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AN ALGEBRAIC TREATMENT OF ALGEBRAICALLY COMPACT GROUPS

M. W. LEGG AND E. A. WALKER*

1. **Introduction.** Algebraically compact groups were defined by Kaplansky in [4]. It is inherent in his definition that a complete set of invariants exists for this class of groups. Kaplansky's definition was topological, and other authors, such as Fuchs [2], who have discussed the structure of these groups have taken a topological approach. In fact, it appears that no non-topological discussion of the structure theory of algebraically compact groups exists in the literature. Such a discussion was motivated as follows. The algebraically compact groups are the injectives relative to the class of pure short exact sequences. The development of the structure theory of these injectives was topological. In attempting to develop structure theories for the injectives with respect to other classes of short exact sequences, no corresponding topological methods were available on the one hand, and on the other, no purely algebraic development of algebraically compact groups seemed to exist. It was with some relief that the authors discovered that an elementary purely algebraic treatment of these groups was possible, and this paper presents that treatment.

The notation and terminology used will mainly conform to that of Fuchs [2]. We recall here a few fundamental notions. In this paper, group means Abelian group. A group G is *divisible* if $nG = G$ for all integers $n \neq 0$. Every group G has a maximum divisible subgroup dG, and dG is a summand of G. If $dG = 0$, G is called *reduced*. The injective Abelian groups are precisely the divisible ones, and injective envelopes are called *divisible hulls*. If p is a prime, then G is *p-primary*, or is a *p-group*, if every element of G has order a power of p. If G_p denotes the maximum p-primary subgroup of G, and tG denotes the torsion subgroup of G, then tG is the direct sum $\sum G_p$ of the p-groups G_p. A group G is *p-divisible* if $p^nG = G$ for all n. The group $(Q/Z)_p$ is denoted $Z(p^\infty)$, where Q is the additive group of rational numbers and Z is the group of integers. $Z(n)$ denotes the cyclic group of order n.

2. **Purity.** Before defining algebraically compact groups, we must introduce the concept of purity.

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Definition. A subgroup A of B is a *pure subgroup* of B if $A \cap nB = nA$ for all positive integers n.

Clearly divisible subgroups are pure. It is easy to see that the direct sum $\sum A_i$ is pure in the direct product $\prod A_i$, and that if G/A is torsion free, then A is pure in G.

The next lemma is our principal tool.

Lemma 2.1. *If A is pure in B, then there exists a group C, containing B, with C/A divisible and A pure in C.*

Proof. Let D be the divisible hull of B. By Zorn's Lemma there exist subgroups of D maximal with respect to the property of containing B and containing A as a pure subgroup. Let C be such a subgroup of D. We show that C/A is divisible. Suppose the contrary. Let $C/A = (X/A) \oplus (R/A)$ where X/A is divisible and R/A is reduced. We consider two cases. If R/A is torsion free, let R'/A be the divisible hull of R/A in D/A, and let $C'/A = (X/A) \oplus (R'/A)$. Then $C \subsetneqq C'$ and A is pure in C' since it is pure in both X and R'. This contradicts the maximality of C. If R/A is not torsion free, it has a finite cyclic summand, say S'/A. Then A is pure in S' so that $S'/A = (S'' \oplus A)/A$ for suitable S''. Choose R' so that $R/A = ((S'' \oplus A)/A) \oplus (R'/A)$. Let S be the divisible hull of S'' in D. Now let $C''/A = (X/A) \oplus ((S \oplus A)/A) \oplus R'/A$. Then $C \subsetneqq C''$ and A is pure in C'' since A is pure in $S \oplus A$, R' and X. Again we contradict the maximality of C. Thus we must have C/A divisible.

3. Definition of Algebraically Compact.

Definition (J. M. Maranda [6]). A group G is called *algebraically compact* if for every group Y, any homomorphism from a pure subgroup of Y to G can be extended to a homomorphism of Y to G.

We note that divisible groups are algebraically compact. It is straightforward to show that summands and products of algebraically compact groups are algebraically compact. The fact that bounded groups are algebraically compact is a classical theorem. Here is a simple proof.

Theorem 3.1. *If X is a bounded group, then X is algebraically compact.*

Proof. Suppose $nX = 0$, n a positive integer. Let A be pure in B and $f: A \to X$. By Lemma 2.1 we may assume that B/A is divisible. If $b \in B$, then $b = nb' + a'$ for suitable $b' \in B$ and $a' \in A$. Define $g: B \to X$ by $g(b) = f(a')$. If also $b = nb'' + a''$, then $n(b'' - b') = a' - a''$. Since A is pure in B there exists $y \in A$ such that $ny = a' - a''$. Thus $0 = nf(y) = f(a') - f(a'')$ so that g is well defined. Clearly g is a homomorphism. Hence X is algebraically compact.

4. **Basic Properties.** In this section we lay the foundation for the presentation of the structure theory. We begin with a result due to Loś [5].

LEMMA 4.1. *Every group can be embedded as a pure subgroup in a product of groups of type* $Z(p^k)$, $1 \leqq k \leqq \infty$, *for various primes* p.

PROOF. Let G be a group, D its divisible hull and $S = (\prod G/nG) \oplus D$, where the product extends over all positive integers n. Define $\phi : G \to S : g \to (\{g + nG\}_n, g)$. Then ϕ is a monomorphism. Suppose $(\{g + nG\}_n, g) = m(\{g_n + nG\}_n, d)$ with $m \in N$, $d \in D$ and $g, g_n \in G$. Then we have $g - mg_m \in mG$, so that $g = mg'$ for suitable $g' \in G$. Thus $(\{g + nG\}_n, g) = m(\{g' + nG\}_n, g')$. Hence $\phi(G)$ is pure in S. Each G/ng is bounded, hence a direct sum of cyclics, hence pure in the product of these cyclics. Also the divisible D is pure in a product of groups of type $Z(p^\infty)$. The result follows.

Since a product of groups of type $Z(p^k)$, $1 \leqq k \leqq \infty$, is algebraically compact, the following is a consequence of Lemma 4.1.

LEMMA 4.2. *Every group can be embedded as a pure subgroup of an algebraically compact group.*

The following results are now immediate.

THEOREM 4.3. *A group is algebraically compact if and only if it is a summand whenever it is a pure subgroup.*

COROLLARY 4.4. *A group is algebraically compact if and only if it is a summand of a product of groups of type* $Z(p^k)$, $1 \leqq k \leqq \infty$.

COROLLARY 4.5. *A reduced group is algebraically compact if and only if it is a summand of a product of cyclic primary groups.*

PROOF. We need only show necessity since the sufficiency is obvious. Let G be reduced algebraically compact. Then G is a summand of a group of the form $P \oplus D$ where P is a direct product of cyclic primary groups and D is divisible. Suppose $G \oplus S = P \oplus D$. Since G and P are reduced, the maximum divisible subgroup dS of S is D. Thus $G \oplus (S/D) \cong P$ and the result follows.

With the aid of Lemma 2.1 we can improve on Lemma 4.2.

THEOREM 4.6. *Every group G can be embedded as a pure subgroup of an algebraically compact group A so that A/G is divisible.*

PROOF. By Lemma 4.2 we can take G to be a pure subgroup of an algebraically compact group, say B. Let $d(B/G) = A/G$. Note that

G is pure in A. Also A/G is a summand of B/G so that A is pure in B. By Lemma 2.1 there exists a group C with $B \subseteq C$, C/A divisible and A pure in C. Since B is algebraically compact, there exists an $f: C \rightarrow B$ that extends the injection of A into B. Since C/A is divisible and B/A is reduced, we have $\eta f = 0$, where $\eta : B \rightarrow B/A$ is the natural homomorphism. Thus $f(C) \subseteq A$, so that A is a summand of C, hence of B. It follows that A is algebraically compact.

Since subgroups of algebraically compact groups need not be algebraically compact, the next theorem is of interest.

THEOREM 4.7. *If A is algebraically compact and S is a subgroup of A such that A/S is reduced, then S is algebraically compact.*

PROOF. By Theorem 4.6, there is an algebraically compact group G with S pure in G and G/S divisible. Since A is algebraically compact, there is an $f: G \rightarrow A$ that extends the injection of S into A. Then if $\eta : A \rightarrow A/S$ is the natural homomorphism, $\eta f = 0$. Hence $f(G) \subseteq S$ so that S is a summand of G and is thus algebraically compact.

5. **Reduction to the adjusted and torsion free cases.** In determining the structure of algebraically compact groups we can restrict our investigation to reduced groups, since the structure of divisible groups is known. A further simplification can be made by introducing adjusted groups.

DEFINITION (HARRISON [3]). A reduced algebraically compact group is called *adjusted* if it has no torsion free direct summands.

THEOREM 5.1. *Let A be a reduced algebraically compact group. Then A is the direct sum of an adjusted algebraically compact group and a torsion free algebraically compact group. The adjusted component of A is unique.*

PROOF. Let G be that subgroup of A such that G/tA is the divisible part of A/tA, where tA denotes the torsion subgroup of A. Then A/G is reduced so that G is algebraically compact. Since A/G is torsion free, G is pure in A and is thus a summand of A. Thus we have $A = G \oplus F$ with F torsion free. Since G/tA is divisible, any torsion free summand of G would be divisible. But G is reduced. Thus G is adjusted.

Suppose $A = G' \oplus F'$ with G' adjusted and F' torsion free. Then $tA \subseteq G'$ and G'/tA is divisible, for if not, G' has a torsion free summand. Hence $G' \subseteq G$. Also, since A is reduced, A/G' is reduced, so that $G \subseteq G'$. Thus $G = G'$ and the uniqueness of the adjusted component is proved.

Note that the above proof shows that a reduced algebraically compact group A is adjusted if and only if A/tA is divisible. We have now separated the study of reduced algebraically compact groups into adjusted and torsion free cases.

6. **Coprimary groups.** A further simplification is provided by the introduction of coprimary groups.

DEFINITION (NUNKE [7]). Let A be reduced algebraically compact, p and q primes, and d_qA the q-divisible part of A. Let $A_p = \bigcap_{q \neq p} d_qA$. We term A_p the *p-coprimary component* of A. If $A = A_p$, then A is said to be *p-coprimary*.

If we view A as a summand of a product of cyclic primary groups, then A_p consists of those elements of A that have components from p-groups only. The subgroup A_p is pure in A, is p-reduced, q-divisible (for $q \neq p$) and contains the p-primary component of A.

THEOREM 6.1. *If A is algebraically compact, then A_p is algebraically compact.*

PROOF. Since it is clear that A/A_p is reduced, the result follows on application of Theorem 4.7.

COROLLARY 6.2. *If A is reduced algebraically compact, then so is $\prod A_p$. (Where the product is taken over all primes.)*

The following lemma will be useful in the proofs of several subsequent theorems.

LEMMA 6.3. *Let G and H be reduced algebraically compact groups and A a pure subgroup of both G and H with G/A and H/A divisible. Then G is isomorphic to H.*

PROOF. Since H is algebraically compact, there exists an $f: G \to H$ that extends the injection of A into H. Let $x \in \operatorname{Ker} f$ and $n \in N$. Since G/A is divisible, $x = ng - a$ for suitable $g \in G$ and $a \in A$. Thus $nf(g) = a$. Then, since A is pure in H, there exists $a' \in A$ such that $na' = a$. It follows that $x = n(g - a')$. Thus $x = 0$, because G is a summand of a product of cyclic primary groups. Hence f is injective.

Assume that $G \subseteq H$. Then G/A is a summand of H/A so that G is pure in H. Hence $H = G \oplus S$ for suitable S. Thus $H/A = G/A \oplus S$. Hence S is divisible and, since H is reduced, must thus be zero. The lemma follows.

THEOREM 6.4. *If A is a reduced algebraically compact group, then $A \cong \prod A_p$, where the product extends over all primes.*

PROOF. We have shown that $\prod A_p$ is algebraically compact. It is clear that $\sum A_p$ is pure in $\prod A_p$. If we view A as a summand of a product of cyclic primary groups it is immediate that $\prod A_p/\sum A_p$ is divisible and that $A/\sum A_p$ is torsion free and divisible. Thus the theorem follows on application of Lemma 6.3.

We are now in a position to separately characterize adjusted and torsion free algebraically compact groups.

7. **The adjusted case.** In this section we show that adjusted groups are characterized by their Ulm invariants.

DEFINITION. Let G be a group, p a prime, and n a non-negative integer. Let $G_p(n)$ denote the dimension of $(p^nG)[p]/(p^{n+1}G)[p]$ as a vector space over the integers modulo p. We call $G_p(n)$ the *n-th Ulm invariant of* G (with respect to p).

We shall need some facts concerning basic subgroups. A subgroup B of a p-primary group A is called a *basic subgroup* of A if:

(i) B is a direct sum of cyclic groups;
(ii) B is pure in A;
(iii) A/B is divisible.

The following two results are well known:

(i) Every p-primary group A contains a basic subgroup B.
(ii) Basic subgroups are characterized by their Ulm invariants.

THEOREM 7.1. *Let A and B be adjusted, p-coprimary, algebraically compact groups and let A_1 and B_1 be basic subgroups of tA and tB, respectively. If $A_1 \cong B_1$, then $A \cong B$.*

PROOF. By the remark following Theorem 5.1, A/tA is divisible. Since tA/A_1 is also divisible, it follows that A/A_1 is divisible. Similarly B/B_1 is divisible. Since A_1 and B_1 are pure in A and B, respectively, we can apply Lemma 6.3. The result follows.

The Ulm invariants of a p-coprimary group A are the same as those of a basic subgroup of tA. Hence Theorem 7.1 and the fact that direct sums of cyclic groups are characterized by their Ulm invariants yield the following.

COROLLARY 7.2. *Let A and B be adjusted, p-coprimary algebraically compact groups. Then $A \cong B$ if and only if $A_p(n) = B_p(n)$ for all non-negative integers n.*

If A is adjusted, then so is A_p since it is a summand of A. Two adjusted groups A and B are isomorphic if and only if $A_p \cong B_p$ for all primes p. Note that the n-th Ulm invariant of A is equal to the n-th Ulm invariant of $t(A_p)$.

THEOREM 7.3. *If A and B are adjusted algebraically compact groups, then $A \cong B$ if and only if $A_p(n) = B_p(n)$ for all primes p and all non-negative integers n.*

PROOF. This follows from Corollary 7.2.

We conclude the work of this section by answering the question: What sets of cardinal numbers can be the invariants for adjusted groups?

Suppose that for each prime p an arbitrary sequence of cardinals, say $\{s(n, p)\}, n = 0, 1, 2, \cdots$ is given. For each p let $B(p) = \sum_n B(n, p)$ be that direct sum of cyclic p-groups having $\{s(n, p)\}$ as its sequence of Ulm invariants, that is, where $B(n, p)$ is a direct sum of $s(n, p)$ copies of $Z(p^{i+1})$.

Let $\bar{B}(p) = t(\prod_n B(n, p))$. Note that $\prod_n B(n, p)$ is reduced algebraically compact. Finally, let A_p be that subgroup of $\prod_n B(n, p)$ such that $A_p/\bar{B}(p)$ is the maximum divisible subgroup of $\prod_n B(n, p)/\bar{B}(p)$. It follows, as in the proof of Theorem 5.1, that each A_p is adjusted. Hence $\prod A_p/t(\prod A_p) = \prod A_p/\sum t(A_p)$ is divisible, making $\prod A_p$ adjusted. We have thus constructed an adjusted group, $\prod A_p$, having for each p the sequence $\{s(n, p)\}$, $n = 0, 1, 2, \cdots$ as its sequence of Ulm invariants with respect to the prime p.

8. **The torsion free case.** We now turn to the torsion free case. Let A be a reduced, torsion free, p-coprimary algebraically compact group. We will show that the dimension, $\dim A/pA$, of A/pa as a vector space over the p-element field is a complete invariant for A. Furthermore, given any cardinal α there is such an A with $\dim A/pA = \alpha$.

THEOREM 8.1. *Let A and B be reduced, torsion free, p-coprimary algebraically compact groups. Then $A \cong B$ if* $\dim A/pA = \dim B/pB$.

PROOF. Let Z_p be the subgroup of Q consisting of those rational numbers with denominators relatively prime to p. For $r/s \in Z_p$ and $x \in A$, we set $(r/s)x = x_1$, where x_1 is the unique element of A such that $x = sx_1$. This definition makes Z_px a subgroup of A.

Choose a basis of A/pA, say $\{x_i + pA\}_{i \in I}$, where $|I| = \dim A/pA$. Then $\sum_i Z_px_i$ is a subgroup of A and for $a \in A$ we can write $a = \sum m_ix_i + pa_1$, where the sum is finite, $a_1 \in A$ and $m_i \in Z$. Thus $A/\sum_i Z_px_i$ is p-divisible, hence divisible.

Suppose q is prime, $q \neq p$, $a \in A$, and $qa = \sum r_ix_i \in \sum_i Z_px_i$. Since each r_i is divisible by q, we conclude that $a \in \sum_i Z_px_i$. Let

$pa = \sum r_i x_i \in \sum_i Z_p x_i$. Since $\{x_i + pA\}_{i \in I}$ is a basis for A/pA, it follows that p divides each r_i, so that $a \in \sum_i Z_p x_i$. Hence $A/\sum_i Z_p x_i$ is torsion free.

Choose an analogous set $\{y_i + pB\}_{i \in I}$ for B. If we identify the subgroups $\sum_i Z_p x_i$ and $\sum_i Z_p y_i$ of A and B, respectively, we can apply Lemma 6.3 to conclude that $A \cong B$.

We now turn to the construction of a reduced, torsion free, p-coprimary algebraically compact group H with $\dim H/pH$ any prescribed cardinal. First, we need a theorem due to Harrison [3], but for which we have an elementary proof using our Lemma 2.1.

THEOREM 8.2. *If T is a torsion group, then* $\mathrm{Hom}(T, G)$ *is algebraically compact for all groups G.*

PROOF. Let A be pure in B and $f: A \to \mathrm{Hom}(T, G)$. By Lemma 2.1 we can assume that B/A is divisible.

Let $b \in B$, $t \in T$, and let m be a multiple of $O(t)$, the order of t. Then $b = mb_m + a_m$ for suitable $b_m \in B$ and $a_m \in A$. Define $g: B \to \mathrm{Hom}(T, G)$ by $g(b)(t) = f(a_m)(t)$. If n is also a multiple of $O(t)$ and $b = nb_n + a_n$, then $nb_n - mb_m = a_m - a_n$. Hence $a_m - a_n = O(t) \cdot x$ for suitable $x \in B$. By purity of A in B we have $a_m - a_n = O(t) \cdot a$ for some $a \in A$. Thus $f(a_m)(t) = f(a_n)(t)$ so that g is well-defined. It is routine to check that $g(b) \in \mathrm{Hom}(T, G)$ and to check that g itself is a homomorphism. Since g extends f we conclude that $\mathrm{Hom}(T, G)$ is algebraically compact.

THEOREM 8.3. *Let α be any cardinal number. Then* $H = \mathrm{Hom}(Z(p^\infty), \sum_\alpha Z(p^\infty))$ *is a reduced, torsion free, p-coprimary algebraically compact group with* $\dim H/pH = \alpha$.

PROOF. The group H is algebraically compact by Theorem 8.2, is torsion free since $Z(p^\infty)$ is divisible, is reduced since $Z(p^\infty)$ is torsion, and is p-coprimary since $Z(p^\infty)$ is p-coprimary.

Let f_i be an element of H that maps $Z(p^\infty)$ isomorphically onto the i-th summand in $\sum_\alpha Z(p^\infty)$. It is straightforward to check that $\{f_i + pH\}_{i \in I}$, where $|I| = \alpha$, is a basis for H/pH. The crucial fact needed is that $f \in H$ is divisible by p if and only if f has non-zero kernel; that is, two elements f and g of H are equal modulo pH if and only if $f - g$ annihilates the socle of $Z(p^\infty)$.

The following definition is not quite what one would expect in light of Theorem 8.1; however in the last section we will see the reason for defining the invariants in this manner.

DEFINITION. For G a group and p a prime, let $G_0(p)$ denote the dimension of $G/(pG + tG)$ as a vector space over the integers modulo p. We call $G_0(p)$ the *p-th torsion free number of G.*

Theorem 8.4. *Two reduced torsion free algebraically compact groups A and B are isomorphic if and only if $A_0(p) = B_0(p)$ for all primes p.*

Proof. Note that if F is torsion free, $F_0(\mathrm{p}) = \dim F/pF$. Thus the result follows from Theorems 8.1 and 6.4.

If for each prime p an arbitrary cardinal α_p is prescribed, we can construct a reduced, torsion free, algebraically compact group A with $A_0(p) = \alpha_p$ by taking $A = \prod_p A_p$, where $A_p = \mathrm{Hom}(Z(p^\infty), \sum_{\alpha_p} Z(p^\infty))$. Hence an arbitrary sequence of cardinals can be the sequence of torsion free numbers for a reduced algebraically compact group.

9. **Conclusion.** The results of this paper are summarized in the following theorem.

Theorem 9.1. *Two algebraically compact groups A and B are isomorphic if and only if*

(i) $dA \cong dB$,
(ii) $A_0(p) = B_0(p)$ *for all primes p, and*
(iii) $A_p(n) = B_p(n)$ *for all primes p and all non-negative integers n.*

Proof. Of course this follows from Theorems 7.3 and 8.4. We must make two observations. Let the reduced algebraically compact group G be decomposed as $G = A \oplus F$ where A is adjusted and F is torsion free. Clearly $G_p(n) = A_p(n)$ for all p and all n. Finally, if we recall that A is adjusted if and only if A/tA is divisible, we see that $A_0(p) = 0$ for all p. Thus $G_0(p) = F_0(p)$.

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Moorhead State College, Moorhead, Minnesota 56560
New Mexico State University, Las Cruces, New Mexico 88001