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COMPLETE ERGODICITY, WEAK MIXING AND STACKING METHODS

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Introduction. The purpose of this paper is to introduce stacking methods for constructing measure-preserving transformations, and to develop sufficient conditions which will insure that the resulting transformations are completely ergodic. It will be shown that while, in general, complete ergodicity does not imply weak mixing, in the case of these transformations it does.

1. **Preliminaries and Notation.** All transformations T considered in this paper will be defined on $[0, 1) = I$. They will be bimeasurable with respect to Lebesgue measure, invertible, and measure-preserving. Furthermore they will be constructed by a stacking method which fits into one of the four categories described in the next section. The following are a few preliminary definitions and theorems:

DEFINITION 1.1. T admits a k-stack if and only if there exists a set A with $\mu(A) > 0$ such that $T^kA = A$ and $\mu(A \cap T^iA) = 0$, $0 < i < k$.

THEOREM 1.1 (BLUM AND FRIEDMAN [1]). *T admits a k-stack if and only if the* kth *roots of unity are eigenvalues of T.*

In this paper, the term eigenvalue of T will be used in place of the eigenvalues of the induced operator U_T where $U_T(f)(x) = f(Tx)$.

LEMMA 1.1. *If T admits a k-stack then T^k is not ergodic.*

THEOREM 1.2. *If T^k is not ergodic, then for some prime $p \leqq k$, T admits a p-stack; furthermore, if k is prime, then T admits a k-stack.*

COROLLARY 1.2. *T^p is ergodic if and only if the* pth *roots of unity are not eigenvalues of T for prime p.*

2. **Stacking Methods.** Stacking methods were originally developed by Von Neumann and Kakutani to show the existence of ergodic transformations. R. V. Chacon generalized the method in [2].

Type 1a. The unit interval is divided into a stack consisting of h_1 subintervals of width w_1 each, and a residual R_1, and T is defined

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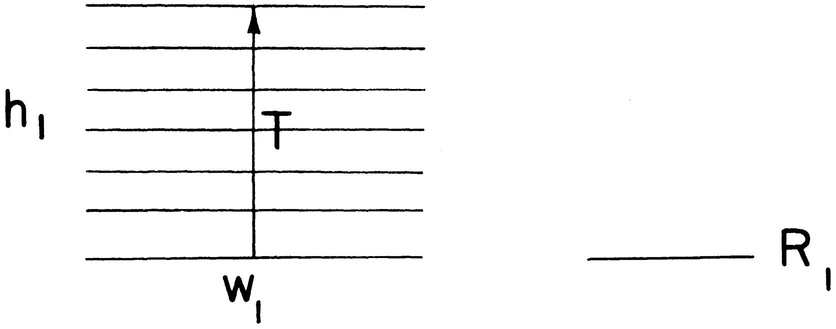

Figure 1.

linearly (T of each point is the point directly above it) except for the top of the stack and the residual, where T is not yet defined. (See Fig. 1.)

At step $n(\geqq 2)$, the stack of height h_{n-1} is cut into $k \geqq 3$ equal substacks and j pieces are cut from the residual and added to column $\ell \neq k$. These pieces are cut from the left end of the residual R_{n-1} in such a manner so as to maintain the measure-preserving property of T. (See Fig. 2.) Now T is extended linearly as in Figure 3. Then the stack is restacked to obtain a single stack and smaller residual. Note that the sequence of heights $\{h_n\}$ satisfies the difference equation $h_n = kh_{n-1} + j$.

Type 1b. The procedure is the same as for type 1a except that $k \geqq 2$ and the pieces are added in column k. Again note that $h_n = kh_{n-1} + j$.

Type 2a. At step n cut the stack of height h_{n-1} into $k_{n-1} \geqq 3$ equal pieces and add j_{n-1} pieces in column $\ell_{n-1} \neq k$. In this case we have $h_n = k_{n-1}h_{n-1} + j_{n-1}$.

Type 2b. The same as type 2a, except that $k_{n-1} \geqq 2$ and $\ell_{n-1} = k$. Again we have the difference equation $h_n = k_{n-1}h_{n-1} + j_{n-1}$. In a stacking method of type 1 (or type 2 if j_n and k_n are constant) the following calculation gives the length of the original base so that T will be everywhere defined. If the original base is $[0, a)$ then $[0, h_1 a)$ is used in the original stack, and $R_1 = [h_1 a, 1)$. At the $n + 1$st step j pieces of length a/k^n were cut from R_{n-1}. Thus to completely use up R_1 we must have $1 - h_1 a = \sum_{n=1}^{\infty} ja/k^n$ or $a = (k - 1)/(j + (k - 1)h_1)$.

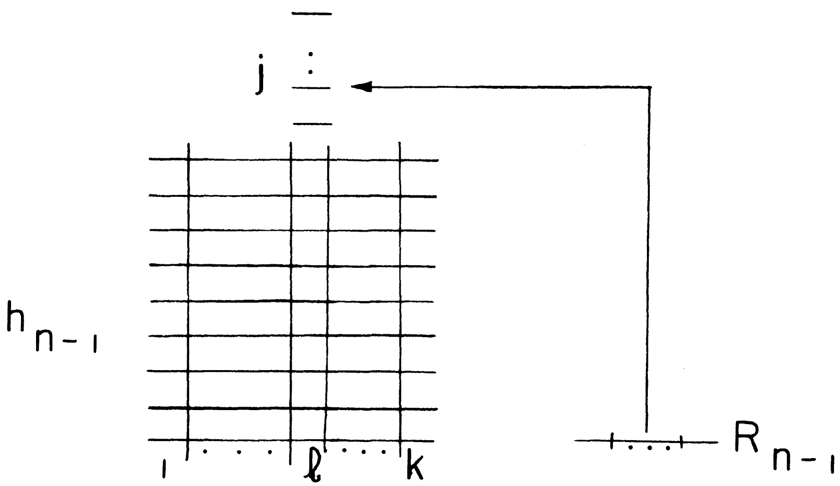

Figure 2.

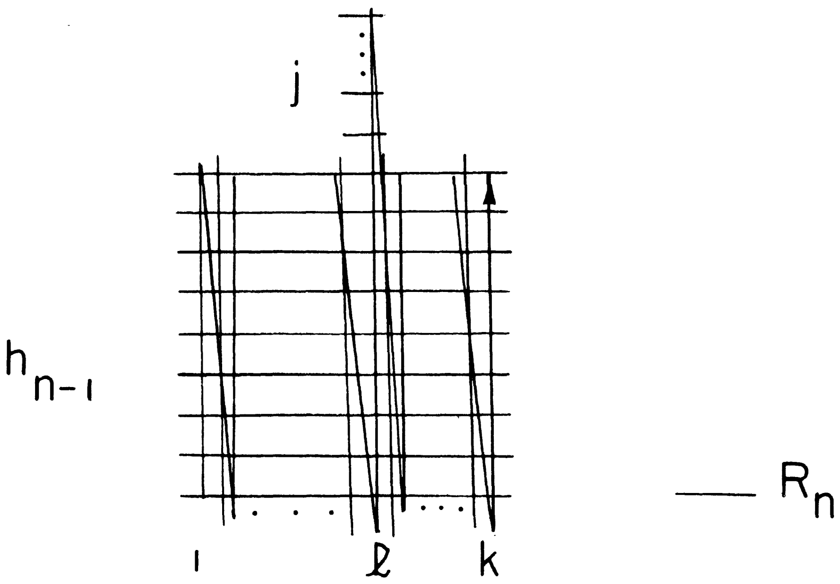

Figure 3.

NOTATION. The stack at the end of step n will be referred to as C_n, and $I_{x,n}$ refers to the interval at height x in C_n. For more information about stacking methods in general, see e.g. Friedman [4]. A modification of the proof of theorem (6.3) in [4] gives the following theorem:

THEOREM 2.1. *If T is constructed by any of the stacking methods of this section, then T is ergodic.*

3. Complete ergodicity.

DEFINITION 3.1. T is said to be *completely ergodic* if T^k is ergodic for every $k \in N$.

The following lemma is immediate.

LEMMA 3.1. *T is completely ergodic if and only if T^p is ergodic for every prime p.*

The next two theorems are the basis for the arguments used in the remainder of this section.

THEOREM 3.1. *Let T be an ergodic measure-preserving transformation constructed by a stacking method. If T^k is ergodic, then for every $I_{x,n}$ there exists $m(k,n)$ such that if $L = \{\ell : I_{\ell,n+m(k,n)} \subset I_{x,n}\}$ then L contains a complete system of residue classes* mod k.

PROOF. By contradiction: Suppose there is no integer m^* such that L contains a complete system of residue classes mod k for some $I_{x,n}$, with T^k ergodic. Then we are always missing at least one residue class mod k, call it j, as we restack. Thus we have:

$$\bigcup_{m=0}^{\infty} T^{km}(I_{x,n}) \cap C_\ell \subset C_\ell \setminus \bigcup_{p=0}^{p^*} I_{j+kp},$$

where $p^* = [(h_\ell - j)/k]$. Now let w_n be the width of C_n. Thus:

$$\mu\left(\bigcup_{m=0}^{\infty} T^{km}(I_{x,n}) \cap C_\ell\right) \leqq \mu(C_\ell) - \left[\frac{h_\ell - j}{k}\right] w_\ell,$$

is true for each ℓ, and therefore in the limit as $\ell \to \infty$. Hence

$$\mu\left(\bigcup_{m=0}^{\infty} T^{km}(I_{x,n})\right) \leqq 1 - \lim_{\ell \to \infty} \left[\frac{h_\ell - j}{k}\right] w_\ell,$$

since $C_\ell \to I$ and $\mu(C_\ell) \to 1$. Note that $\lim_{\ell \to \infty} [(h_\ell - j)/k] w_\ell = 1/k$ since $h_\ell w_\ell \to 1$ and $w_\ell \to 0$. Thus $\mu(\bigcup T^{km}(I_{x,n})) \leqq 1 - 1/k < 1$, which contradicts the fact that T^k is ergodic.

THEOREM 3.2. *Let T be an ergodic measure-preserving transformation constructed by stacking procedure. Let p be prime. If for every $I_{x,n}$ there exists $m(p,n)$ such that $L = \{\ell : I_{\ell,n+m(p,n)} \subset I_{x,n}\}$ contains two different residue classes* mod p, *then T^p is ergodic.*

The following proof is based on an argument due to Chacon [3], which is included here in its entirety for completeness.

PROOF. If T^p is not ergodic, then by the corollary to theorem 1.2, $\exp(2\pi i/p)$ is an eigenvalue of T. Let f be its associated eigenfunction. Suppose f is constant (and not 0) say $f \equiv r$ on some interval J. Note that there exists x and n such that $J \supset I_{x,n}$. By hypothesis $L \supset \{\ell_1, \ell_2\}$, where $\ell_1 < \ell_2$ and $\ell_1 \bmod p \neq \ell_2 \bmod p$. Let $d = \ell_2 - \ell_1 \bmod p$. Therefore $\ell_2 = d + \ell_1 + qp$ for some integer q. Let $x \in I_{\ell_1,n+m(p,n)}$ and let $y = T^{d+pq}(x) \in I_{\ell_2,n+m(p,n)}$. Then $\{x, y\} \subset I_{x,n}$, so $f(x) = f(y) = r$. But

$$f(y) = f(T^{d+pq}(x)) \exp\left[\left(\frac{2\pi i}{p}\right)(pq + d)\right] f(x) = e^{2\pi i d/p} r.$$

Thus $r = e^{2\pi i d/p}\, r$ but this is impossible as $0 < d < p$. In general proceed by approximation to the previous case. Since T is ergodic, we may assume $|f| = 1$, thus $f(x) = e^{i\theta(x)}$ where θ is a measurable function of x. Let $\mathcal{C}_k = \bigcup_{n>k} \mathcal{C}_n$ hence $\mathcal{C}_k$ is a class of intervals. Our construction implies that for every $k \in N$, $\mathcal{C}_k$ generates $\mathcal{A}$— the Lebesgue measureable sets. It follows that if $A \in \mathcal{A}$ and $\mu(A) > 0$ then for any $\epsilon > 0$ there exists $I \in \mathcal{C}_k$ such that $\mu(A \cap I) > (1 - \epsilon)\mu(I)$. Now by Lusin's theorem there exists a closed set F of measure arbitrarily close to 1, such that θ is uniformly continuous on F. Therefore given $\eta > 0$ there exists $\delta > 0$ such that $x, y \in F$ and $|x - y| < \delta$ imply $|\theta(x) - \theta(y)| < \eta$. Let k be such that $\mu(I) < \delta$ if $I \in \mathcal{C}_k$.

Choose $I \in \mathcal{C}_k$ such that $\mu(I \cap F) > (1 - \epsilon)\mu(I)$. If ϵ is sufficiently small then $I \cap F$ splits into two sets, one of which we label J such that there exists $x, y \in F \cap J$ as above. We therefore have

$$e^{i\theta(y)} = e^{\frac{2\pi i}{p}(pq+d)} \cdot e^{i\theta(x)} \doteq e^{\frac{2\pi i d}{p} + \theta(x)}$$

or alternately $\theta(y) = 2\pi d/p + \theta(x) \bmod 2\pi$. But this contradicts the uniform continuity of θ in F. Therefore T^p is ergodic.

COROLLARY 3.2. *Let T be an ergodic transformation constructed by a stacking procedure. Then T is completely ergodic if and only if for every $k \in N$ and every $I_{x,n}$ there exists $m(k,n)$ such that $L = \{\ell : I_{\ell,n+m(k,n)} \subset I_{x,n}\}$ contains a complete system of residue classes* mod k.

LEMMA 3.2. *If* $(j, h_1) = 1$ *and* $(j, k) = 1$ *and* j *is prime then* $(j, h_n) = 1$ *where* $h_n = kh_{n-1} + j$.

PROOF. The proof is an immediate consequence of the fact that the difference equation $h_n = kh_{n-1} + j$ has the unique solution (see Goldberg [5]) $h_n = k^n h_1 + j(1 - k^n)/(1 - k)$.

THEOREM 3.3. *If* T *is a measure-preserving transformation defined by a stacking procedure of type* 2 (a *or* b) *such that there exists a subsequence* $\{n_\ell\}$ *such that* $j_{n_\ell} = 1$ *or* j_{n_ℓ} *is prime and* $(j_{n_\ell}, h_{n_\ell}) = 1$, *then* T *is completely ergodic.*

PROOF. By lemma 1.2 it will suffice to show that T^p is ergodic for every prime p. By theorem 3.2, it will suffice to show that for every $I_{x,n}$ there exists $m(p, n)$ such that $L = \{\ell : I_{\ell, n+m(p,n)} \subset I_{x,n}\}$ contains representatives of at least two distinct residue classes mod p. (Indeed there exists $m^*(p, n)$ so that L will contain a complete residue system mod p).

Fix p and n. Now consider $I_{x,n}$ for some arbitrary but fixed x. Now restack to step $n_\ell^* + 1$ where $n_\ell^* = \min\{n_m > n\}$. Let $a = n_\ell^*$. Recall that ℓ_a is the column in which the pieces from the residual are added in step a. If $\ell_a \neq 1$ we have as a minimum the following residue classes: $x, x + h_a, x + h_{a+1}$. If $h_a \not\equiv 0 \bmod p$ then $x + h_a$ is a new residue class mod p, and we have our two classes. If $h_a \equiv 0 \bmod p$, recall that $h_{a+1} = k_a h_a + j_a$. Therefore $x + h_{a+1} = x + k_a h_a + j_a \equiv x + j_a \bmod p$. Now if $j_a = j_{n_\ell}^* = 1$, clearly we have a new residue class. If j_a is not 1, then j_a is prime. If $x + j_a$ is not new, then we must have $j_a \equiv 0 \bmod p$, which implies $j_a = p$ since both are prime. Thus we would obtain $h_a \equiv 0 \bmod j_a$ contradicting our hypothesis. So in this case we do have our two residue classes.

If $\ell_a = 1$, then restacking to step $n_\ell^* + 1$ we have at least the following residue classes:

$$x, \quad x + h_a + j_a, \quad x + 2h_a + j_a \quad (\text{since } k_a \geqq 3).$$

Then either $x + h_a + j_a$ is a new residue class and we have our two classes, or not. If not then $h_a + j_a \equiv 0 \bmod p$, i.e., $h_a \equiv -j_a \bmod p$. In this case $x + 2h_a + j_a \equiv x - j_a \bmod p$, and the same argument as above shows that this is a new residue class. Hence in any case L contains at least 2 distinct residue classes mod p, and therefore T^p is ergodic, note $m(p, n) = n_\ell^* + 1 - n$.

The following theorem can be proved directly, in the same manner as the preceding theorem, but is also a corollary to the above theorem.

THEOREM 3.4. *Let T be a measure-preserving transformation constructed by a stacking method of type* 1 (a *or* b) *such that either* $j = 1$ *or* $(j, h_1) = 1$, $(j, k) = 1$ *and* j *is prime. Then* T *is completely ergodic.*

PROOF. Type 1 is a special case of type 2. Note by lemma 3.2, j fills all the hypotheses of theorem 3.3. If T is constructed by type 1a $m(p, n) = 1$, if by type 1b, $m(p, n) = 2$.

THEOREM 3.5. *If T is a transformation constructed by a stacking procedure of type* 2 (a *or* b) *such that there exists a subsequence* $\{n_k\}$ *with* h_{n_k} *prime then* T *is completely ergodic.*

PROOF. Without loss of generality we may assume that all h_n are prime and $\ell_n \neq 1$. Again it will suffice to show that for every prime p and $I_{x,n}$ there exists $m(p, n)$ such that $L = \{\ell : I_{\ell, n+m(p,n)} \subset I_{x,n}\}$ contains at least two distinct residue classes mod p.

Fix p and $I_{x,n}$. We may assume $h_n > p$. If $\ell_n \neq 1$, $I_{x,n} \supset I_{x,n+1} \cup I_{x+h_n,n+1}$ since $k_n \geqq 2$. Note that $h_n \not\equiv 0 \bmod p$, since h_n is prime and $h_n > p$. Hence L contains at least 2 distinct residue classes mod p.

If $\ell_n = 1$, then $k_n \geqq 3$ and therefore

$$I_{x,n} \supset I_{x,n+1} \cup I_{x+h_n+j_n,\, n+1} \cup I_{x+2h_n+j_n,\, n+1}\,.$$

Either $x + h_n + j_n$ is a new residue class mod p or else $h_n + j_n \equiv 0 \bmod p$. In the latter case $x + 2h_n + j_n \equiv x + h_n \bmod p$ which must be new since $h_n > p$ and h_n is prime. Thus in any case for $m(p, n) = n_\ell + 1 - n$ where $n_\ell = \min\{n_\ell > n, n_\ell$ in sequence with h_{n_ℓ} prime$\}$ L contains at least 2 residue classes mod p.

4. Weak Mixing.

DEFINITION. T is weakly mixing if and only if for every pair of measurable sets A and B

$$\frac{1}{n} \sum_{i=0}^{n-1} |\mu(T^i A \cap B) - \mu(A)\mu(B)| = 0.$$

The following characterization of weakly mixing is due to Halmos [**6**].

THEOREM 4.1. *T is weakly mixing if and only if* 1 *is the only eigenvalue of T.*

THEOREM 4.2. *If T is a completely ergodic, measure-preserving transformation constructed by a stacking method defined in this paper, then T is weakly mixing.*

The proof is again a modification of the one used earlier and due to Chacon. Most of the details will be omitted.

PROOF. If T is not weakly mixing, then T has an eigenvalue $\lambda \neq 1$. $\lambda^n = 1$ implies T has an n stack and hence T^n is not ergodic. Since T is completely ergodic $\lambda^n \neq 1$ for all n.

In the case where the eigenfunction is constant on some interval $I_{x,n}$ choosing 3 points appropriately and restacking once or twice if necessary, one obtains the contradiction $\lambda^{j_n} = 1$.

In the general case, by using Lusin's theorem and by an appropriate choice of points one obtains the following: For any $\eta > 0$ there exists n such that $|j_n\Psi| < 2\eta \mod 2\pi$, and this implies $\lim_{n\to\infty}|j_n\Psi| = 0 \mod 2\pi$ or $|\Psi|\lim_{n\to\infty}|j_n| = 0 \mod 2\pi$. Now either $\Psi = 0 \mod 2\pi$, in which case $\lambda = e^{i\psi} = 1$ which is a contradiction, or else $\lim |j_n|$ exists. Since $j_n > 0$ and j_n is an integer, the only possibility is that there exists an integer j such that $\lim j_n = j$. But then $|\Psi| \lim |j| = 0$ becomes $j\Psi = 0$. Thus $\lambda^j = e^{j\psi} = 1$, which is again a contradiction. Hence T must be weakly mixing.

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