{\infty} (1 - x^n)^{g(n)/n}
\]

and \(h(m) = \sum_{d|m} g(d)\).
We shall give a simple alternate proof in this case. It is well known ([4], theorem 5, special case) that, if \( h(n) = \sum_{d|n} g(d) \), then

\[
(2.5) \quad h(kM) = \sum_{d|k,d|M} h(kd)h(Md)g(d)\mu(d).
\]

We also recall that

\[
(2.6) \quad \eta(k, m) \equiv \sum_{r=0}^{k-1} \omega^m = \begin{cases} 0, & k \not| m \\ k, & k \mid m \end{cases}
\]

and that

\[
(2.7) \quad \sum_{m=1}^{\infty} \frac{g(m)x^m}{1 - x^m} = \sum_{i=1}^{\infty} h(i)x^i.
\]

From (2.5) and (2.6), we have

\[
\sum_{m=1}^{\infty} h(m)\eta(k, m)x^m = k \sum_{M=1}^{\infty} h(kM)x^{kM}
\]

\[
= \sum_{M=1}^{\infty} \sum_{d|k,d|M} kh(kd)\mu(d)g(d)h(n)x^{kM}
\]

\[
= \sum_{d|k} kh(kd)\mu(d)g(d) \sum_{n=1}^{\infty} h(n)x^{kd_n},
\]

which on using (2.6) and (2.7) can be written as

\[
\sum_{r=0}^{k-1} \sum_{m=1}^{\infty} \frac{g(m)\omega^mx^m}{1 - x^m\omega^m} = \sum_{d|k} h(kd)\mu(d)g(d) \sum_{n=1}^{\infty} \frac{g(n)x^{kn}}{1 - x^{kd_n}}.
\]

We now restrict \( x \) to be such that \( 0 < x < 1 \) (we can at the end extend the result to \( |x| < 1 \) by analytic continuation). Dividing both sides by \( x \) and integrating with respect to \( x \), we obtain

\[
\sum_{r=0}^{k-1} \sum_{m=1}^{\infty} \frac{g(m)}{m} \log (1 - \omega^mx^m)
\]

\[
= \sum_{d|k} h(kd)\mu(d) \frac{g(d)}{d} \sum_{n=1}^{\infty} \frac{g(n)}{n} \log (1 - x^{kn}),
\]

the constant of integration being zero as can be seen by setting \( x = 0 \). Thus we have
\[
\sum_{r=0}^{k-1} \log f(\omega^r x) = \sum_{d \mid k} h(k/d) \mu(d) \frac{g(d)}{d} \log f(x^d),
\]

which is the same as relation (2.3).

3. Proof of the identities (1.5) to (1.8). Choosing \( g(n) = n, a = 1, k = 4, \) (1.2) yields

\[
(3.1) \quad \phi(x)\phi(ix)\phi(-x)\phi(-ix) = \frac{\phi^7(x^4)}{\phi^7(x^8)},
\]

\( i \) being an imaginary square root of \(-1\). Also

\[
(3.2) \quad 4 \sum_0^\infty p(4n + \ell) x^{in + \frac{1}{2}} = \frac{1}{\phi(x)} + \frac{i^{-\ell}}{\phi(ix)}
\]

\[
+ \frac{i^{-2\ell}}{\phi(-x)} + \frac{i^{-3\ell}}{\phi(-ix)} \quad (\ell = 0, 1, 2, 3).
\]

We shall also need the well-known identity of Jacobi:

\[
(3.3) \quad \sum_{k=-\infty}^\infty y^k z^k = \phi(z^2) \prod_{m=1}^\infty (1 + yz^{2m-1})(1 + y^{-1}z^{2m-1}).
\]

Using Euler's identity

\[
(3.4) \quad \phi(x) = \sum_{-\infty}^\infty (-1)^n x^{3n + 1/2},
\]

we can write

\[
(3.5) \quad \phi(x) = g_0(x) + g_1(x) + g_2(x) + g_3(x),
\]

where

\[
(3.6) \quad n(3n + 1)/2 \equiv \ell(\text{mod } 4).
\]

Then

\[
\phi(-x) = g_0(-x) + g_1(-x) + g_2(-x) + g_3(-x)
\]

\[
= g_0(x) - g_1(x) + g_2(x) - g_3(x),
\]

in view of (3.6), so that on using (3.6) and (3.4),
\[ \phi(x) + \phi(-x) = 2\{g_0(x) + g_2(x)\} \]

\[ = 2 \sum_{k=0}^{\infty} x^{2k+12k+1} - 2 \sum_{k=0}^{\infty} x^{(4k+1)(6k+2)} \]

\[ = 2\phi(x^{18}) \left\{ \prod_{1}^{\infty} (1 + x^{18m-26})(1 + x^{18m-22}) \right\} \]

\[ - x^2 \prod_{1}^{\infty} (1 + x^{18m-38})(1 + x^{18m-10}) \}

\[ = 2\phi(x^{18})A_1(x^2). \]

Hence

\[ \phi(ix) + \phi(-ix) = 2\phi(x^{18})A_1((ix)^2) \]

\[ = 2\phi(x^{18})A_1(-x^2) \]

\[ = 2\phi(x^{18})A_2(x^2) \]

in terms of \(A_1(x)\) and \(A_2(x)\) given in (A) of §1.

With \(g(n) = n, k = 2, a = 1\) (1.2) yields

\[ \phi(x)\phi(-x) = \frac{\phi^3(x^2)}{\phi(x^4)}, \]

and so, from (3.1),

\[ \phi(ix)\phi(-ix) = \frac{\phi^3(x^4)}{\phi^5(x^8)\phi^3(x^2)}. \]

Hence, from (3.2) and (3.7) to (3.10), we obtain

\[ \sum_{n=0}^{\infty} p(4n)x^n = \frac{1}{4} \left\{ \frac{\phi(x) + \phi(-x)}{\phi(x)\phi(-x)} + \frac{\phi(ix) + \phi(-ix)}{\phi(ix)\phi(-ix)} \right\} \]

\[ = \frac{1}{2} \frac{\phi(x^4)}{\phi^3(x^2)} \phi(x^{18})A_1(x^2) + \frac{1}{2} \frac{\phi^3(x^2)\phi^3(x^8)}{\phi^5(x^4)} \phi(x^{18})A_2(x^2), \]

which is the same as (1.5). (1.6) to (1.8) follow on similar lines.

4. Proof of the identities (1.9) to (1.11). The generating function \(\psi(x)^{-1}\) of \(p^{(3)}(n)\) is given by

\[ \psi(x) = \prod_{n=1}^{\infty} (1 - x^n)^{\min(3,n)} \]
If $\omega$ is a primitive cube root of unity, then

\[
3 \sum_{n=0}^{\infty} p^{(3)(3n+1)} x^{3n+2} = \frac{1}{\psi(x)} + \frac{\omega^{2x}}{\psi(\omega x)} + \frac{\omega^x}{\psi(\omega^2 x)} = f(x) + f(x) + \omega f(x),
\]

for $\ell = 0, 1, 2$.

Also,

\[
\phi^{(3)}(x) = h_0(x) + h_1(x) + h_2(x)
\]

with

\[
h_z(x) = \sum_{n=0}^{\infty} (-1)^n (2n+1)x^{n+1} y^z.
\]

so that

\[
(4.4) \quad \phi^{(3)}(\omega^z x) = h_0(x) + \omega^z h_1(x) + \omega^{2z} h_2(x).
\]

In fact,

\[
h_0(x) = \phi^{(3)}(x) + 3x\phi^{(2)}(x),
\]

\[
h_1(x) = -3x\phi^{(2)}(x),
\]

\[
h_2(x) = 0 \quad \text{(See Kolberg [1] p. 82)}.
\]

From (4.1) and (1.3), we obtain

\[
(4.5) \quad \psi(x)\psi(\omega x)\psi(\omega^2 x) = \frac{\phi^{12}(x)}{\phi^{12}(x)} = \frac{\phi^{12}(x^3)(1 - x^3)(1 - x^6)}{\phi^{12}(x^3)(1 - x^3)(1 - x^6)}.
\]

Further, from (4.1) and (4.4) we obtain

\[
\psi(\omega x)\psi(\omega^2 x) = \frac{(h_0^2(x) + h_1^2(x) - h_0(x)h_1(x))(1 - 2x + 2x^3 - x^4)}{(1 - x^3)(1 - x^6)}
\]

and similar expressions for $\psi(\omega^2 x)\psi(x)$ and $\psi(x)\psi(\omega x)$.

Taking $\ell = 0$ in (4.2) and using (4.5) and the above expressions for $\psi(\omega x)\psi(\omega^2 x)$ etc., we obtain

\[
3 \sum_{n=0}^{\infty} p^{(3)(3n+1)} x^{3n+2} = 3(h_0^2(x)(1 + 2x^3) - h_1^2(x)(2x + x^4) \frac{\phi^{12}(x^3)}{\phi^{12}(x^3)}).
\]
which yields (1.9) on substituting the above Kolberg’s expressions for $h_0(x)$ and $h_1(x)$. (1.10) and (1.11) follow on the same lines.

5. Proof of (1.13) and (1.14). Choosing

$$g(n) = \begin{cases} n^2, & \text{if } n = 2^a (\alpha \geq 0), \\ 0, & \text{otherwise} \end{cases}, \text{ and } a = 1, k = 4,$$

we have from (1.2)

$$(5.1) \quad f(x)f(ix)f(-x)f(-ix) = \frac{f^2(x^2)}{f^{10}(x^8)}.$$ 

We can also verify that in this case $f(x)$ satisfies

$$(5.2) \quad (1 - x)f^2(x^2) = f(x)$$

or

$$f^2(x^2) = f(x)(1 + x + x^2 + \cdots),$$

so that if we put $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $(a_0 = 1)$, it is easily seen that the coefficients $a_n$ are given by the recursion formulae

$$(5.3) \quad a_n = - \sum_{r=0}^{n-1} a_r, \quad \text{if } n \text{ is odd},$$

and

$$(5.4) \quad a_n = \sum_{j=0}^{n/2} a_j a_{n/2-j}, \quad \text{if } n \text{ is even}.$$ 

These equations (5.3) and (5.4) determine $f(x)$. However, these are not required for the proof of (1.13) and (1.14).

We shall indicate the proof of the first half of (1.13). First we note that

$$(5.5) \quad 4 \sum_{\ell=0}^{\infty} P(4n + \ell)x^{n+\ell} = \frac{1}{f(x)} + \frac{i^{-\ell}}{f(ix)} + \frac{i^{-2\ell}}{f(-x)} + \frac{i^{-3\ell}}{f(-ix)}$$ \quad (\ell = 0, 1, 2, 3).$$

From (5.2) we have $f(-x) = (1 + x)f^2(x^2)$, so that

$$(5.6) \quad f(x) + f(-x) = 2f^2(x^2),$$

$$(5.7) \quad f(x)f(-x) = f^4(x^2)(1 - x^2),$$

and

$$(5.8) \quad \frac{f(-x^2)}{f(x^2)} = \frac{(1 + x^2)}{(1 - x^2)}.$$
Taking $t = 0$ in (5.5), and using (5.6), (5.7), (5.1) and similarly (5.8) we obtain

\[
\sum P(4n)x^{in} = \frac{1}{4} \left\{ \frac{f(x) + f(-x)}{f(x)f(-x)} + \frac{f(ix) + f(-ix)}{f(ix)f(-ix)} \right\}
\]

\[
= \frac{1}{4} \left( \frac{(f(x) + f(-x))f(ix)f(-ix) + (f(ix) + f(-ix))f(x)f(-x)}{f(x)f(ix)f(-x)f(-ix)} \right)
\]

\[
= \frac{f^{10}(x^8)}{2f^{21}(x^4)} \left\{ f^2(x^2)f^4((ix)^2)(1 - i^2x^2) \right\}
\]

\[
+ f^2((ix)^2f^4(x^2)(1 - x^2) \right\}
\]

\[
= \frac{f^{10}(x^8)}{2f^{21}(x^4)} f^6(x^2) \frac{(1 + x^2)^2}{(1 - x^2)^4} \left\{ (1 + x^2)^3 + (1 - x^2)^3 \right\}.
\] 

(5.9)

But by repeated use of (5.2) raised to the suitable exponents, we obtain

\[
\frac{f^{10}(x^8)f^{10}(x^2)}{f^{21}(x^4)} \frac{f^3(x^2)}{f^3(x^4)} \frac{f^{16}(x^4)}{f^2(x^2)} \frac{1}{f(x)} = \frac{(1 - x^2)^8}{(1 - x^4)^5} \frac{(1 - x)}{f(x)} = \frac{(1 - x^2)^3(1 - x)}{(1 + x^2)^5f(x)}
\].

The first half of the identity (1.13) follows on substituting this in (5.9). The other half of (1.13) and (1.14) follow on using similar arguments.

**References**

3. V. V. Subrahmanyasastri, *A result concerning the Euler function $f(x) = \prod_n (1 - x^n)$*, Math. student 35 (1967), 85-87.