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ON QUASI-COMPLETE ABELIAN p-GROUPS

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1. **Introduction.** An Abelian p-group (henceforth the word group means Abelian p-group) is quasi-complete if the closure (relative to the p-adic topology) of every pure subgroup is again pure. Information on quasi-complete groups (also called quasi-closed groups) may be found in [3], [4] and [5]. The subject of this paper is a refinement of the above concept.

A subgroup H of a group G is said to be (topologically) imbedded in G if the natural p-adic topology on H coincides with the relative topology on H induced by the p-adic topology on G. Properties of imbedded subgroups have been studied in [6]. The following criterion is easily obtained.

Criterion for Imbeddedness. H is an imbedded subgroup of G if and only if there exists a function $\ell : N \rightarrow N$ (N denotes the non-negative integers) satisfying $H \cap p^{\ell(n)}G \subset p^nH$ for each $n \in N$. (H is then said to be ℓ-imbedded.)

The I-imbedded subgroups (I is the identity function) are, simply, the pure subgroups. Some examples of non-pure imbedded subgroups are given in § 2, where the results from [6] that are required in this paper are summarized.

In § 3 we derive criteria for the closure of an ℓ-imbedded subgroup to be, again, ℓ-imbedded. We also establish some properties of ℓ-imbedded subgroups that generalize some familiar properties of pure subgroups.

We say that a reduced group is ℓ-quasi-complete if the closure of every ℓ-imbedded subgroup is, again, imbedded. The quasi-complete groups of [3], [4] and [5] are now the I-quasi-complete groups. In § 4 we give characterizations of ℓ-quasi-complete groups that generalize results of [3] and [5].

In § 5 we establish results that lead to a conjecture on the characterization of torsion complete groups.

2. **Imbedded subgroups.** The notation and terminology used in this paper is generally that of [1] and [2]. The reader is also referred to [1] for a discussion of linear topologies in Abelian groups.

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We are concerned here only with Abelian p-groups and the p-adic topology of such groups. The p-adic topology on a group G (denoted by $T(G)$) is obtained by taking the set $\{p^nG \mid n \in N\}$ as a filter-base for the open subgroups. Then the relative topology (denoted by $T(G; H)$) induced on a subgroup H by the p-adic topology on G has the set $\{H \cap p^nG \mid n \in N\}$ as a filter-base for the open subgroups. Since $p^nH \subset H \cap p^nG$ for each $n \in N$, it follows that $T(H)$ is finer than $T(G; H)$. In general, however, $T(H)$ and $T(G; H)$ do not coincide. For example, let G have infinite length and let $H = G[p]$. Then $T(H)$ is discrete, but $T(G; H)$ is not discrete.

In order for H to be imbedded (see § 1) in G it is necessary and sufficient that $T(G; H)$ be finer than $T(H)$; that is, p^nH must be open relative to $T(G; H)$ for each $n \in N$. Thus H is imbedded if and only if for each $n \in N$ there exists $k_n \in N$ such that $H \cap p^{k_n}G \subset p^nH$. Setting $\mathfrak{L}(n) = k_n$ we have the criterion of § 1. We say that $\mathfrak{L}$ is an imbedding function for H in G. If $\mathfrak{L} \leqq \mathfrak{L}'$ (relative to pointwise ordering) then evidently $\mathfrak{L}'$ is also an imbedding function for H. The pointwise infimum of the set of imbedding functions for H is again an imbedding function; so each imbedded subgroup has a unique minimal imbedding function.

The minimal imbedding function fails to be strictly increasing if and only if the reduced part of H is contained in a bounded summand of G. For technical reasons it is convenient to consider only imbedding functions that are strictly increasing. Thus if H is $\mathfrak{L}$-imbedded in G and $\mathfrak{L}$ is not strictly increasing, then we agree to replace $\mathfrak{L}$ by the function $\mathfrak{L}'$ defined inductively by $\mathfrak{L}'(n) = \max\{\mathfrak{L}'(n-1) + 1, \mathfrak{L}(n)\}$. This convention holds also for the minimal imbedding function. Hence the identity function I is the smallest (relative to pointwise ordering) possible imbedding function.

Since I is an imbedding function for each pure subgroup, the concept of imbeddedness may be regarded as a generalization of purity, and the degree of "impurity" of an imbedded subgroup may be measured by the extent that its minimal imbedding function differs from I. Thus the *difference function*, $\delta(n) = \mathfrak{L}(n) - n$, for an imbedded subgroup H with minimal imbedding function $\mathfrak{L}$ is of interest. Examples are given in [**6**] of imbedded subgroups with unbounded minimal difference functions. If H is imbedded in G with a bounded minimal difference function δ, then $\mathfrak{L}(n) = k + n$ is an imbedding function for H (but not necessarily the minimal one) where k is an upper bound for $\{\delta(n) \mid n \in N\}$. In this case H satisfies the condition $H \cap p^{k+n}G \subset p^nH$ for each $n \in N$. There is a class of imbedded subgroups that satisfy the stronger condition $H \cap p^{k+n}G \subset p^n(H \cap p^kG)$

for each $n \in N$ with k fixed. Such a subgroup is said to be regularly imbedded with index k. This type of imbedded subgroup is very close to being pure and, indeed, the pure subgroups are precisely the regularly imbedded subgroups with index zero. The following characterization of regularly imbedded subgroups is found in [6].

PROPOSITION 2.1. *A subgroup H of the p-group G is regularly imbedded in G if and only if there exist a pure subgroup S of G and an integer m such that $p^m S \subset H \subset S$.*

The following results from [6] will be used in the sequel. The proofs are quite straightforward.

LEMMA 2.2. *Let K be $\mathfrak{L}$-imbedded in G and let $K \subset H \subset G$. Then K is $\mathfrak{L}$-imbedded in H.*

LEMMA 2.3. *Let K be $\mathfrak{L}_1$-imbedded in H and let H be $\mathfrak{L}_2$-imbedded in G. Then K is $\mathfrak{L}_2 \circ \mathfrak{L}_1$-imbedded in G (i.e., $K \cap p^{\mathfrak{L}_2(\mathfrak{L}_1(n))}G \subset p^n K$ for each $n \in N$).*

LEMMA 2.4. *Let H be $\mathfrak{L}$-imbedded in G and let $K \subset H$. Then H/K is $\mathfrak{L}$-imbedded in G/K.*

LEMMA 2.5. *Let K be $\mathfrak{L}_1$-imbedded in G and H/K be $\mathfrak{L}_2$-imbedded in G/K. Then H is $\mathfrak{L}_2 \circ \mathfrak{L}_1$-imbedded in G.*

The next items show that several elementary properties of pure subgroups are valid for arbitrary imbedded subgroups.

LEMMA 2.6. *If H is $\mathfrak{L}$-imbedded in G, then $H \cap G^1 = H^1$ (G^1 is the first Ulm subgroup of G).*

PROOF. Clearly $H^1 \subset H \cap G^1$, and for each $n \in N$

$$H \cap G^1 \subset H \cap p^{\mathfrak{L}(n)}G \subset p^n H,$$

so $H \cap G^1 \subset \bigcap_{n \in N} p^n H = H^1$, as desired.

COROLLARY 2.7. *Let $H \subset G^1$. Then H is imbedded in G if and only if H is divisible.*

PROOF. If H is divisible (and, hence, pure), then H is I-imbedded. Conversely $H = H \cap G^1 = H^1$ by 2.6, so H is divisible.

COROLLARY 2.8. *Let H be imbedded in G. Then $\bar{H}$ is imbedded if and only if $(G/H)^1$ is divisible.*

PROOF. Note that $\bar{H}/H = (G/H)^1$ and apply 2.5 and 2.7.

Recall from § 1 that G is an $\mathfrak{L}$-quasi-complete group if the closure of each $\mathfrak{L}$-imbedded subgroup of G is imbedded. It doesn't follow directly from the definition that an I-quasi-complete group is the same as a quasi-complete group (i.e., the closure of a pure subgroup might be imbedded but not pure). This problem is resolved, however, by the following proposition.

PROPOSITION 2.9. *Let H be $\mathfrak{L}$-imbedded in G. If $\bar{H}$ is imbedded in G, then $\bar{H}$ is again $\mathfrak{L}$-imbedded.*

PROOF. $\bar{H}/H$ is divisible by 2.8. Thus $\bar{H}/H$ is I-imbedded in G/H. Hence $I \circ \mathfrak{L} = \mathfrak{L}$ is an imbedding function for $\bar{H}$ in G by 2.5, as desired.

3. **The closure of an imbedded subgroup.** In this section we establish criteria for the closure of an $\mathfrak{L}$-imbedded subgroup to be imbedded (and, hence, $\mathfrak{L}$-imbedded). We need two definitions.

DEFINITION 3.1. Let $K \subset G$. An $\mathfrak{L}$-imbedded subgroup H of G is said to be an $\mathfrak{L}$-hull of K if $K \subset H$ and H does not properly contain an $\mathfrak{L}$-imbedded subgroup of G that contains K.

It can be shown that not every subgroup has an $\mathfrak{L}$-hull.

DEFINITION 3.2. Let H be $\mathfrak{L}$-imbedded in G. H is said to be semi-strongly $\mathfrak{L}$-imbedded if for each subsocle S containing $H[p]$ there exists a subsocle T satisfying

(1) $\bar{H}[p] \cap S \subset T \subset S$

(2) T is dense in S (in the relative topology from G).

(3) T supports an $\mathfrak{L}$-imbedded subgroup of G containing H.

The objective of this section is to prove the following theorem.

THEOREM 3.3. *Let H be an $\mathfrak{L}$-imbedded subgroup of G. The following are equivalent:*

(1) $\bar{H}$ is $\mathfrak{L}$-imbedded.

(2) H is semi-strongly $\mathfrak{L}$-imbedded.

(3) If $H \subset K \subset \bar{H}$, then K has an $\mathfrak{L}$-imbedded $\mathfrak{L}^2$-hull.

In (3) $\mathfrak{L}^2$ is the composite $\mathfrak{L} \circ \mathfrak{L}$. Clearly an $\mathfrak{L}$-imbedded subgroup is $\mathfrak{L}^2$-imbedded since $\mathfrak{L}^2 \geqq \mathfrak{L}$, but obviously the converse is false. In addition we note that an $\mathfrak{L}$-imbedded $\mathfrak{L}^2$-hull of K is also an $\mathfrak{L}$-hull of K. The equivalence of (1) and (3) generalizes a result in [3] for pure subgroups. Condition (2) is new as far as the author knows.

In order to establish the equivalence of (1) and (2) we need some facts about $\mathfrak{L}$-imbedded subgroups that generalize some properties of pure subgroups. For example, if S is dense in $G[p]$ and N is a neat subgroup supported by S, then N is pure in G and dense in G. This result is obtained by taking $m = 0$ in Proposition 3.5.

LEMMA 3.4. *If H is any subgroup of G, then clearly $\overline{H[p^k]} \cap G[p^k] \subset \bar{H}[p^k]$ for each $k \in N$. If H is imbedded in G, then the other inclusion holds also.*

PROOF. The proof is routine upon noting that if H is ℓ-imbedded, then

$$(H + p^{\ell(n+k)-k}G)[p^k] \subset H[p^k] + p^nG$$

for every $n \in N$.

PROPOSITION 3.5. *Let $H[p]$ be dense in $G[p]$ and let $H \cap p^{m+1}G \subset pH$ for some $m \in N$. Then*

(1) *H is regularly imbedded with index m in G.*
(2) *$p^mG \subset \bar{H}$.*

PROOF. (1). $H \cap p^{m+1}G \subset pH$ implies that

$$H \cap p^{m+1}G \subset p(H \cap (G[p] + p^mG)),$$

and the density of $H[p]$ in $G[p]$ implies that $G[p] \subset H[p] + p^mG$. Thus $H \cap (G[p] + p^mG) \subset H \cap (H[p] + p^mG) = H[p] + (H \cap p^mG)$. Hence $H \cap p^{m+1}G \subset p(H \cap p^mG)$. This shows that $H \cap p^{m+n}G \subset p^n(H \cap p^mG)$ for $n = 1$. The induction step follows similarly.

(2). We show inductively that $p^mG[p^k] \subset \bar{H}$ for each $k \in N$. The validity for $k = 1$ follows from the density of $H[p]$ in $G[p]$. Thus assume $p^mG[p^k] \subset \bar{H}$ and let $x \in p^mG[p^{k+1}]$. Then $px \in \bar{H}$; so for each $n \in N$ there exist $h \in H$ and $g \in G$ such that $px = h + p^{m+n+1}g$. Now $h \in H \cap p^{m+1}G \subset p(H \cap p^mG)$; so it follows that $x \in H + p^{m+n}G + G[p]$ for each $n \in N$. But $G[p] \subset \bar{H}$; so we conclude that $x \in \bar{H}$, as desired.

The following results also generalize properties of pure subgroups.

LEMMA 3.6. *Let H be ℓ-imbedded in G and let $H[p]$ be closed in $G[p]$. Then $\bar{H} \cap p^{\ell(1)-1}G \subset H$.*

PROOF. The proof consists of showing inductively that $\bar{H}[p^k] \cap p^{\ell(1)-1}G \subset H$ for every $k \in N$. Now the closure of $H[p]$ in $G[p]$ is $\overline{H[p]} \cap G[p]$, which equals $\bar{H}[p]$ by Lemma 3.4. Thus $\bar{H}[p] = H[p]$; so the case $k = 1$ is trivially true. Assume the validity for arbitrary k and let $x \in \bar{H}[p^{k+1}] \cap p^{\ell(1)-1}G$. Then $px \in \bar{H}[p^k] \cap p^{\ell(1)}G$; so $px \in H \cap p^{\ell(1)}G \subset pH$. Thus $x \in H + \bar{H}[p]$; that is, $x \in H$, as desired.

Corollary 3.7. *If H is $\mathfrak{L}$-imbedded in G and $H[p]$ is closed in $G[p]$, then $\bar{H}$ is $\mathfrak{L}$-imbedded in G.*

Proof. Let H be $\mathfrak{L}$-imbedded. Then

$$\bar{H} \cap p^{\mathfrak{L}(n)}G \subset \bar{H} \cap p^{\mathfrak{L}(1)-1}G \subset H.$$

Thus $\bar{H} \cap p^{\mathfrak{L}(n)}G \subset H \cap p^{\mathfrak{L}(n)}G \subset p^nH \subset p^n\bar{H}$, as desired.

Lemma 3.8. *Let $H \subset G^1$ and L an $\mathfrak{L}$-hull of H in G. Then L is divisible.*

Proof. If L is not divisible, then $L = B \oplus K$, where B is non-zero, bounded and, hence, $K^1 = L^1$. Thus $H \subset G^1 \cap L = L^1 = K^1 \subset K \subset L$, so K is an $\mathfrak{L}$-imbedded subgroup (by 2.3) containing H. This contradicts that L is an $\mathfrak{L}$-hull, so L must be divisible.

Now we are ready for the proof of Theorem 3.3.

(1) implies (3). By 2.8 $\bar{H}/H$ is divisible. Let L/H be the divisible hull of K/H in $\bar{H}/H$. Then L/H is I-imbedded in G/H and, since H is $\mathfrak{L}$-imbedded in G, L is $\mathfrak{L}$-imbedded in G by 2.5. To show that L is an $\mathfrak{L}^2$-hull of K in G, let us suppose that M is an $\mathfrak{L}^2$-imbedded subgroup satisfying $K \subset M \subset L$. Then M/H is $\mathfrak{L}^2$-imbedded in G/H and $K/H \subset M/H \subset L/H \subset (G/H)^1$, so M/H is, indeed, divisible by 2.7. Since L/H is the divisible hull of K/H, we conclude that $M = L$, as desired.

(3) implies (1). Take $K = \bar{H}$ and let L be an $\mathfrak{L}$-imbedded $\mathfrak{L}^2$-hull of $\bar{H}$. Then L/H is an $\mathfrak{L}$-imbedded subgroup of G/H containing $\bar{H}/H$. Suppose $\bar{H}/H \subset M/H \subset L/H$ where M/H is $\mathfrak{L}$-imbedded. Then M is $\mathfrak{L}^2$-imbedded by 2.5 and $\bar{H} \subset M \subset L$. Thus $M = L$ since L is an $\mathfrak{L}^2$-hull of $\bar{H}$ and, hence, $M/H = L/H$. Therefore, L/H is an $\mathfrak{L}$-hull of $\bar{H}/H$, so L/H is divisible by 3.8. Thus $\bar{H}/H = (G/H)^1$ is divisible, so $\bar{H}$ is $\mathfrak{L}$-imbedded by 2.8, as desired.

(1) implies (2). Let S be a subsocle of G containing $H[p]$. We wish to find a subsocle T that satisfies (1), (2) and (3) of Definition 3.2. Let $\mathcal{L}$ be the collection of subgroups K of G satisfying

(i) K is $\mathfrak{L}$-imbedded.
(ii) $K[p] \subset S$.
(iii) $H \subset K$.

Clearly $\mathcal{L}$ is inductive, so let L be a maximal element of $\mathcal{L}$. Let $T = L[p]$. Then 3.2(3) is satisfied since T supports L. To verify 3.2(1) we show that $\bar{H}[p] \cap S \subset L[p]$. Let $x \in \bar{H}[p] \cap S$. If $x \in H$, then $x \in L$, as desired. Thus assume $x \notin H$. Let M/H be the divisible hull of $\langle x + H\rangle$ in $(G/H)^1$ (clearly $\langle x + H\rangle \subset (G/H)^1$ and $(G/H)^1$ is divisible since H is $\mathfrak{L}$-imbedded). Then $M/H \simeq Z(p^\infty)$. Now $(M + L)/L$ is a homomorphic image of M/H, so either $(M + L)/L = 0$ or $(M + L)/L \simeq Z(p^\infty)$. In the first case $x \in L$, as desired. In the second case, $M + L$ is $\mathfrak{L}$-imbedded in G and $(M + L)[p] = \langle x \rangle + L[p] \subset S$. Thus $M + L$ is in $\mathcal{L}$, so $M + L = L$ since L is maximal. Hence, again, $x \in L$, as desired. Thus $\bar{H}[p] \cap S \subset L[p]$, which establishes 3.2(1). For 3.2(2) we must show that $S \subset \overline{L[p]} \cap G[p]$. But $\overline{L[p]} \supset \bar{L}[p]$ by 3.4, so it suffices to show that $S \subset \bar{L}$. Let $x \in S$. If $x + L \in (G/L)^1$, then $x + L \in \bar{L}/L$. Thus $x \in \bar{L}$, as desired. If $x + L \notin (G/L)^1$, then $x + L$ has finite height, say m, in G/L. Let $x + L = p^m(y + L)$. Then $\langle y, L \rangle/L$ is pure in G/L, so $\langle y, L \rangle$ is $\mathfrak{L}$-imbedded by 2.5. Now $\langle y, L \rangle[p] = \langle x \rangle + L[p]$, so $\langle y, L \rangle \in \mathcal{L}$. This contradicts the maximality of L, so, in fact, the case $x + L \notin (G/L)^1$ cannot happen. Thus $S \subset \bar{L}$, as desired.

(2) IMPLIES (1). Let $S = \bar{H}[p]$. Let T be the corresponding subsocle of Def. 3.2. Since $\bar{H}[p] \cap S \subset T$, then, in fact, $T = S$ in this case. Thus $\bar{H}[p]$ supports an $\mathfrak{L}$-embedded subgroup L which contains H and $H[p]$ is dense in $L[p]$ (in the relative topology from L). Therefore $p^m L \subset \bar{H} \cap L \subset \bar{H}$ by 3.5 ($m = \mathfrak{L}(1) - 1$). Since $L[p]$ is closed in $G[p]$, $\bar{L}$ is $\mathfrak{L}$-imbedded by 3.7 and $p^m \bar{L} \subset L$ by 3.6. Thus $p^{2m}\bar{L} \subset \bar{H} \subset \bar{L}$. This implies that $\bar{H}$ is imbedded in $\bar{L}$, which is imbedded in G. Thus $\bar{H}$ is imbedded (and, thus, $\mathfrak{L}$-imbedded) in G by 2.3, as desired.

4. **The global case.** In this section we are concerned with the reduced groups in which the closure of *every* $\mathfrak{L}$-imbedded subgroup is again $\mathfrak{L}$-imbedded. Recall that such a group is said to be $\mathfrak{L}$-quasi-complete. Since $\bar{0} = G^1$, it follows from 2.7 that $G^1 = 0$ if G is $\mathfrak{L}$-quasi-complete. We need the following concept.

DEFINITION 4.1. Let H be $\mathfrak{L}$-imbedded in G. H is said to be strongly $\mathfrak{L}$-imbedded if every subsocle of G containing $H[p]$ supports an $\mathfrak{L}$-imbedded subgroup of G containing H.

Note that a strongly $\mathfrak{L}$-imbedded subgroup is also semi-strongly $\mathfrak{L}$-imbedded. Note also that this concept coincides with the strong purification property of [5] when $\mathfrak{L} = I$. The following characterization of the $\mathfrak{L}$-quasi-complete groups generalizes Theorem 1 of [5].

THEOREM 4.2. *Let G be an abelian p-group. The following are equivalent:*

(1) *G is $\mathfrak{L}$-quasi-complete.*

(2) *Every $\mathfrak{L}$-imbedded subgroup of G is strongly $\mathfrak{L}$-imbedded.*

(3) *If H is $\mathfrak{L}$-imbedded in G and $H \subset K \subset \bar{H}$, then K has an $\mathfrak{L}$-imbedded $\mathfrak{L}^2$-hull in G.*

PROOF. The equivalence of (1) and (3) follows immediately from Theorem 3.3. It is also a consequence of 3.3 that (2) implies (1), since a strongly $\mathfrak{L}$-imbedded subgroup is semi-strongly $\mathfrak{L}$-imbedded. Thus we must show that (1) implies (2). Let H be $\mathfrak{L}$-imbedded in G and let $H[p] \subset S \subset G[p]$. We must show that S supports an $\mathfrak{L}$-imbedded subgroup containing H. Since $\bar{H}$ is $\mathfrak{L}$-imbedded (G is $\mathfrak{L}$-quasi-complete), H is semi-strongly $\mathfrak{L}$-imbedded by 3.3. Let T be the subsocle of Def. 3.2, and let L be the $\mathfrak{L}$-imbedded subgroup supported by T and containing H. Now $\bar{L}$ is $\mathfrak{L}$-imbedded and S is dense in $\bar{L}[p]$. Let M be a neat subgroup of $\bar{L}$ supported by S. Then M is pure (I-imbedded) in $\bar{L}$ by 3.5. Thus M is $\mathfrak{L}$-imbedded in G by 2.3. Since M may be chosen such that $H \subset M$, the proof is now complete.

5. **A conjecture on torsion-complete groups.** Let $C(\mathfrak{L})$ denote the class of $\mathfrak{L}$-quasi-complete groups for arbitrary $\mathfrak{L}$. We note that $C(\mathfrak{L}_1) \subset C(\mathfrak{L}_2)$ if and only if $\mathfrak{L}_2 \leqq \mathfrak{L}_1$. In particular $C(I)$ contains every $\mathfrak{L}$-quasi-complete group for every $\mathfrak{L}$. Let $\mathcal{T}$ denote the class of torsion-complete p-groups. It is known that $\mathcal{T} \subset C(I)$ (see [**2**], for example). Our main theorem in this section is the rather interesting result that $\mathcal{T} \subset C(\mathfrak{L})$ for every $\mathfrak{L}$.

THEOREM 5.1. *Let G be torsion complete. Then G is $\mathfrak{L}$-quasi-complete for every $\mathfrak{L}$.*

PROOF. Let H be $\mathfrak{L}$-imbedded in G. We show that $\bar{H} \cap p^{\mathfrak{L}(n)}G \subset p^n\bar{H}$ for each $n \in N$. Thus let $x \in \bar{H} \cap p^{\mathfrak{L}(n)}G$. Then $x \in \bar{H}[p^t]$ for some $t \in N$. By 3.4, $\bar{H}[p^t] \subset \overline{H[p^t]}$. Thus x is the limit of a Cauchy sequence in $H[p^t]$. Let $\{x_k\} \subset H[p^t]$ be a subsequence of that sequence satisfying

(1) $x_1 \in p^{\mathfrak{L}(n)}G$

(2) $x_{k+1} - x_k \in p^{\mathfrak{L}(n+k)}G$ for $k \in N$.

Since H is $\mathfrak{L}$-imbedded, $x_1 \in p^nH$ and $x_{k+1} - x_k \in p^{n+k}H$. Let $x_1 = p^ny_1$ and $x_{k+1} - x_k = p^{n+k}w_k$, where $y_1 \in H$ and $\{w_k\} \subset H$. Let $y_k = y_1 + \sum_{i=1}^{k-1} p^iw_i$ for $k > 1$. Then $\{y_k\}$ is a bounded Cauchy sequence in H ($\{y_k\} \subset H[p^{n+t}]$), so there exists $y \in \bar{H}$ such that $\lim y_k = y$, since G is torsion complete. Now $\lim x_k = x$ and, since $x_k = p^ny_k$, $\lim x_k = p^ny$. But limits are unique since $G^1 = 0$. Thus $x = p^ny \in p^n\bar{H}$, as desired. Hence $\bar{H}$ is $\mathfrak{L}$-imbedded and G is $\mathfrak{L}$-quasi-complete.

Let $C = \bigcap_{\mathfrak{L}} C(\mathfrak{L})$. Then every group in C has the property that the closure of an $\mathfrak{L}$-imbedded subgroup is again $\mathfrak{L}$-imbedded for every $\mathfrak{L}$. Let us call such a group a *totally quasi-complete* group. Then the preceding theorem says that a torsion complete group is totally quasi-complete. There exist examples of I-quasi-complete groups that are not torsion complete (see [4]). The author has attempted to construct totally quasi-complete groups that are not torsion complete. His failure to do so has prompted the following conjecture on the characterization of torsion complete groups.

CONJECTURE 5.2. An Abelian p-group is torsion complete if and only if it is totally quasi-complete.

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