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BOUNDS FOR RIEMANN-STIELTJES INTEGRALS

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ABSTRACT. Let f, g, h be real valued functions on a compact interval $[a, b]$, where h is of bounded variation with total variation V on $[a, b]$, and such that $\int_a^b f\,dg$ and $\int_a^b hf\,dg$ both exist. If $m = \inf\{h(x) : a \leqq x \leqq b\}$ it is shown that

$$\int_a^b hf\,dg \leqq m \int_a^b f\,dg + V \sup_{a \leqq \alpha < \beta \leqq b} \int_\alpha^\beta f\,dg,$$

$$\int_a^b hf\,dg \geqq m \int_a^b f\,dg + V \inf_{a \leqq \alpha < \beta \leqq b} \int_\alpha^\beta f\,dg.$$

Corresponding bounds hold for improper Riemann-Stieltjes integrals. The first of the inequalities above extends a result of R. Darst and H. Pollard, who dealt with the case $f(x) \equiv 1$, and g continuous on $[a, b]$.

In a recent paper [2], Darst and Pollard proved that *if h is real and of bounded variation on the interval $[a, b]$ and g is continuous there, then*

$$\text{(1)} \qquad \int_a^b h\,dg \leqq (\inf h)[g(b) - g(a)] + V(h; [a, b])S_g(a, b),$$

where V is the total variation of h on $[a, b]$, and

$$\text{(2)} \qquad S_g(a, b) = \sup_{a \leqq \alpha < \beta \leqq b} \int_\alpha^\beta dg.$$

Although it was not pointed out in [2], by replacing g in (1) by $(-g)$, one also obtains

$$\text{(1')} \qquad \int_a^b h\,dg \geqq (\inf h)[g(b) - g(a)] + V(h; [a, b])s_g(a, b),$$

where

$$\text{(2')} \qquad s_g(a, b) = \inf_{a \leqq \alpha < \beta \leqq b} \int_\alpha^\beta dg.$$

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The inequalities (1), (1′) thus give upper and lower bounds for $\int_a^b h\,dg$, analogous to those given in the "second integral mean value theorem" of R. P. Boas [1, p. 4]. (See also Widder [4, p. 18].)

It is the purpose of this note to show that the bounds (1), (1′) remain valid even if g is *not* continuous on $[a,b]$, provided only that g is *bounded* on $[a,b]$ and $\int_a^b h\,dg$ exists. A careful examination of the proof in [2] shows that the continuity of g was only used at two points of the proof: first, to justify the assumption that $h(\xi)=0$ for some $\xi \in [a,b]$ in the second reduction step of the proof; finally, to justify the existence of the integral $\int_a^b h\,d\varphi$ (since the continuity of φ follows from the continuity of g).

We observe first that when g is bounded with $g(a)=0$, and if

$$\varphi(t) \equiv \inf_{a \leqq \xi \leqq t} g(\xi), \qquad a \leqq t \leqq b,$$

it follows that φ is monotone decreasing on $[a,b]$. Also one easily verifies that φ is left-continuous, right-continuous, or continuous at each point $t \in [a,b]$ at which g has the corresponding property. Now the existence of $\int_a^b h\,dg$ implies that h and g have no common points of left- or of right-discontinuity on $[a,b]$ if $\int_a^b h\,dg$ is defined in the Pollard-Moore sense as a limit under successive refinement of partitions, or that h and g have no common points of discontinuity if $\int_a^b h\,dg$ is defined as a limit taken as the norm of partitions tends to zero. It follows that h and φ also have no common points of discontinuity of the same character, and since h and φ are of bounded variation on $[a,b]$, $\int_a^b h\,d\varphi$ exists. (See, for example Hildebrandt [3, pp. 50, 56, 66].) The continuity of g is thus not essential for the final steps of the proof in [2].

In order to complete the proof of our assertion, it suffices to rearrange the proof in [2] somewhat in order to avoid the necessity of assuming that h vanishes at some point of $[a,b]$ when $m = \inf h(x) = 0$. As in [2], the general case of (1) follows from the case $m=0$, so we are to prove that

$$\int_a^b h\,dg \leqq V(h;\,[a,b])S_g(a,b) \tag{3}$$

when $\inf h(x)=0$. Given any integer $n \geqq 1$ there exists $\xi_n \in [a,b]$ such that $\lim h(\xi_n)=0$. We now write

$$\int_a^b h\,dg = \int_a^{\xi_n} h\,dg + \int_{-b}^{-\xi_n} h_1\,d\mu,$$

$$(h_1(x) \equiv h(-x),\ \mu(x) \equiv -g(-x)),$$

and note that h and h_1 are nonnegative on their respective intervals of integration, and that $V(h_1;\ [-b,\ -\xi_n]) = V(h;\ [\xi_n, b])$ and $S_\mu(-b, -\xi_n) = S_g(\xi_n, b)$, just as in [2]. Noting that we may assume that $g(a) = 0$ since $\int_a^b h\,dg = \int_a^b h\,d(g - g(a))$, and defining φ on $[a, \xi_n]$ as above, and $\psi(t) \equiv g(t) - \varphi(t)$, it follows as in [2] that

$$\int_a^{\xi_n} h\,dg \leqq h(\xi_n)\psi(\xi_n) + S_g(a, \xi_n)V(h;\ [a, \xi_n])$$
$$\leqq h(\xi_n)S_g(a, \xi_n) + S_g(a, \xi_n)V(h;\ [a, \xi_n]).$$

Proceeding in the same way on $[-b, -\xi_n]$, we similarly obtain

$$\int_{-b}^{-\xi_n} h_1\,d\mu \leqq h_1(-\xi_n)S_\mu(-b, -\xi_n)$$
$$+ S_\mu(-b, -\xi_n)V(h_1;\ [-b, -\xi_n])$$
$$= h(\xi_n)S_g(\xi_n, b) + S_g(\xi_n, b)V(h;\ [\xi_n, b]).$$

It follows that for each $n \geqq 1$,

$$\int_a^b h\,dg \leqq 2h(\xi_n)S_g(a, b) + S_g(a, b)V(h;\ [a, b]),$$

so that (3) follows on letting $n \to \infty$.

Because of the usefulness of bounds of the form (1), (1′), it may be worthwhile pointing out the following extensions.

COROLLARY 1. *Let h be of bounded variation on $[a, b]$, and let f and g be any functions such that $\int_a^b f\,dg$ and $\int_a^b hf\,dg$ both exist. If $m = \inf\{h(x) : a \leqq x \leqq b\}$, then*

$$\int_a^b hf\,dg \leqq m\int_a^b f\,dg + V(h;\ [a, b]) \sup_{a \leqq \alpha < \beta \leqq b} \int_\alpha^\beta f\,dg,$$

$$\int_a^b hf\,dg \geqq \int_a^b f\,dg + V(h;\ [a, b]) \inf_{a \leqq \alpha < \beta \leqq b} \int_\alpha^\beta f\,dg.$$

This follows from (1) and (1′) applied to $\int_a^b h\,dG$, where $G(x) \equiv \int_a^x f\,dg$; note that by [3, p. 53] $\int_a^b h\,dG$ exists and equals $\int_a^b hf\,dg$.

This result may also be extended to *improper integrals*,

$$\int_a^{b-} F\,d\mu \equiv \lim_{c \to b-} \int_a^c F\,d\mu,$$

where $-\infty < a < b \leqq +\infty$, and $\int_a^c F\,d\mu$ exists for each $c \in (a, b)$. We define $V(h; [a, b)) \equiv \lim_{c\to b-} V(h; [a, c])$, and say that h *is of bounded variation on* $[a, b)$ if this limit is *finite.*

Corollary 2. *Suppose that h is of bounded variation on $[a, b)$, that $\int_a^{b-} f\,dg$ exists, and that $\int_a^c hf\,dg$ exists for each $c \in (a, b)$. If $G(x) \equiv \int_a^x f\,dg$ is bounded on $[a, b)$, then $\int_a^{b-} hf\,dg$ exists, and*

$$\int_a^{b-} hf\,dg \leqq m \int_a^{b-} f\,dg + V(h; [a, b)) \sup_{a \leqq \alpha < \beta < b} \int_\alpha^\beta f\,dg,$$

$$\int_a^{b-} hf\,dg \geqq m \int_a^{b-} f\,dg + V(h; [a, b)) \inf_{a \leqq \alpha < \beta < b} \int_\alpha^\beta f\,dg,$$

where $m = \inf\{h(x) : a \leqq x < b\}$ is (necessarily) finite.

It is easy to see that h is bounded on $[a, b)$, and even that $h(b-)$ exists and is finite. By writing

$$\int_a^c hf\,dg = \int_a^c h\,dG = h(c)G(c) - \int_a^c G\,dh,$$

we see that $\int_a^{b-} hf\,dg$ exists, the improper integral $\int_a^{b-} G\,dh$ being absolutely convergent. We now apply Corollary 1 on the interval $[a, c]$ for $a < c < b$ to obtain (for example) the upper bound

$$\int_a^c hf\,dg \leqq m(c) \int_a^c f\,dg + V(h; [a, c]) \sup_{a \leqq \alpha < \beta \leqq c} \int_\alpha^\beta f\,dg,$$

where $m(c) = \inf\{h(x) : a \leqq x \leqq c\}$. By considering the two cases $m = h(b-)$, $m < h(b-)$, it can be shown that $\lim_{c\to b-} m(c) = m$, and the result now follows readily.

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