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ON CELLS IN EUCLIDEAN SPACE THAT CANNOT BE SQUEEZED

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1. **Introduction.** Let K be a k-cell in Euclidean n-space E^n. Loosely speaking, we say that a map f of E^n onto itself squeezes K to an m-cell provided that f is a homeomorphism off K and $f|K$ is related to a canonical projection of a round k-cell to a round m-cell. In case $n = 3$ it is known that for each 3-cell K in E^3 there exist many maps squeezing K to 2-cells and many maps squeezing K to 1-cells [**6**], and whenever $n \geqq 3$ it is known that for each 2-cell D in E^n there exist many maps squeezing D to 1-cells ([**6**], [7], [**15**]). In this paper we point out counterexamples to generalizations of these results: there exists a k-cell K in E^n $(3 \leqq k < n)$ for which there is no map squeezing K to a lower dimensional cell, and there exists an n-cell K^* in E^n $(n \geqq 4)$ for which there is no map squeezing K^* to an m-cell $(m \leqq n - 2)$. These counterexamples are embedded as everywhere wild subsets of E^n with properties that easily eliminate the possibility of a squeezing map. However, this paper is not concerned primarily with such examples; instead, the purpose is to prove that for some relatively simple k-cells in E^n $(n \geqq 4)$, each one locally tame modulo a Cantor set, there is no map squeezing any one of them to either a 2-cell or a 1-cell.

2. **Definitions.** For each positive integer k let B^k denote the set $\{(x_1, \cdots, x_k) \in E^k \mid x_1^2 + \cdots + x_k^2 \leqq 1\}$. Clearly for $m \leqq k$, B^m can be regarded as a subset of B^k. Let π denote the projection map of B^k to B^m that sends $(x_1, \cdots, x_k)$ to $(x_1, \cdots, x_m)$.

Suppose K is a k-cell in E^n. A map f of E^n onto itself is said to *squeeze K to an m-cell* iff there exist homeomorphisms g of B^k onto K and h of B^m onto $f(K)$ such that f carries $E^n - K$ homeomorphically onto $E^n - f(K)$ and $fg = h\pi$. In particular, we say that such a map f squeezes K to the m-cell $f(K)$. Alternatively, if there is no map f that squeezes K to an m-cell, then we say that K *cannot be squeezed to an m-cell.*

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A metric space X is uniformly locally simply connected, or 1-ULC, iff to each $\epsilon > 0$ there corresponds a $\delta > 0$ such that any map from the boundary of the disk B^2 into a δ-subset of X can be extended to a map of B^2 into an ϵ-subset of X. Similarly, given a subset Y of X, we say that Y is 1-ULC in X iff to each $\epsilon > 0$ there corresponds a $\delta > 0$ such that each map of the boundary of B^2 into a δ-subset of Y can be extended to a map of B^2 into an ϵ-subset of X.

Let X be a compact subset of E^n. Following [1] we say that X has Property UV^∞ iff for each open set U containing X there exists an open set V containing X that is contractible in U. This concept has been studied by other authors (see [8]) under a variety of names.

Given such a set X, we say that X satisfies the Cellularity Criterion iff for each open set U containing X there exists a set V containing X such that each map of the boundary of B^2 into $V - X$ extends to a map of B^2 into $U - X$.

We use the symbols Bd and Int to denote the boundary and interior of a manifold-with-boundary, and we use Cl to denote topological closure.

For definitions of other terms used here the reader is referred to such papers as [3], [9], [10].

3. Cells that cannot be squeezed to arcs.

PROPOSITION 3.1. *If e is an embedding of B^k in E^n ($n \geqq 3$) such that $e(B^{k-1})$ satisfies the Cellularity Criterion, then each UV^∞ continuum X in $e(B^{k-1})$ also satisfies the Cellularity Criterion. Thus, X is cellular provided $n \neq 4$.*

PROOF. Let U be a neighborhood of X in E^n. Since X has property UV^∞, there exists a closed neighborhood N of X in $e(B^{k-1})$ that is contractible in $U \cap e(B^{k-1})$. Define $Z = N \cap \mathrm{Cl}(e(B^{k-1}) - N)$. Use the structure of $e(B^k)$ to lift the induced contraction (obtained by restriction) of Z off X, defining a contraction of Z in $U \cap (e(B^k) - X)$. Apply Tietze's Extension Theorem to extend this contraction to one having domain a neighborhood W_2 of Z in E^n and range $U - X$.

Let W_1 be an open subset of U such that $W_1 \cap e(B^{k-1}) = N - Z$, and W_3 an open subset of E^n containing $e(B^{k-1}) - N$ such that $W_1 \cap W_3 = \emptyset$. The hypothesis that $e(B^{k-1})$ satisfies the Cellularity Criterion implies the existence of a neighborhood V^* of $e(B^{k-1})$ such that each loop in $V^* - e(B^{k-1})$ is contractible in $(W_1 \cup W_2 \cup W_3) - e(B^{k-1})$. Define $V' = V^* \cap W_1$.

We assume that if $k = n$, then $X \cap e(\mathrm{Bd}\, B^{k-1}) \neq \emptyset$, for otherwise the Corollary to Theorem 8 of [9], which applies to UV^∞ continua as well as to compact absolute retracts, implies that X is cellular. In this case there exists a neighborhood V of X in E^n such that $V \subset V'$ and each point of $V \cap (e(B^{k-1}) - X)$ can be joined to a point of $V \cap e(\text{Bd}\, B^{k-1})$ by an arc contained in $V' \cap (e(B^{k-1}) - X)$; in case $k < n$ define $V = V'$.

We show that any loop in $V - X$ is contractible in $U - X$. Let f be a map of $\text{Bd}\, B^2$ into $V - X$. Then f is homotopic in $V' - X$ to a map f' of $\text{Bd}\, B^2$ into $V - e(B^{k-1})$: if $k = n$ we adjust f slightly so that $f(\text{Bd}\, B^2)$ meets $e(B^{k-1})$ at just a finite number of points and perform a homotopy in $V' - X$ that pushes each such point along an arc in $V' - X$ out over the boundary of $e(B^{k-1})$; if $k < n$ we can perform a slight adjustment of f to move $f(\text{Bd}\, B^2)$ away from $e(B^{k-1})$. By hypothesis f' can be extended to a map

$$F : B^2 \to (W_1 \cup W_2 \cup W_3) - e(B^{k-1}) \subset (W_1 \cup W_2 \cup W_3) - X.$$

However, $F(B^2)$ may contain points of W_3 outside U. To remedy this, remove the interiors of finitely many pairwise disjoint 2-cells in B^2 to obtain a disk with holes H in B^2 such that

$$\text{Bd}\, B^2 \subset \text{Bd}\, H, \qquad F(H) \subset W_1 \cup W_2 \subset U,$$
$$F(\text{Bd}\, H - \text{Bd}\, B^2) \subset W_2.$$

Redefine F on each component Y of $B^2 - H$ by restricting the contraction of W_2 in $U - X$ to $\text{Bd}\, Y$. This produces the required contraction of $f(\text{Bd}\, B^2)$ in $U - X$. The second part of the proposition follows, of course, from [9, Theorem 1].

Corollary 3.2. *If e is an embedding of B^k in E^n ($n \geqq 5$) such that $e(B^{k-1})$ is cellular, then each Cantor set C in $e(B^{k-1})$ is tame.*

Proof. Select an arc X in $e(B^{k-1})$ containing C. By the preceding proposition X is cellular, and by [**10**, Lemma 3] C is tame.

Corollary 3.2 also holds when $n = 3$, in which case it is a direct consequence of McMillan's collapsing theorem [**11**, Theorem 1].

A compact 0-dimensional subset C of a cell K is said to be *tame relative to* K iff $C \cap \text{Bd}\, K$ is tame in $\text{Bd}\, K$ and $C \cap \text{Int}\, K$ is locally tame in $\text{Int}\, K$. In addition, a 0-dimensional F_σ-set F in K is said to be *tame relative to* K iff F can be expressed as a countable union of compact subsets that are tame relative to K.

Proposition 3.3. *If K denotes a k-cell in E^n ($3 \leqq k \leqq n$, $n \geqq 4$) that is locally tame modulo a Cantor set C and that can be squeezed to an arc, then there exists a 0-dimensional F_σ-subset F of K such that F is tame relative to K and $E^n - K$ is 1-ULC in $(E^n - K) \cup F$.*

Proof. Let f be a map of E^n onto itself that squeezes K to an arc and $g: B^k \to K$ and $h: B^1 \to f(K)$ the accompanying homeomorphisms, such that $fg = h\pi$. Enumerate the rational numbers in $(-1, 1)$ as $r_1, r_2, \cdots$, and for $i = 1, 2, \cdots$ define a $(k-1)$-cell Q_i as $g\pi^{-1}(r_i) = f^{-1}h(r_i)$. Since the nondegenerate point inverses under f are $(k-1)$-cells like these Q_i's, it follows from [1, Lemma 5.2] that each Q_i satisfies the Cellularity Criterion.

Case 1. $3 \leqq k \leqq n - 2$. By Corollary 3.2 each Q_i is locally tame modulo the tame Cantor set $Q_i \cap C$. The dimension restriction for this case implies $\dim Q_i \leqq n - 3$, from which one can show easily that $E^n - Q_i$ is 1-ULC. Thus, Q_i is tame ([3, Theorem 2], [12, Theorem 1]). Then, for any map s of B^2 into E^n such that $s(\text{Bd } B^2) \subset E^n - K$, the map can be altered slightly, pushing $s(B^2)$ off the Q_i's one at a time, to define a map s' on B^2 such that (i) $s'|\text{Bd } B^2 = s|\text{Bd } B^2$, (ii) s' is close to s, and (iii) $s(B^2) \cap K$ is a 0-dimensional subset of $K - \bigcup Q_i$.

Essentially (up to a short homotopy) there are just countably many maps of $\text{Bd } B^2$ into $E^n - K$ requiring extension. Thus, using the property established in the preceding paragraph, we can find a 0-dimensional F_σ-set F in $K - \bigcup Q_i$ such that $E^n - K$ is 1-ULC in $(E^n - K) \cup F$. Accordingly, the set F can be decomposed into closed (relative to K) subsets $F_1, F_2, \cdots$, and since each F_j misses $\bigcup Q_i$, Corollary 4 of [2] implies that F_j is a subset of a Cantor set that is tame relative to K.

Remark. Whenever $k < n - 2$ we may assume that F is a subset of C.

Case 2. $k = n$. Certainly $E^n - K$ is 1-ULC in $(E^n - K) \cup C$. Let s denote a map of B^2 into $(E^n - K) \cup C$ such that $s(\text{Bd } B^2) \subset E^n - K$. Since $Q_i \cap C$ satisfies the cellularity criterion, s can be modified near points of $s^{-1}(s(B^2) \cap Q_i \cap C)$ so that $s(B^2) \cap Q_i \cap C = \emptyset$. But $Q_i \cap C$ is in a tame arc in K, which implies that $\text{Bd } K - (Q_i \cap C)$ is 1-ULC. Thus, the modification of s can be chosen with range $(E^n - K) \cup (\text{Bd } K - (Q_i \cap C))$, and using the local tameness of $\text{Bd } K - C$, we can improve this further to $(E^n - K) \cup (C - Q_i)$. Furthermore, by repeating this process carefully we find a map s' of B^2 such that (i) $s|\text{Bd } B^2 = s'|\text{Bd } B^2$, (ii) s' is close to s, and (iii) $s'(B^2) \subset (E^n - K) \cup (C - \bigcup Q_i)$.

As in Case 1, there exists an F_σ-set F in $C - \bigcup Q_i$ such that $E^n - K$ is 1-ULC in $(E^n - K) \cup F$, and, by [2], F can be expressed as the countable union of tame closed subsets.

Case 3. $k = n - 1$. The argument for this case requires some technical variations on the argument for Case 2, and we leave it to the interested reader.

PROPOSITION 3.4. *Suppose K is a k-cell in E^n ($n \geqq 3$), F is a 0-dimensional F_σ-set in K such that F is tame relative to K and $E^n - K$ is 1-ULC in $(E^n - K) \cup F$, M is a $(k-1)$-cell spanning K, and $\epsilon > 0$. Then there exists an ϵ-push θ of K onto itself such that each loop in $E^n - K$ is contractible in $E^n - \theta(M)$.*

PROOF. As in the proof of Proposition 3.3, each map f of Bd B^2 into $E^n - K$ can be extended to a map g of B^2 into E^n such that $g^{-1}(g(B^2) \cap K)$ is 0-dimensional. Then, since $E^n - K$ is 1-ULC in $(E^n - K) \cup F$, we perform modifications of g near points of $g^{-1}(g(B^2) \cap K)$ to define a map g' of B^2 into $(E^n - K) \cup F$ that extends f.

To complete the argument we need only push M off F with an ϵ-push of K. The set F can be decomposed into compact sets $F_1, F_2, \cdots$ that are tame relative to K. We can construct a sequence $\{\theta_n\}$ of pushes of K, where θ_n first pushes Bd M off Bd $K \cap (\bigcup_{i=1}^n F_i)$ and then pushes Int M off $(\bigcup_{i=1}^n F_i)$ and keeps the adjusted Bd M fixed, with sufficient care to guarantee that $\theta = \lim \theta_n$ is an ϵ-push of K and $\theta(M) \cap F = \emptyset$.

REMARK. In case $3 \leqq k \leqq n-2$, one can easily show that $E^n - \theta(M)$ is 1-ULC, which implies that $\theta(M)$ is tame ([3], [12]). In case $k = n \geqq 5$, if M is locally tame at each point of $M \cap$ Int K, it is also possible to show that $E^n - \theta(M)$ is 1-ULC, and Theorem 9 of [13] implies that $\theta(M)$ is tame.

Propositions 3.3 and 3.4 combine to imply that the cells of [4] cannot be squeezed to arcs.

THEOREM 3.5. *For $3 \leqq k \leqq n$ and $n \geqq 4$ there exists a k-cell in E^n that is locally tame modulo a Cantor set but that cannot be squeezed to a 1-cell.*

PROOF. The k-cells K described in [4] are locally tame modulo Cantor sets, but each contains a 2-cell D (with D in Bd K if $k = n$) such that there is no small push θ of K onto itself such that every loop in $E^n - K$ is contractible in $E^n - \theta(D)$.

4. **The composition of squeezes.**

PROPOSITION 4.1. *If f_r is a map of E^n onto itself that squeezes the r-cell R to the s-cell S and f_s is a map of E^n onto itself that squeezes S onto the t-cell T, then $f_s f_r$ squeezes R onto T.*

PROOF. The only problem occurs in showing that $f_s f_r \mid R$ is conjugate to the canonical projection of B^r onto B^t. Let π_r denote the projection of B^r onto B^s and π_s the projection of B^s onto B^t. First we establish the following claim:

Any homeomorphism λ of B^s onto itself extends to a homeomorphism L of B^r onto itself such that $\lambda\pi_r = \pi_r L$.

Each $b \in B^r$ can be uniquely represented as $b = (x, y)$ where $x \in B^s$ and y is a $(r - s)$-tuple. Define

$$L(b) = (\lambda(x), m(x) \cdot y)$$

where $m(x) = [(1 - |\lambda(x)|^2)/(1 - |x|^2)]^{1/2}$. (It is to be understood that $m(x) \cdot y = 0$ if $|x| = |\lambda(x)| = 1$.) Verifying that L is the required homeomorphism is routine and is left to the reader.

We now consider the proof of the proposition. Let $g_r : B^r \to R$ and $h_s : B^s \to S$ denote the homeomorphisms such that $f_r g_r = h_s \pi_r$, and let $g_s : B^s \to S$ and $h_t : B^t \to T$ denote the homeomorphisms such that $f_s g_s = h_t \pi_s$. Using the claim above we find a homeomorphism L of B^r onto itself such that $\pi_r L = (h_s^{-1} g_s)\pi_r$.

Define $g : B^r \to R$ as $g = g_r L$. Then

$$(f_s f_r)g = f_s f_r g_r L = f_s h_s \pi_r L = f_s h_s h_s^{-1} g_s \pi_r = f_s g_s \pi_r = h_t \pi_s \pi_r.$$

Thus, $f_r f_g$ squeezes R to T.

THEOREM 4.2. *For $3 \leqq k \leqq n$ and $n \geqq 4$ there exists a k-cell in E^n that is locally tame modulo a Cantor set and that cannot be squeezed to a 1-cell or a 2-cell.*

Since any 2-cell in E^n can be squeezed to a 1-cell ([**5**, Theorem 2], [**7**, Theorem 1], [**15**, Theorem 3]), Proposition 4.1 implies that no cell satisfying Theorem 3.5 can be squeezed to a 2-cell.

5. Cells that cannot be squeezed.

PROPOSITION 5.1. *If K is a k-cell in E^n $(3 \leqq k < n)$ and f is a map of E^n to itself squeezing K to an m-cell $(m < k)$, then K contains a 2-cell D that satisfies the Cellularity Criterion. Thus, if $n \geqq 5$, then K contains a cellular 2-cell.*

PROOF. In case $2 \leqq m < k$, then by [**14**, Theorem 3] or [**15**, Theorem 2] there exists a tame arc A in Int $f(K)$. Certainly A must satisfy the Cellularity Criterion, and consequently $f^{-1}(A)$ also must satisfy it [**1**, Lemma 5.2]. Let $g : B^k \to K$ and $h : B^m \to f(K)$ be homeomorphisms such that $fg = h\pi$. Note that $f^{-1}(A) = g\pi^{-1}h^{-1}(A)$, which implies that $f^{-1}(A)$ is a $(k - m + 1)$ cell. Since

$(k - m + 1) \geqq 2$, $f^{-1}(A)$ collapses to a 2-cell D, and such a cell satisfies the Cellularity Criterion [11, Theorem 1]. As before, the second statement of the proposition follows immediately from [9, Theorem 1].

An analogous proof can be given for the following result about codimension 0 cells.

PROPOSITION 5.2. *If K is an n-cell in E^n ($n \geqq 4$) and f a map of E^n onto itself squeezing K to an m-cell ($m \leqq n - 2$), then* Bd K *contains a 2-cell D that satisfies the Cellularity Criterion. Thus, if $n \geqq 5$,* Bd K *contains a cellular 2-cell.*

These results immediately imply that the cells constructed in [5] satisfy the following theorem.

THEOREM 5.3. *For $3 \leqq k < n$ there exists a k-cell in E^n that cannot be squeezed to an m-cell ($m < k$) and there exists an n-cell in E^n that cannot be squeezed to a j-cell ($j \leqq n - 2$).*

PROOF. Examples are described in [5] of k-cells in E^n such that for no 2-cell D in K (or in Bd K if $k = n \geqq 4$) is $E^n - D$ simply connected. In particular, no 2-cell D in K (or in Bd K) satisfies the Cellularity Criterion.

QUESTION. Can each n-cell in E^n be squeezed to an $(n - 1)$-cell?

REFERENCES

1. Steve Armentrout, *Homotopy properties of decomposition spaces,* Trans. Amer. Math. Soc. **143** (1969), 499–507. MR **42** #8214.

2. J. L. Bryant, *On embeddings with locally nice cross-sections,* Trans. Amer. Math. Soc. **155** (1971), 327–332. MR **43** #2721.

3. J. L. Bryant and C. L. Seebeck III, *Locally nice embeddings of polyhedra,* Quart. J. Math. Oxford Ser. (2) **19** (1968), 257–274. MR **38** #2751.

4. R. J. Daverman, *On the scarcity of tame disks in certain wild cells,* Fund. Math. **79** (1973), 63–77.

5. ———, *On the absence of tame disks in certain wild cells* (to appear).

6. R. J. Daverman and W. T. Eaton, *An equivalence for the embeddings of cells in a 3-manifold,* Trans. Amer. Math. Soc. **145** (1969), 369–381. MR **40** #3519.

7. ———, *Each disk in E^n can be squeezed to an arc* (manuscript).

8. R. C. Lacher, *Cell-like spaces,* Proc. Amer. Math. Soc. **20** (1969), 598–602. MR **38** #2754.

9. D. R. McMillan, Jr., *A criterion for cellularity in a manifold,* Ann. of Math. (2) **79** (1964), 327–337. MR **28** #4528.

10. ———, *Taming Cantor sets in E^n,* Bull. Amer. Math. Soc. **70** (1964), 706–708. MR **29** #1628.

11. ———, *Piercing a disk along a cellular set,* Proc. Amer. Math. Soc. **19** (1968), 153–157. MR **36** #3332.

12. R. T. Miller, *Approximating codimension three embeddings*, Ann. of Math. (2) **95** (1972), 406–416.

13. C. L. Seebeck III, *Collaring an* $(n - 1)$*-manifold in an* n*-manifold*, Trans. Amer. Math. Soc. **148** (1970), 63–68. MR 41 #2692.

14. ———, *Tame arcs on wild cells*, Proc. Amer. Math. Soc. **29** (1971), 197–201. MR 43 #6896.

15. R. B. Sher, *Tame polyhedra in wild cells and spheres*, Proc. Amer. Math. Soc. **30** (1971), 169–174. MR 43 #6887.

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