SOME DISCRETE SUBSPACES OF $\beta m$

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ABSTRACT. By considering some discrete subspaces of the Stone-Čech compactification $\beta m$ of a discrete space, we show that a nondiscrete door space which is not maximal door can be embedded in $\beta m$ for every infinite discrete space $m$. This provides a counterexample to the converse of a theorem of Y. Kim. Maximal door spaces are characterized in terms of their embedding in $\beta m$.

By a space we shall mean a Hausdorff topological space. An infinite cardinal number $m$ and a discrete space of cardinality $m$ will be denoted by the same symbol, and $\beta m$ will represent its Stone-Čech compactification. The cardinality of a set $A$ will be denoted by $|A|$, $\text{Cl}_X A$ is the closure of $A$ in $X$, and $\mathbb{N}$ is the set of natural numbers. See [1] for a general reference.

A door space is a space in which every subset is either open or closed. A nondiscrete door space is called maximal door if the only finer door topology for the set is discrete. Kim [2] characterized nondiscrete door spaces and maximal door spaces as follows. A Hausdorff space $X$ is nondiscrete door (maximal door) if and only if $X = S \cup \{p\}$ where $S$ is an infinite discrete set and $p$ is a point such that the restriction of its neighborhoods to $S$ forms a filter (an ultrafilter) in $S$. Kim also showed that for every maximal door space $X$ there is a discrete space $m$ such that $X$ can be embedded in $\beta m$; and, furthermore $m$ may be taken to be $|X|$. He left open the question of whether every door space which can be embedded in $\beta m$ for some $m$ must be maximal door. We answer this question in the negative, and supply a stronger condition which does characterize maximal door spaces.

**Theorem 1.** For every infinite cardinal $m$ there exists a nondiscrete door space $X$ with $|X| > m$ so that $X$ can be embedded in $\beta m$, but $X$ is not maximal door. In particular, there is a nondiscrete door space of cardinality $2^{\aleph_0}$ which is not maximal door, but can be embedded in $\beta \aleph_0$.

**Proof.** We first construct for each infinite cardinal $m$ a certain discrete subspace of $\beta m$ which is of cardinal $n > m$. When $m = \aleph_0$, we have $n = 2^{\aleph_0}$. By [1; 12B] every $m$ can be taken to be the union of a

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collection \( \{ A_\alpha \mid \alpha \in I \} \) where \( |A_\alpha| = m \) for each \( \alpha \in I \), \( |I| > m \), and \( |A_\alpha \cap A_\beta| < m \) for \( \alpha \neq \beta \). It follows from [1; 6.9(a)] that for each \( \alpha \), \( \text{Cl}^\beta_m A_\alpha = \beta A_\alpha \), which is homeomorphic to \( \beta m \). By [1; 121] we may choose \( x_\alpha \in \text{Cl}^\beta_m A_\alpha \) such that every neighborhood of \( x_\alpha \) intersects \( A_\alpha \) in a set of cardinality \( m \). Now by [1; 6.9(c)] \( \text{Cl}^\beta_m A_\alpha \) is open in \( \beta m \) and is therefore a neighborhood of \( x_\alpha \). Suppose \( x_\beta \in \text{Cl}^\beta_m A_\alpha \) for \( \alpha \neq \beta \). Then \( x_\beta \in \text{Cl}^\beta_m A_\alpha \cap \text{Cl}^\beta_m A_\beta \) which must be a neighborhood of \( x_\beta \) in the relative topology on \( \text{Cl}^\beta_m A_\beta \). But \( A_\beta \cap \text{Cl}^\beta_m A_\alpha \cap \text{Cl}^\beta_m A_\beta = A_\alpha \cap A_\beta \) is a neighborhood of \( x_\beta \) in \( A_\beta \) which has cardinality less than \( m \), contradicting the choice of \( x_\beta \). Hence \( S = \{ x_\alpha \mid \alpha \in I \} \) is a discrete collection. In case \( m = \aleph_0 \) we may choose \( |I| = 2^{\aleph_0} \) by [1; 6Q.1].

For each \( p \in \text{Cl}^\beta_m S \setminus S \) we see that \( S \cup \{ p \} \) is a nondiscrete door space embedded in \( \beta m \). We now show that not every such \( S \cup \{ p \} \) can be maximal door. Suppose \( S \cup \{ p \} \) is maximal door for each \( p \in \text{Cl}^\beta_m S \setminus S \). Consider the extension \( f : \beta S \rightarrow \text{Cl}^\beta_m S \) of the inclusion map of \( S \) into \( \text{Cl}^\beta_m S \). We shall now show that \( f \) is one-to-one and onto, and hence a homeomorphism.

If \( p \in \text{Cl}^\beta_m S \setminus S \), then \( p \) is a cluster point of the ultrafilter \( \mathcal{G} \) of the restrictions of its neighborhoods to \( S \). This ultrafilter is a z-ultrafilter, and so has a unique limit \( x \) in \( \beta S \), and \( f(x) = p \).

On the other hand, \( \mathcal{G} \) is the only ultrafilter in \( S \) of which \( p \) is a cluster point. For suppose \( \mathcal{G} \) is a filter in \( S \) which clusters to \( p \) in \( S \cup \{ p \} \). Then each element of \( \mathcal{G} \) intersects \( S \), and \( \mathcal{G} \) must be in \( \mathcal{G} \) for each \( G \in \mathcal{G} \). But each \( x \in \beta S \setminus S \) is the limit of an ultrafilter in \( S \), so \( f(x) = p \) for only one \( x \in \beta S \setminus S \).

Thus \( \text{Cl}^\beta_m S = \beta S \). But \( |\text{Cl}^\beta_m S| \leq |\beta m| < |\beta n| = |\beta S| \). Hence there must exist some \( p \in \text{Cl}^\beta_m S \setminus S \) such that \( S \cup \{ p \} \) is not maximal door.

A subset of \( S \) of \( \beta m \) is said to be strongly discrete if for each \( s \in S \) there is a neighborhood \( U_s \subset \beta m \) of \( s \) such that if \( s \neq t \), then \( U_s \cap U_t \cap m = \emptyset \). This definition is equivalent to that in [3].

**Theorem 2.** A nondiscrete door space \( S \cup \{ p \} \) is maximal door if and only if it can be embedded in some \( \beta m \) in such a way that \( S \) is strongly discrete.

**Proof.** Kim [2] showed that every maximal door space could be embedded in such a way.

Suppose, then, that \( S \cup \{ p \} \) can be embedded in \( \beta m \) for some \( m \) and that \( S \) is strongly discrete in \( \beta m \). Let \( f \) be a continuous function from \( S \) to \([0, 1]\). We shall show that \( f \) can be extended to \( \beta m \), so that \( \text{Cl}^\beta_m S = \beta S \). For each \( s \in S \) let \( U_s \) be a neighborhood of \( s \) so that the \( U_s \)'s illustrate that \( S \) is strongly discrete. Extend \( f \) to \( S \cup m \) by defining \( f(x) = f(s) \) if \( x \in U_s \), and \( f(x) = 0 \) otherwise. Now \( f \) is a continuous
function on \( m \), and hence it has a unique extension \( F \) to \( \beta m \). But \( m \) is dense in \( S \cup m \), and \( f \) and \( F \) agree on \( m \). Thus \( f \) and \( F \) must agree on all of \( S \cup m \) and, in particular, on \( S \). Therefore \( S \) is \( C^* \) embedded in \( \text{Cl}_{\beta m} S \), so \( \text{Cl}_{\beta m} S = \beta S \) [1; 6.9]. Now \( p \in \beta S \setminus S \) and, since \( S \) is discrete, the unique \( z \)-ultrafilter \( \mathcal{G} \) in \( S \) which converges to \( p \) is an ultra-filter of open subsets of \( S \). Thus \( F \cup \{ p \} \) is open in \( S \cup \{ p \} \) for each \( F \in \mathcal{G} \), and \( \mathcal{G} \) is exactly the restriction to \( S \) of the neighborhoods of \( p \). Hence \( S \cup \{ p \} \) is maximal door.

In light of Theorem 1 it is natural to ask whether a countable non-discrete door space which is embedded in \( \beta m \) must be maximal door. Theorem 4 gives an affirmative answer to this question.

**Lemma 3.** Every countable discrete subset of \( \beta m \) is strongly discrete.

**Proof.** Let \( S = \{ x_i \mid i \in \mathbb{N} \} \) be a countable discrete subset of \( \beta m \), and let \( U_i \) be an open set which contains \( s_i \) but not \( s_j \) for \( i \neq j \). By the regularity of \( \beta m \), for each \( i \) there is an open set \( V_i \) such that \( s_i \in V_i \subset \text{Cl}_{\beta m} V_i \subset U_i \).

We now define a neighborhood \( W_i \) for each \( i \) so that \( \{ W_i \mid i \in \mathbb{N} \} \) illustrates that \( S \) is strongly discrete. Let \( W_1 = V_1 \), and for each \( i > 1 \), let \( W_i = V_i \setminus \bigcup_{j=1}^{i-1} \text{Cl}_{\beta m} V_j \). It is clear that each \( W_i \) is an open neighborhood of \( x_i \), and that \( W_i \cap W_j = \emptyset \) when \( i \neq j \).

**Theorem 4.** Every countable nondiscrete door space which can be embedded in \( \beta m \) is a maximal door space.

**Proof.** This follows directly from Theorem 2 and Lemma 3.

**References**


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