# SOME OSCILLATION CRITERIA FOR FOURTH ORDER DIFFERENTIAL EQUATIONS

### DAVID LOWELL LOVELADY

1. **Introduction.** In [2, Theorem 11.4, p. 374], W. Leighton and Z. Nehari showed that if $q$ is a continuous function from $R^+ = [0, \infty)$ to $(0, \infty)$, and if

$$(1) \qquad \int_0^\infty t^2 q(t)\, dt = \infty,$$

then every solution of

$$(2) \qquad u'''' + qu = 0$$

is oscillatory. (We call a continuous function $f$ from $R^+$ to $R = (-\infty, \infty)$ *oscillatory* if and only if the set $\{t : t \text{ is in } R^+ \text{ and } f(t) = 0\}$ is unbounded. See the book of C. A. Swanson [4] for an excellent discussion of the work of Leighton and Nehari and many other authors). We shall give herein an oscillation criterion for

$$(3) \qquad (p_3(p_2(p_1 u')')')' + qu = 0$$

which includes [2, Theorem 11.4]. In particular, with respect to

$$(4) \qquad (ru'')'' + qu = 0,$$

our results generalize [2, Theorem 11.4] by showing that if

$$(5) \qquad \int_0^\infty r(s)^{-1}\, ds = \infty$$

and

$$(6) \qquad \int_0^\infty \left( \int_0^t (t - s) r(s)^{-1}\, ds \right) q(t)\, dt = \infty,$$

then every solution of (4) is oscillatory.

We shall also show that (6) can be weakened to

$$(7) \qquad \int_0^\infty \left( \int_0^t (t - s) s r(s)^{-1}\, ds \right) q(t)\, dt = \infty,$$

if we hypothesize the existence of a bounded solution of (4). In particular, this says that (1) can be weakened to

$$(8) \qquad \int_0^\infty t^3 q(t)\, dt = \infty$$

and still ensure oscillation for (2) if we hypothesize the existence of a bounded solution of (2). Finally, we shall point out that results of Leighton and Nehari can be coupled with work of the present author [3] to obtain a condition ensuring that every solution of (2) is unbounded and nonoscillatory.

2. **Results.** Let $q$ be a continuous function from $R^+$ to $R^+$, and if $k$ is in $\{1, 2, 3\}$ let $p_k$ be a continuous function from $R^+$ to $(0, \infty)$ with

$$(9) \qquad \int_0^\infty p_k(t)^{-1}\, dt = \infty.$$

By a solution of (3) we mean a differentiable function $u$ from $R^+$ to $R$ such that $p_1 u'$ is differentiable, $p_2(p_1 u')'$ is differentiable, $p_3(p_2(p_1 u')')'$ is differentiable, and (3) is true.

THEOREM 1. *Suppose that each of* (H1), (H2), *and* (H3) *is true.*

(H1):

$$\int_0^\infty \left( \int_0^s \left( p_3(\xi)^{-1} \int_0^\xi \left( p_2(\sigma)^{-1} \int_0^\sigma p_1(\tau)^{-1}\, d\tau \right) d\sigma \right) d\xi \right) q(s)\, ds = \infty.$$

(H2): *If*

$$(10) \qquad \int_0^\infty q(s)\, ds < \infty,$$

*and if*

$$(11) \qquad \int_0^\infty \left( p_3(\xi)^{-1} \int_\xi^\infty q(\sigma)\, d\sigma \right) d\xi < \infty,$$

*then*
$$\int_0^\infty \left( p_2(s)^{-1} \int_s^\infty \left( p_3(\xi)^{-1} \int_\xi^\infty q(\sigma)\, d\sigma \right) d\xi \right) ds = \infty.$$

(H3): $\quad \int_0^\infty \left( \int_0^s \left( p_1(\xi)^{-1} \int_0^\xi p_2(\sigma)^{-1}\, d\sigma \right) d\xi \right) q(s)\, ds = \infty.$

*Then every solution of* (3) *is oscillatory.*

COROLLARY 1. *Suppose that* (H3) *holds and*

$$(12) \qquad \int_0^\infty \left( \int_0^s \left( p_3(\xi)^{-1} \int_0^\xi p_2(\sigma)^{-1} \, d\sigma \right) d\xi \right) q(s) \, ds = \infty .$$

*Then every solution of* (3) *is oscillatory.*

COROLLARY 2. *Suppose that* $r$ *is a continuous function from* $R^+$ *to* $(0, \infty)$ *such that* (5) *and* (6) *hold. Then every solution of* (4) *is oscillatory.*

THEOREM 2. *Suppose that* $r$ *is a continuous function from* $R^+$ *to* $(0, \infty)$ *such that* (5) *and* (7) *hold, and suppose that* $q$ *has only positive values. Suppose also that there exists a bounded nontrivial solution of* (4). *Then every solution of* (4) *is oscillatory.*

COROLLARY 3. *Suppose* (8) *holds,* $q$ *has only positive values, and there exists a bounded nontrivial solution of* (2). *Then every solution of* (2) *is oscillatory.*

THEOREM 3. *If*

$$(13) \qquad \limsup_{t \to \infty} t^4 q(t) < 1,$$

*and* (8) *holds, then every nontrivial solution of* (2) *is unbounded and nonoscillatory.*

COROLLARY 4. *If* (13) *holds and*

$$\liminf_{t \to \infty} t^4 q(t) > 0,$$

*then every nontrivial solution of* (2) *is unbounded and nonoscillatory.*

The proof of Theorem 1 is our longest and most involved proof, so we shall defer it to the end of the paper. Note that if (10) fails, or if (10) holds and (11) fails, then (H2) is trivially satisfied. Also, if (10) fails, then (H1) and (H3) follow immediately from (9), so we have an extension of a classical conclusion of W. B. Fite [1]. If (10) and (11) hold, and $t$ is in $R^+$, then two successive applications of integration-by-parts give

$$\int_0^t \left( p_2(s)^{-1} \int_s^\infty \left( p_3(\xi)^{-1} \int_\xi^\infty q(\sigma) \, d\sigma \right) d\xi \right) ds$$

$$= \left( \int_0^t p_2(s)^{-1}\, ds \right) \left( \int_t^\infty \left( p_3(\xi)^{-1} \int_\xi^\infty q(\sigma)\, d\sigma \right)\, d\xi \right)$$

$$+ \int_0^t \left( \int_0^s p_2(\xi)^{-1}\, d\xi \right) p_3(s)^{-1} \left( \int_s^\infty q(\xi)\, d\xi \right) ds$$

$$\geqq \int_0^t \left( \int_0^s p_2(\xi)^{-1}\, d\xi \right) p_3(s)^{-1} \left( \int_s^\infty q(\xi)\, d\xi \right) ds$$

$$= \left( \int_0^t \left( p_3(s)^{-1} \int_0^s p_2(\xi)^{-1}\, d\xi \right)\, ds \right) \left( \int_t^\infty q(\xi)\, d\xi \right)$$

$$+ \int_0^t \left( \int_0^s \left( p_3(\xi)^{-1} \int_0^\xi p_2(\sigma)^{-1}\, d\sigma \right)\, d\xi \right) q(s)\, ds$$

$$\geqq \int_0^t \left( \int_0^s \left( p_3(\xi)^{-1} \int_0^\xi p_2(\sigma)^{-1}\, d\sigma \right)\, d\xi \right) q(s)\, ds.$$

Thus (12) implies (H2). Now (9) and two applications of L'Hôpital's Rule say that

$$\lim_{s \to \infty} \frac{\displaystyle\int_0^s \left( p_3(\xi)^{-1} \int_0^\xi p_2(\sigma)^{-1}\, d\sigma \right)\, d\xi}{\displaystyle\int_0^s \left( p_3(\xi)^{-1} \int_0^\xi \left( p_2(\sigma)^{-1} \int_0^\sigma p_1(\tau)^{-1}\, d\tau \right)\, d\sigma \right) d\xi} = 0,$$

so there is $c$ in $R^+$ such that

$$\int_0^s \left( p_3(\xi)^{-1} \int_0^\xi \left( p_2(\sigma)^{-1} \int_0^\sigma p_1(\tau)^{-1}\, d\tau \right)\, d\sigma \right) d\xi$$

$$\geqq \int_0^s \left( p_3(\xi)^{-1} \int_0^\xi p_2(\sigma)^{-1}\, d\sigma \right) d\xi$$

whenever $s \geqq c$, and we see that (12) implies (H1), and Corollary 1 is immediate. Note that if $p_1 = p_3$ then (12) is the same as (H3), so, in this case, (12) implies that every solution of (3) is oscillatory. If $r$ is as in Corollary 2, then let $p_1 = p_3 = 1$ and $p_2 = r$. Now (9) follows from (5) and (12) is the same as (6), so Corollary 2 is immediate from the above observations.

Note that Corollary 2 says that if $\sigma < 1$ and

$$(14) \qquad\qquad \int_0^\infty t^{2-\sigma} q(t)\, dt = \infty,$$

then every solution of $(t^\sigma u''(t))'' + q(t)u(t) = 0$ on $(0, \infty)$ is oscillatory. To see this let $a > 0$ and put $r(t) = a^\sigma$ if $0 \leq t \leq a$ and $r(t) = t^\sigma$ if $t > a$. Now (6) and (14) are clearly equivalent. In particular, this says that if $\rho$ is a number and $\rho \geq \sigma - 3$, then every solution of $(t^\sigma u''(t))'' + t^\rho u(t) = 0$ on $(0, \infty)$ is oscillatory.

In Theorem 2 and Corollary 3 it *is possible* that there are no bounded solutions; compare Theorem 3 and Corollary 4. In particular, if there is $a > 0$ such that $q(t) = \beta t^{-4}$ whenever $t \geq a$, where $0 < \beta < 1$, then (8) holds and all nontrivial solutions of (2) are nonoscillatory and unbounded. On the other hand, as is evidenced by $u'''' + u = 0$, it is certainly not the case that the hypotheses of Theorem 2 and Corollary 3 preclude the existence of bounded nontrivial solutions.

PROOF OF THEOREM 2. It follows from [3, Theorem 1] and the hypotheses of Theorem 2 that every bounded solution of (4) is oscillatory. Since there exists a bounded nontrivial solution of (4), we thus see that there exists a nontrivial oscillatory solution of (4). But [2, Corollary 9.10, p. 367] says that if one nontrivial solution of (4) is oscillatory then every nontrivial solution of (4) is oscillatory, and the proof is complete.

Corollary 3 is an obvious consequence of Theorem 2.

PROOF OF THEOREM 3. According to (13) and [2, Theorem 11.1, p. 371], (2) has no oscillatory solutions. But according to (8) and [3, Theorem 1], every bounded solution of (2) is oscillatory. Thus every nontrivial solution of (2) is not only unbounded but also nonoscillatory, and the proof is complete.

Corollary 4 is now obvious. It remains to prove Theorem 1.

PROOF OF THEOREM 1. Let $u$ be a nonoscillatory solution. If $u$ is eventually negative, we can replace $u$ by $-u$, so we assume that $u$ is eventually positive. Find $a \geq 0$ such that $u(t) > 0$ if $t \geq a$. On $[a, \infty)$, let $v_1 = u$, $v_2 = p_1 v_1'$, $v_3 = p_2 v_2'$, and $v_4 = p_3 v_3'$. Now the system

(15)
$$\begin{cases} v_1' = v_2/p_1 \\ v_2' = v_3/p_2 \\ v_3' = v_4/p_3 \\ v_4' = -qv_1 \end{cases}$$

is satisfied. Clearly $v_4$ is nonincreasing. If there is $b \geq a$ such that $v_4(b) < 0$, then

(16) $$v_{k-1}(t) = v_{k-1}(a) + \int_a^t (v_k(s)/p_{k-1}(s))\, ds$$

and (9) say that $v_3(t) \to -\infty$, $v_2(t) \to -\infty$, and $v_1(t) \to -\infty$ as $t \to +\infty$, a contradiction. Thus $v_4 \geqq 0$ on $[a, \infty)$, so $v_4(\infty) = \lim_{t \to \infty} v_4(t)$ exists and $v_4(\infty) \geqq 0$. Also, $v_4(b) > 0$ if $b$ is in $[a, \infty)$. For if $v_4(b) = 0$, then $v_4(t) = 0$ whenever $t \geqq b$. Thus, from (15), $v_4'(t) = 0$ and $q(t) = 0$ whenever $t \geqq b$. But this violates (H1), so $v_4 > 0$ on $[a, \infty)$. Thus $v_3$ is increasing on $[a, \infty)$. Now we take cases. Suppose $v_3 < 0$ on $[a, \infty)$. Now $v_3(\infty) \leqq 0$, and if $v_3(\infty) < 0$ then (16) again gives a contradiction, so $v_3(\infty) = 0$. Now $v_2$ is decreasing on $[a, \infty)$, and $v_2(\infty) < 0$ is impossible, so $v_2(\infty) \geqq 0$. If $s \geqq t \geqq a$, then

$$v_4(s) - v_4(t) = -\int_t^s q(\xi)u(\xi)\, d\xi,$$

so

$$v_4(\infty) - v_4(t) = -\int_t^\infty q(\xi)u(\xi)\, d\xi,$$

$$v_4(t) \geqq \int_t^\infty q(\xi)u(\xi)\, d\xi.$$

Since $v_2 > 0$, $u$ is increasing, so

$$v_4(t) \geqq u(a) \int_t^\infty q(\xi)\, d\xi$$

whenever $t \geqq a$. If (10) fails, this is a contradiction, so assume (10) holds. Since $v_3(\infty) = 0$,

$$v_3(t) = -\int_t^\infty (v_4(s)/p_3(s))\, ds$$

whenever $t \geqq a$. But the preceding inequality says that if (11) fails this is a contradiction, so assume (11) holds. If $t \geqq a$, then

$$v_2(t) - v_2(a) = \int_a^t (v_3(\xi)/p_2(\xi))\, d\xi$$

$$= -\int_a^t \left( p_2(\xi)^{-1} \int_\xi^\infty (v_4(\sigma)/p_3(\sigma))\, d\sigma \right) d\xi,$$

so

$$-v_2(a) \leqq - \int_a^t \left( p_2(\xi)^{-1} \int_\xi^\infty (v_4(\sigma)/p_3(\sigma))\, d\sigma \right) d\xi,$$

$$v_2(a) \geqq \int_a^t \left( p_2(\xi)^{-1} \int_\xi^\infty (v_4(\sigma)/p_3(\sigma))\, d\sigma \right) d\xi$$

$$\geqq u(a) \int_a^t \left( p_2(\xi)^{-1} \int_\xi^\infty \left( p_3(\sigma)^{-1} \int_\sigma^\infty q(\tau)\, d\tau \right) d\sigma \right) d\xi.$$

This contradicts (H2), and we are through with the case "$v_3 < 0$ on $[a, \infty)$".

Since $v_3$ is increasing, the case "$v_3 < 0$ is false" ensures that there is a number $b \geqq a$ with $v_3$ positive in $[b, \infty)$. Now $v_2$ is increasing on $[b, \infty)$. If $v_2 \leqq 0$ on $[b, \infty)$, then $u$ is bounded. But (H1) and [3, Theorem 1] say that every bounded solution of (3) is oscillatory, so there is $c \geqq b$ with $v_2$ positive on $[c, \infty)$. Now, if $t \geqq c$,

$$u(t) = u(c) + \int_c^t (v_2(s)/p_1(s))\, ds$$

$$\geqq \int_c^t (v_2(s)/p_1(s))\, ds$$

$$= \int_c^t p_1(s)^{-1} \left( v_2(c) + \int_c^s (v_3(\xi)/p_2(\xi))\, d\xi \right) ds$$

$$\geqq \int_c^t \left( p_1(s)^{-1} \int_c^s (v_3(\xi)/p_2(\xi))\, d\xi \right) ds$$

$$\geqq v_3(c) \int_c^t \left( p_1(s)^{-1} \int_c^s p_2(\xi)^{-1}\, d\xi \right) ds.$$

If $t \geqq c$,

$$0 < v_4(t) = v_4(c) + \int_c^t v_4{}'(s)\, ds$$

$$= v_4(c) - \int_c^t q(s)u(s)\, ds,$$

so

$$(17) \quad \begin{cases} v_4(c) \geqq \displaystyle\int_c^t q(s)u(s)\,ds \\[2em] \qquad \geqq v_3(c) \displaystyle\int_c^t \left( \int_c^s \left( p_1(\xi)^{-1} \int_c^\xi p_2(\sigma)^{-1}\,d\sigma \right) d\xi \right) q(s)\,ds. \end{cases}$$

But, according to L'Hôpital's Rule,

$$\lim_{s \to \infty} \frac{\displaystyle\int_c^s \left( p_1(\xi)^{-1} \int_c^\xi p_2(\sigma)^{-1}\,d\sigma \right) d\xi}{\displaystyle\int_0^s \left( p_1(\xi)^{-1} \int_0^\xi p_2(\sigma)^{-1}\,d\sigma \right) d\xi} = 1,$$

so (H3) implies the divergence of the integral in (17) as $t \to \infty$, we have a contradiction, and the proof is complete.

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FLORIDA STATE UNIVERSITY, TALLAHASSEE, FLORIDA 32306