ENTIRE FUNCTIONS WITH PRESCRIBED ASYMPTOTIC BEHAVIOR

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Abstract. A sufficient condition for a canonical product to be of bounded index is given, from which most of the well known results can be obtained as easy corollaries. Let \( f \) be an entire function of exponential type with order \( \rho \) and lower order \( \lambda \). If \( \rho - \lambda < 1 \) then there exists an entire function \( g \) of bounded index such that \( \log M(r,f) \sim \log M(r,g) \). This solves a conjecture of S. M. Shah except for the extremal case of \( \rho = 1 \) and \( \lambda = 0 \).

1. Introduction. An entire function \( f(z) \) is said to be of bounded index if there exists a non-negative integer \( N \) such that

\[
\max_{0 \leq i \leq N} \left\{ \frac{|f^{(i)}(z)|}{i!} \right\} \leq \frac{|f^{(n)}(z)|}{n!} \quad \text{for all } n \text{ and all } z.
\]

The least such integer \( N \) is called the index of \( f \) (see [4]).

It is well known that a canonical product having geometrically increasing zeros is of bounded index. We now prove a strong generalization of this result.

Theorem 1. Let \( f(z) = \prod_{j=1}^{\infty} (1 + \{z/t_j\}^q) \) be an entire function with \( t_j \in \mathbb{C} \setminus \{0\} \), \( q_j \in \mathbb{N} \) and \( \sum_{j=1}^{\infty} (q_j/|t_j|) < \infty \). If \( \sum_{j=n}^{\infty} 1/|t_n - t_j| = o(1) \) as \( n \to \infty \), then \( f \) is of bounded index.

The condition \( \sum_{j=1}^{\infty} (q_j/|t_j|) < \infty \) can be replaced by \( \limsup_{j \to \infty} (Q_j/|t_j|) < \infty \), where \( Q_j = \sum_{i=1}^{j} q_i \), provided \( f \) is entire, i.e., the infinite product converges uniformly on every bounded region. However let us remark that the conditions in Theorem 1 are only sufficient and not necessary, (see [2], Theorem 3).

As a direct consequence we obtain the following result of B. S. Lee and S. M. Shah [3].

Corollary 2. Let \( f(z) = \prod_{n=1}^{\infty} (1 - z/a_n) \), where \( a_n \in \mathbb{R}^+ \) and \( (a_{n+1}/a_n) \geq \alpha > 1 \), then \( f \) is an entire function of bounded index.

In 1970 W. J. Pugh and S. M. Shah [5] showed that for any transcendental entire function \( f \) of finite order it is always possible to
find an entire function \( g \) of unbounded index such that
\[
\log M(r, f) \sim \log M(r, g) \quad (r \to \infty).
\]

In [6] S. M. Shah conjectured: If \( f \) is an entire function of exponential type then there exists an entire function \( g \) of bounded index such that \( \log M(r, f) \sim \log M(r, g) \). We now prove this conjecture for functions of exponential type with non-extremal asymptotic behavior.

**Theorem 3.** Let \( f \) be an entire function of exponential type with order \( \rho \) and lower order \( \lambda \). If \( \rho - \lambda < 1 \), then there exists an entire function \( g \) of bounded index such that
\[
N \left( r, \frac{1}{g} \right) \sim \log M(r, g) \sim \log M(r, f) \quad (r \to \infty).
\]

**Theorem 4.** Let \( \phi(t) \) be an increasing, positive function of \( t \geq 1 \) with \( \lim_{t \to \infty} \phi(t)/t < \infty \). If there exists an integer \( n > 0 \) such that \( \phi(t + 1) - \phi(t) \leq \phi(t)^{(n-1)/n} \) for \( t \) sufficiently large, then there exists an entire function \( f \) of bounded index such that
\[
N \left( r, \frac{1}{f} \right) \sim \log M(r, f) \sim \int_{1}^{r} \frac{\phi(t)}{t} dt \quad (r \to \infty).
\]

As a straightforward consequence we have

**Corollary 5.** Let \( 0 \leq \lambda \leq \rho \leq 1 \) be given. Then there exists an entire function \( f \) of bounded index and order \( \rho \) with lower order \( \lambda \).

2. **Proof of Theorem 1.** Let \( q \) be a positive integer and let \( t \) be a complex number with \( |t| > q \).

(i) Let \( z \in \mathbb{C} \) with \( |z| = |t| + a, a > 0 \). Then,
\[
\frac{|qz^{q-1}|}{|z|^q + |z|^q} \leq \frac{|qz|^{q-1}}{|z|^q - |t|^q} = \frac{q}{|z|(1 - (|t|/|z|)^q)}
\]
\[
\leq \frac{q}{(|t| + a)(1 - |t|^q/|t|^q + aq|t|^{q-1})}
\]
\[
= \frac{|t| + qa}{(|t| + q)a} \leq \frac{1}{a} + \frac{q}{|t| + a} = \frac{1}{a} + \frac{q}{|z|}.
\]

(ii) Let \( z \in \mathbb{C} \) with \( |t| = |z| + b, b > 0 \). Then,
\[
\frac{|qz^q - 1|}{|z^q + t^q|} \leq \frac{q|z|^q - 1}{(|z| + b)^q - |z|^q} \leq \frac{q|z|^q - 1}{bq|z|^q - 1} = \frac{1}{b}.
\]

(iii) Let \( z \in \mathbb{C} \) such that \(|t| - 1 < |z| \leq |t| + 1 \). Furthermore let \( a_1, \ldots, a_q \) denote the zeros of \( z^q + t^q \) then

\[
\frac{qz^q - 1}{z^q + t^q} = \sum_{n=1}^{\infty} \frac{1}{z - a_n}.
\]

Clearly there exists \( z' \) such that \(|z'| + 1 = |t| \) and \(|z' - z| \leq 2 \). Thus, by (iii), \( |1/(z' - a_n)| \leq 1 \).

Let \( d > 0 \) be given and let \(|z - a_n| \geq d \) for \( n = 1, 2, \ldots, q \). Obviously the length of the arc from \( a_n \) to \( a_{n+1} \) for the circle of radius \(|t|\) is exactly \( 2\pi |t|/q \). The distance of two points on a circle is at least the shortest arc length between those points divided by \( \pi \). Thus, by renumbering the \( a_i \)'s (so that \( a_1 \) is closest to \( z \)), we have,

\[
|z - a_n| \geq |a_n - a_2| \geq \frac{(n - 2)|t|}{q} \geq 2(n - 2) \text{ for } 2 < n \leq (q + 1)/2
\]

and

\[
|z - a_n| \geq |a_n - a_q| \geq \frac{(q - n)|t|}{q} \geq 2(q - n) \text{ for } (q + 1)/2 < n < q.
\]

The same result holds for \( z' \) and therefore,

\[
\left| \sum_{n=1}^{q} \frac{1}{z - a_n} \right| \leq \left| \sum_{n=1}^{q} \frac{z - z'}{(z - a_n)(z' - a_n)} \right| + \left| \sum_{n=1}^{q} \frac{1}{z' - a_n} \right|
\]

\[
\leq \sum_{n=1}^{q} \frac{2}{|z - a_n| |z' - a_n|} + 1
\]

\[
\leq \frac{6}{d} \sum_{n=3}^{q-1} \frac{2}{|z - a_n| |z' - a_n|} + 1
\]

\[
\leq \frac{6}{d} \sum_{j=1}^{(q+1)/2} \frac{4}{j^2} + 1
\]

\[
\leq \frac{6}{d} + \sum_{j=1}^{\infty} \frac{4}{j^2} + 1 = M.
\]

Obviously, \( M = M(d) \) is independent of \( t \) and \( q \).
We now proceed with the proof of Theorem 1.

For \( f(z) = \prod_{j=1}^{\infty} (1 + \{z/t_j\}^{q_j}) \) we have,

\[
\left| \frac{f'(z)}{f(z)} \right| = \left| \sum_{j=1}^{\infty} \frac{q_j z^{q_j-1}}{t_j^{q_j} + z^{q_j}} \right| \leq \sum_{j=1}^{\infty} \frac{q_j |z|^{q_j-1}}{|z^{q_j} + t_j^{q_j}|}.
\]

Let \( d > 0 \) be given and let \( z \in \mathcal{C} \) such that \( |z - b_i| \geq d \) for \( i = 1, 2, \ldots \), where the \( b_i \)'s denote the zeros of \( f \). There exists an integer \( n \geq 0 \) such that \( |t_n| \geq |z| < |t_{n+1}| \). Since \( \sum_{j=1}^{\infty} q_j |t_j| < \infty \), there exists an integer \( n_0 > 0 \) such that \( |t_j| > q_j \) for all \( j \geq n_0 \). Let \( \alpha_j = |t_j| - |z| \), then, by (i), (ii), and (iii), we have, for \( |z| \) sufficiently large,

\[
\left| \frac{f'(z)}{f(z)} \right| \leq \sum_{j=1}^{n_0} \frac{q_j |z|^{q_j-1}}{|z^{q_j} + t_j^{q_j}|} + \sum_{j=n_0+1}^{n-1} \frac{q_j |z|^{q_j-1}}{|z^{q_j} + t_j^{q_j}|} + 2(M + 1) + \sum_{j=n+2}^{\infty} \frac{q_j |z|^{q_j-1}}{|z^{q_j} + t_j^{q_j}|}
\]

(2.1)

\[
\leq K + \sum_{j=n_0+1}^{n-1} \left( \frac{1}{\alpha_j} + \frac{q_j}{|t_j| + \alpha_j} \right) + 2(M + 1) + \sum_{j=n+2}^{\infty} \frac{1}{\alpha_j}
\]

\[
\leq K + \sum_{j\neq n} \left| \frac{1}{|t_n| - |t_j|} \right| + \sum_{j=1}^{\infty} \frac{q_j}{|t_j|}
\]

\[
+ 2(M + 1) + \sum_{j\neq n+1} \left| \frac{1}{|t_n| - |t_j|} \right|
\]

\[
\leq T.
\]

Clearly \( f'(z)/f(z) \) is bounded for bounded \( z \) with \( |z - b_i| \geq d \) for all \( i \).

Therefore, for each \( d > 0 \) there exists a constant \( L > 0 \) such that \( |f'(z)| \leq L |f(z)| \), whenever \( |z - b_j| \geq d \) for all \( j \). Hence, by [2, Theorem 2], \( f \) is of bounded index.

Suppose we replace the condition \( \sum_{j=1}^{\infty} (q_j/|t_j|) < \infty \) by demanding the infinite product \( f \) to be entire and by \( \limsup_{z \to \infty} Q_j/|t_j| < \infty \), where \( Q_j = \sum_{i=1}^{j} q_n \). It is well known that \( f(z) \) of bounded index implies \( f(\alpha z) \) is also of bounded index for any \( \alpha \in \mathcal{C} \). Thus, without loss of generality we may assume \( \limsup_{z \to \infty} Q_j/|t_j| < 1 \). Hence, \( a_n \leq Q_n < |t_n| \) for \( n \) sufficiently large and therefore we can use the same argument to obtain the inequality (2.1). Since,
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we similarly obtain, \( f \) is of bounded index. q.e.d.

3. In this section we assume familiarity with the most elementary results and notations of Nevanlinna's theory of meromorphic functions.

For a transcendental entire function \( f \) we have,

\[
\log M(r, f) = \log M(r_0, f) + \int_{r_0}^{r} \frac{\Psi(t)}{t} \, dt \quad (r \geq r_0),
\]

where \( r_0 > 0 \) and \( \Psi(t) \) is a non-negative, non-decreasing function of \( t \).

A. Edrei and W. H. T. Fuchs [1] proved that given a positive, non-decreasing function \( \Phi(t) \) with \( \int_0^t \Phi(t) \, dt \leq r^K \) (for some \( K > 0 \) and \( r \) sufficiently large), then there exists an entire function \( g \) of finite order such that

\[
N \left( r, \frac{1}{g} \right) \sim \log M(r, g) \sim \int_1^{r} \frac{\Phi(t)}{t} \, dt \quad (r \to \infty).
\]

We will rely heavily on their construction of the function \( g(z) \).

**Proof of Theorem 3.** Let \( f \) be an entire function satisfying the hypothesis of Theorem 3. It is easy to see that the function \( \Psi(t) \) we obtain from \( f \) according to (3.1) can be replaced by the function \( \Phi(t) \) satisfying,

(i) \( \Phi(t) \) is continuous,

(ii) \( \Phi(1) = 0 \) and \( \Phi(t) \) is strictly increasing and unbounded,

(iii) \( \log M(r, f) \sim \Lambda(r) = \int_1^{r} \Phi(t) \, dt \quad (r \to \infty) \).

Let us now define the function \( B(r) \) by the condition \( B(r) = \Lambda(r)/\log r \) for \( r > 1 \) and \( B(1) = 0 \). Since \( B'(r) = \Phi(r) \log r - \Lambda(r)/r(\log r)^2 > 0 \) we have \( B(r) \) is continuous and strictly increasing. Furthermore, \( f \) transcendental implies \( B(r) \) is unbounded.

Let \( \eta \) be a fixed constant with \( 0 < \eta < 1/2 \) and define a sequence of positive numbers \( \{r_n\}_{n=1}^{\infty} \) by

\[
n = B^2 \eta(r_n) \log r_n \quad \text{for } n = 1, 2, \ldots.
\]

Since \( B^2 \eta(x) \log x \) is continuous, strictly increasing, and unbounded and since \( B^2 \eta(1) \log 1 = 0 \), the sequence \( \{r_n\}_{n=1}^{\infty} \) is uniquely determined, strictly increasing, and unbounded.

Now set

\[
k_j = \exp \left( \frac{j}{B^\eta(r_j)} \right) = \exp(B^\eta(r_j) \log r_j),
\]
and notice that the sequence \( \{k_j\}_{j=1}^{\infty} \) is increasing and unbounded, whereas the sequence \( \{k_j\}_{j=1}^{\infty} \) is decreasing and
\[
\lim_{j \to \infty} k_j = 1.
\]

Denoting by \([y]\) the greatest integer not exceeding \(y\), we define the sequence \( \{q_j\}_{j=1}^{\infty} \) by
\[
q_j = [2jk_1k_2k_3 \cdots k_j] + 1, \quad \text{for} \quad j = 1, 2, \ldots.
\]

It is easily shown that the \(q_j\)'s satisfy the four following relations:

(3.2) \[ q_j > k_j \geq \exp(\sqrt{j}) \quad (j \geq 1), \]

(3.3) \[ q_{j+1} > q_j \quad (j \geq 1), \]

(3.4) \[ \lim_{j \to \infty} \frac{q_{j+1}}{q_j} = 1, \]

(3.5) \[ \lim_{j \to \infty} \frac{q_j}{Q_j} = 0, \text{where} \ Q_j = \sum_{i=1}^{j} q_i. \]

Define the sequence \( \{t_j\}_{j=1}^{\infty} \) of positive, strictly increasing numbers by
\[
\Phi(t_j) = Q_j = \sum_{i=1}^{j} q_i \quad (j = 1, 2, \ldots).
\]

The existence and uniqueness of \( \{t_j\}_{j=1}^{\infty} \) is assured by (ii).

Set \( s_j = t_j + j(j - 1)/2 \) and define \( n_1(t) \) and \( n(t) \) by
\[
n_1(t) = \begin{cases} 
0 & \text{for} \ 0 \leq t < t_1 \\
Q_j & \text{for} \ t_j \leq t < t_{j+1}, \ j = 1, 2, \ldots,
\end{cases}
\]
\[
n(t) = \begin{cases} 
0 & \text{for} \ 0 \leq t < s_1 \\
Q_j & \text{for} \ s_j \leq t < s_{j+1}, \ j = 1, 2, \ldots.
\end{cases}
\]

Clearly,
\[
1 \leq \frac{\Phi(t)}{n_1(t)} < 1 + \frac{q_{j+1}}{Q_j} \text{for} \ t_j \leq t < t_{j+1}, j \geq 1,
\]

and therefore, by (3.4) and (3.5),
\[
\lim_{t \to \infty} \frac{\Phi(t)}{n_1(t)} = 1.
\]

Hence,
\[ \int_1^r \frac{n_1(t)}{t} \, dt \sim \Lambda(r) \quad (r \to \infty). \]

We will now show that under the hypothesis of Theorem 3 we also have,

\[ \int_1^r \frac{n(t)}{t} \, dt \sim \Lambda(r) \quad (r \to \infty). \]

Since \( f \) is of exponential type we have, for some \( A > 0 \),

\[ n(t) \leq n_1(t) \leq \Phi(t) < At \quad \text{for} \quad t \geq 1. \]

Thus, \( t_j > 1/A q_j \geq 1/A \exp(\sqrt{j}) \) and for \( \gamma > 0 \),

\[ j^2 = o(t\gamma) \quad (j \to \infty). \]

Therefore,

\[ \int_{s_j}^{s_{j+1}} \frac{n_1(t)}{t} \, dt = \int_{s_j}^{s_{j+1}-j} \frac{n(t)}{t - j(j - 1)/2} \, dt = \{1 + o(1)\} \int_{s_j}^{s_{j+1}-j} \frac{n(t)}{t} \, dt \quad (j \to \infty). \]

Hence, for \( s_j \leq r < s_{j+1} \) and \( j \to \infty \),

\[ \int_1^{r-j^2} \frac{n_1(t)}{t} \, dt \leq \{1 + o(1)\} \int_1^r \frac{n(t)}{t} \, dt \leq \{1 + o(1)\} \int_1^r \frac{n_1(t)}{t} \, dt. \]

This leaves to show,

\[ \int_{r-j^2}^r \frac{n_1(t)}{t} \, dt = o \left( \int_1^r \frac{n_1(t)}{t} \, dt \right) \quad (j \to \infty). \]

(a) Suppose \( f \) is of lower order \( \lambda > 0 \) then, since \( \Lambda(r) \sim \int_1^r n_1(t)/t \, dt \), we have for \( r \) sufficiently large,

\[ \int_1^r \frac{n_1(t)}{t} \, dt > r^{\lambda/2}. \]

Thus, by (3.6) and (3.7),

\[ \int_{r-j^2}^r \frac{n_1(t)}{t} \, dt \leq j^2 A = o(r^{\lambda/2}) = o \left( \int_1^r \frac{n_1(t)}{t} \, dt \right). \]

(b) Suppose \( f \) is of order \( \rho < 1 \), then there exists \( \gamma, 0 < \gamma < 1 - \rho \) such that, for \( t \) sufficiently large,

\[ \frac{n_1(t)}{t} \leq \frac{\Phi(t)}{t} < t^{-\gamma}. \]
Therefore, by (3.7), we have for \( s_j \leq r < s_{j+1} \)

\[
\int_{r-j}^{r} \frac{n_1(t)}{t} \, dt = o(1) \quad (j \to \infty).
\]

Hence, since \( f \) is either of order \( \rho < 1 \) or of lower order \( \lambda > 0 \),

\[
\int_{1}^{r} \frac{n_1(t)}{t} \, dt \sim \int_{1}^{r} \frac{n(t)}{t} \, dt \sim \Lambda(r) \quad (r \to \infty).
\]

We consider next the infinite product

\[
g(z) = \prod_{j=1}^{\infty} \left( 1 + \left\{ \frac{z}{s_j} \right\}^{q_j} \right).
\]

Let \( |z| = r \) with \( r < R \) and define \( p \) by \( s_p \leq R < s_{p+1} \).

By (3.3), \( q_m - q_n \geq m - n \) and therefore,

\[
\sum_{s_j > R} \left| \frac{z}{s_j} \right|^{q_j} \leq \sum_{s_j > R} \left\{ \frac{r}{R} \right\}^{q_j} < \sum_{j=p}^{\infty} \left\{ \frac{r}{R} \right\}^{q_j} \leq \left\{ \frac{r}{R} \right\}^{q_p} \frac{R}{R-r}.
\]

This shows that the infinite product in (3.9) converges uniformly in every bounded region. Hence \( g(z) \) is an entire function.

Now, \( n(r, 1/g) = n(r) \) and therefore by (3.8),

\[
N\left( r, \frac{1}{g} \right) = \int_{1}^{r} \frac{n(t)}{t} \, dt \sim \Lambda(r) \quad (r \to \infty).
\]

For \( r < R \) and \( s_p \leq R < s_{p+1} \),

\[
\log M(r, g) = \sum_{s_j \leq r} q_j \log \frac{r}{s_j} + \sum_{s_j \leq r} \log \left( 1 + \left\{ \frac{s_j}{r} \right\}^{q_j} \right) + \sum_{r < s_j \leq R} \log \left( 1 + \left\{ \frac{s_j}{r} \right\}^{q_j} \right) + \sum_{s_j > R} \log \left( 1 + \left\{ \frac{s_j}{r} \right\}^{q_j} \right) \leq N\left( r, \frac{1}{g} \right) + p \log 2 + \sum_{s_j > R} \left\{ \frac{r}{s_j} \right\}^{q_j}.
\]

Hence, by (3.10) and elementary inequalities of Nevalinna's theory
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(3.11)
\[ N\left(r, \frac{1}{g}\right) \leq \log M(r, g) \leq N\left(r, \frac{1}{g}\right) \]
\[ + p \log 2 + \left\{ \frac{r}{R} \right\}^{q_p} \frac{R}{R - r} \quad (r < R). \]

Now, let \( R = 2r \) and \( p \) defined by \( s_p \leq R < s_{p+1} \). Then,
\[ q_p \leq Q_p \leq \Phi(2r) < \int_{2r}^{2er} \frac{\Phi(t)}{t} \, dt < \Lambda(2er) \]
and thus, \( q_p = o(r) \) \( (p = p(2r), r \to \infty) \).

By (3.2),
\[ q_p > k_p = \exp(\Phi(r_p) \log r_p). \]
Hence,
\[ B^\eta(r_p) \log r_p = 0(\log r) \quad (r \to \infty), \]
and since \( B^\eta(x) \) is strictly increasing,
\[ r_p < r \]
for \( r \) sufficiently large.

Thus, since \( B^{2\eta}(x) \log x \) is strictly increasing,
\[ p = B^{2\eta}(r_p) \log r_p \leq B^{2\eta}(r) \log r = \Lambda(r) B^{1-2\eta}(r). \]
Since \( \lim_{r \to \infty} B^{1-2\eta}(r) = 0 \), we have,
\[ p = o(\Lambda(r)) \quad (r \to \infty, p = p(2r)). \]
Thus, we obtain for (3.11),
\[ N(r, 1/g) \leq \log M(r, g) \leq N(r, 1/g) + o(\Lambda(r)) + o(1) \quad (r \to \infty). \]
Hence,
\[ \log M(r, g) \sim N(r, 1/g) \sim \Lambda(r) \sim \log M(r, f) \quad (r \to \infty). \]
Since \( s_{j+1} - s_j \geq j + 1 \), we have
\[ \lim_{n \to \infty} \sum_{j \neq n} \left| \frac{1}{s_n - s_j} \right| = 0. \]
Clearly, \( \lim \sup_{t \to \infty} \Phi(t)/t < \infty \) and therefore, by Theorem 1, \( g(z) \) is of bounded index.  \text{q.e.d.}

4. Proof of Theorem 4. Without loss of generality we may assume \( \Phi(1) = 0 \), \( \Phi(t) \) continuous, strictly increasing and unbounded. The condition \( \lim \sup_{t \to \infty} \Phi(t)/t < \infty \) assures us that the function \( f \) we are
about to construct is of exponential type. Let $B(r)$, $\{q_j\}_1^\infty$, $\{t_j\}_1^\infty$, and $n_1(t)$ be defined as in the proof of Theorem 3.

Now let $f(z) = \prod_{j=1}^\infty (1 + \{z/t_j\}^{q_j})$, then by the same argument as in the proof of Theorem 3, $f(z)$ is an entire function and

$$N\left(\frac{r}{g} - \frac{1}{g}\right) = \int_1^r \frac{n_1(t)}{t} \, dt \sim \log M(r, f) \sim \int_1^r \frac{\Phi(t)}{t} \, dt \quad (r \to \infty).$$

Then, for $t \geq t_j$ and $j$ sufficiently large,

$$\Phi(t + 1) - \Phi(t) \leq \Phi(t) \Phi(t) - 1/n \leq Q_j^{1/n} \Phi(t) \leq e^{-\sqrt{n} \Phi(t)} \leq 1/j^3 \Phi(t).$$

Hence, for $j$ sufficiently large,

$$\Phi(t_j + j) \leq (1 + 1/j^3) \Phi(t_j) \leq (1 + 1/j) \Phi(t_j).$$

Therefore,

$$\Phi(t_{j+1}) - \Phi(t_j) = q_{j+1} > (1/j)Q_j = (1/j) \Phi(t_j),$$

and thus $t_{j+1} - t_j > j$ for $j$ sufficiently large.

Now, $\sum_{j=n}^{\infty} 1/|t_j - t_j| = o(1)$ and, by Theorem 1, $f$ is of bounded index. q.e.d.

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