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SMOOTHNESS THEOREMS FOR THE PRINCIPAL CURVATURES AND PRINCIPAL VECTORS OF A HYPERSURFACE

DONALD H. SINGLEY

1. **Introduction.** The most important invariants of a C^∞ hypersurface, M^n, immersed in R^{n+1} are its principal curvatures, the elementary symmetric functions of these principal curvatures (the so-called higher mean curvatures), and the principal vectors associated to these principal curvatures. The higher mean curvatures are clearly C^∞ everywhere on M^n, but the smoothness of the principal curvatures and the associated principal vectors is a more difficult matter. The following facts are generally known: First, the principal curvatures are continuous on all of M and differentiable on an open, dense subset of M. Second, in a neighborhood of any point in this open, dense subset, the principal vectors can be chosen to be C^∞. However, complete proofs of these facts do not seem to be available. The purpose of this paper is to give these proofs, in a somewhat more general setting than that described above. The proofs employ only elementary analysis, linear algebra, and point-set topology.

The general situation that we examine is a manifold, M^n, together with a pair of symmetric 2-covariant C^∞ tensor fields on M, one of which is positive definite. For the remainder of this paper, we denote the positive definite field by G and the other field by G'. In the case of an immersed hypersurface, these tensor fields are the first fundamental form, I, which—being a Riemannian metric—is positive definite, and the second fundamental form, II. From this pair of tensor fields, we form a $(1, 1)$ tensor field, or linear transformation field, on M; the field may be thought of as $(G)^{-1}(G')$. For an immersed hypersurface, this map is called the Weingarten map. The eigenvalues and eigenvectors of this linear transformation field are the obvious generalizations of the principal curvatures and principal vectors of a hypersurface, and it is these quantities for which we prove the smoothness theorems stated above.

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2. **Simultaneous Diagonalization of G and G'.** We shall begin by describing the process of obtaining these eigenvalues and eigenvectors in somewhat more detail. At each point $p \in M$, both G and G' are symmetric bilinear forms on the tangent space at p, $T_p(M)$. Hence, each of these bilinear forms gives a map from $T_p(M)$ to its dual, $T_p{}^*(M)$; moreover, since G is positive definite, the map given by G is an isomorphism. So, it has an inverse, which we denote by G^{-1}. Thus, given a vector $v \in T_p(M)$, we may map it into $T_p{}^*(M)$ using G' and then map the resulting vector back into $T_p(M)$ using G^{-1}. We denote this linear transformation on $T_p(M)$ by $G^{-1}G'(p)$, and we also denote the resulting linear transformation field on M by $G^{-1}G'$.

Let $P(G^{-1}G')(p)$ be the characteristic polynomial for this linear transformation on $T_p(M)$; thus, $P(G^{-1}G')(p) =$ determinant $(tI - G^{-1}G'(p))$. We write

$$P(G^{-1}G')(p) = \sum_{i=0}^{n} (-1)^i \sigma_i(G, G')(p)\, t^{n-i}.$$

Here, the $\sigma_i(G, G')(p)$ are the elementary invariants of this linear transformation; thus, for instance, $\sigma_0(G, G')(p) = 1$; $\sigma_1(G, G')(p) = \text{trace}\,(G^{-1}G')(p)$; and $\sigma_n(G, G')(p) = \text{determinant}\,(G^{-1}G')(p)$. So, for each value of i between 0 and n, where n is the dimension of M, we may define an invariant function on M, which we call $\sigma_i(G, G')$. These functions are sometimes known as the mixed invariants of G and G'.

Each of the mixed invariants is C^∞, because if we pick C^∞ vector fields $E_1, \cdots, E_n$ in a neighborhood of a point p such that $E_1(q), \cdots, E_n(q)$ are a basis for $T_q(M)$ for all q in this neighborhood of p, the entries of the matrices for G and G' with respect to this basis, considered as functions on a neighborhood of p, are C^∞, by the definition of C^∞ tensor fields. Now, if the matrices for G and G' at p are denoted by $\|G(p)\|$ and $\|G'(p)\|$ respectively, the matrix of $G^{-1}G'$ is clearly given by the matrix product $\|G(p)\|^{-1} \cdot \|G'(p)\|$. Hence, the entries of this matrix are rational functions of the entries of the matrices for G and G' whose denominators are $\neq 0$, since G is non-singular. So, the entries of this matrix are also C^∞ in a neighborhood of p. But each $\sigma_i(G, G')$ is a polynomial in the entries of the matrix for the linear transformation $G^{-1}G'$ and so is also C^∞ in a neighborhood of each $p \in M$.

The linear transformation $G^{-1}G'(p)$ has n real eigenvalues, with n linearly independent eigenvectors. This follows from a standard theorem in linear algebra: If G and G' are two symmetric bilinear forms on a vector space V, with G positive definite, there exists a basis for V which is orthonormal for G and orthogonal for G'. ([8], page 380). In this basis, the matrix for G is clearly the identity, and the matrix for G' is diagonal, with diagonal entries $\{g_1, \cdots, g_n\}$. Hence, $G^{-1}G'$ has a diagonal matrix as well, with diagonal entries $\{g_1, \cdots, g_n\}$. So, the $\{g_i\}$'s are the n real eigenvalues of $G^{-1}G'$, and the basis provided by this theorem gives n independent eigenvectors for $G^{-1}G'$. Moreover, in this basis

$$P(G^{-1}G') = \prod_{i=1}^{n} (t - g_i),$$

so $\sigma_i(G, G')$ = the i-th elementary symmetric polynomial in the eigenvalues of $G^{-1}G'$. So, the $\sigma_i(G, G')$ take an extremely simple form with respect to this basis. But we pay a price for this simplification: the $\{g_i\}$'s are not C^∞ on M. In fact, if we are to be guaranteed that the $\{g_i\}$'s are even continuous on M, we must permute the numbering of the eigenvectors, and hence the numbering of the $\{g_i\}$'s, so that after the re-ordering of the $\{g_i\}$'s we have $g_1(p) \geqq g_2(p) \geqq \cdots \geqq g_n(p)$. (The re-ordering of the $\{g_i\}$'s at each point is necessary because, since we obtained the $\{g_i\}$'s pointwise, we may have labeled the same continuous root of $P(G^{-1}G')$ in two different ways at nearby points. The re-ordering by size assures that we do, in fact, label a root consistently from point to point.)

3. **Smoothness Theorems.** We first show that the $\{g_i\}$'s are continuous on M and then discuss their differentiability.

THEOREM 1. *After the $\{g_i\}$'s have been ordered as above, they are continuous functions on M.*

PROOF. Our task is to show that if a polynomial $P_t(G^{-1}G')(p) = \sum_{i=0}^{n} a_i(p)t^i$, whose coefficients, $a_i(p)$, are C^∞, has roots $g_1(p), \cdots, g_n(p)$ which are always real, then the roots are continuous, after they have been put in order of increasing size. Our main tool will be Rouché's Theorem; to apply it, we must extend the polynomial $P_t(p)$ to a complex polynomial with the same coefficients, $P_z(p)$. Thus, $P_z(G^{-1}G')(p) = \sum_{i=0}^{n} a_i(p)z^i$, where z is complex-valued.

Since the problem is local, we may assume that the coefficients, $a_i(p)$, are C^∞ on an open subset of R^n. We must then show that for each g_i at each point p_0, $\forall\epsilon, \exists\delta : \|p - p_0\| < \delta \Rightarrow |g_i(p) - g_i(p_0)| < \epsilon$. Suppose that k of the roots $\{g_i(p_0)\}$ are distinct, with multiplicity $m_1, \cdots, m_k$. Thus, $\sum_{j=1}^{k} m_j = n$. We choose $\epsilon' <$ half the distance between the two roots of $P_z(p_0)$ which are closest together. Then, given ϵ as above, we set $\epsilon'' = \min(\epsilon, \epsilon')$, and we draw circles with radius

ϵ'' around each distinct root. By our choice of ϵ'', these circles do not intersect. Call the circles $\{C_1, \cdots, C_k\}$.

We will show that there is a δ such that if $|p - p_0| < \delta$, $P_z(p)$ has the same number of roots as $P_z(p_0)$ in every circle C_j. Our only remaining difficulty will be to prove that, for each i, $g_i(p)$ lies in the same circle as $g_i(p_0)$. If this is true, clearly $|g_i(p) - g_i(p_0)| < \epsilon'' \leqq \epsilon$, for our choice of δ, and g_i is continuous at p_0.

We will first find a δ_j such that, if $\|p - p_0\| < \delta_j$, $P_z(p)$ and $P_z(p_0)$ have the same number of roots within the one circle C_j. Then if we set $\delta = \min_j(\delta_j)$, if $\|p - p_0\| < \delta$, clearly $P_z(p)$ and $P_z(p_0)$ have the same number of roots within *every* circle, C_1 through C_k. Now, for each p, $P_z(p)$ is an analytic function of z, since it is a polynomial in z. Thus, by Rouché's Theorem ([1], page 124), finding the above δ_j is equivalent to finding a δ_j such that $\|p - p_0\| < \delta_j$ implies $\|P_z(p) - P_z(p_0)\| \leqq \|P_z(p_0)\|$ for all z on C_j.

But on C_j, we have

$$\|P_z(p) - P_z(p_0)\| = \Big\| \sum_{i=0}^{n} (a_i(p) - a_i(p_0))z^i \Big\|$$

$$\leqq \sum_{i=0}^{n} |a_i(p) - a_i(p_0)| \cdot \|z^i\| \leqq M_j \left(\sum_{i=0}^{n} |a_i(p) - a_i(p_0)| \right),$$

where $M_j = \max_i \|z\|^i$ on the circle C_j, $i = 0, 1, \cdots, n$. Also, $\|P_z(p_0)\|$ is continuous, so it assumes its minimum, say m_j, on the circle C_j; moreover, $m_j > 0$, since no roots of $P_z(p_0)$ lie on C_j. We now choose $\gamma \leqq [m_j]/[M_j(n + 1)]$. Since the coefficients of $P_z(p)$ are continuous as functions of p, for each i there are positive numbers d_i such that, if $\|p - p_0\| < d_i$, $|a_i(p) - a_i(p_0)| < \gamma$. Let $\delta_j = \min_i(d_i)$, $i = 0, \cdots, n$. Then for all p such that $\|p - p_0\| < \delta_j$,

$$\|P_z(p) - P_z(p_0)\| \leqq M_j \left(\sum_{i=0}^{n} |a_i(p) - a_i(p_0)| \right)$$

$$< M_j(n + 1) \left[\frac{m_j}{M_j(n + 1)} \right] = m_j \leqq \|P_z(p_0)\|$$

on the circle C_j. So, for this choice of δ_j the hypotheses of Rouché's Theorem are satisfied, and $P_z(p)$ has the same number of roots as $P_z(p_0)$ within C_j. Again, if $\delta = \min_j(\delta_j)$, $\|p - p_0\| < \delta$ implies that $P_z(p)$ and $P_z(p_0)$ have the same number of roots in *every* circle, C_1 through C_k.

We now show that, for each i, $g_i(p)$ lies in the same circle as $g_i(p_0)$, if $\|p - p_0\| < \delta$. Denote the circle containing $g_i(p_0)$ by C_i. Since $P_z(p)$ and $P_z(p_0)$ have the same number of roots, and since all the roots of $P_z(p_0)$ lie within the circles C_1 through C_k, all the roots of $P_z(p)$ also lie within these circles. Now, suppose that $g_i(p_0)$ has multiplicity m_i, so that it is one of m_i identical roots $g_{i_R}(p_0), \cdots, g_{i_S}(p_0)$. The roots of $P_z(p_0)$ that lie in the circles to the left of C_i must be the roots $g_i(p_0), \cdots, g_{i_R-1}(p_0)$. So, $P_z(p)$ must also have $i_R - 1$ roots in the circles to the left of C_i, and these are the smallest $i_R - 1$ roots of $P_z(p)$. Because the roots of $P_z(p)$ are also numbered in increasing order, these roots must be $g_1(p), \cdots, g_{i_R-1}(p)$. Similarly, the roots $g_{i_S+1}(p_0), \cdots, g_n(p_0)$ of $P_z(p_0)$ and the corresponding roots $g_{i_S+1}(p), \cdots, g_n(p)$ of $P_z(p)$ lie to the right of C_i. Hence the roots of $P_z(p)$ lying in C_i are $g_{i_R}(p), \cdots, g_{i_S}(p)$. Since $i_R \leqq i \leqq i_S$, $g_i(p)$ lies in the same circle as $g_i(p_0)$.

Even after the $\{g_i\}$'s have been re-ordered by size, they are still not C^∞ on M. As an example, let g_1 and g_2 be the principal curvatures on a surface immersed in R^3. Then $\sigma_1(G, G')$, the mean curvature of M, and $\sigma_2(G, G')$, the Gaussian curvature of M, are C^∞ everywhere. But, even though both the sum and the product of the principal curvatures are C^∞ everywhere on M, the principal curvatures themselves fail to be C^∞ at umbilics, where $g_1 = g_2$. (See [5], pp. 38–39.) In general, for higher dimensions, if any two of the g_i's are equal, there is no guarantee that either of them will be C^∞. However, the $\{g_i\}$'s are always C^∞ "almost everywhere" on M, as the next theorem shows.

Theorem 2. *The* $\{g_i\}$'s, *as defined above, are* C^∞ *on an open, dense subset of* M.

Proof. We begin by recalling the definition of a partition of the numbers 1 through n. This is a division of the set of numbers $\{1, \cdots, n\}$ into j subsets, P_1 through P_j, where j can be any number between 1 and n, and where the set $\{1, \cdots, n\}$ is the disjoint union of the $\{P_i\}$, $i = 1, \cdots, j$. The ordering of the numbers within each set P_i is immaterial, as is the ordering of the sets $\{P_i\}$. We denote a partition of the numbers 1 through n by a string of parentheses, where the numbers in P_i are contained in the i-th parenthesis. Thus, for instance, (1) (35) (24) (6) is a partition of the numbers 1 through 6; it is the same partition as (1) (6) (53) (42).

Given a partition $\mathcal{P} = (P_1)(P_2) \cdots (P_j)$ of the numbers 1 through n, we denote by $M_{\mathcal{P}}$ the subset of points q in the above manifold M such that if two subscripts, ℓ and m, lie in the same subset P_j, the numbers $g_\ell(q)$ and $g_m(q)$ are equal; and if ℓ and m lie in different subsets of the partition, the numbers $g_\ell(q)$ and $g_m(q)$ are unequal. Thus, for instance, if M is a 3-manifold, $M_{(13)(2)}$ is the set where $g_1 = g_3 \neq g_2$. We note that $M =$ the disjoint union of the $\{M_{\mathcal{P}}\}$'s. Moreover, since the $\{g_i\}$'s are continuous, the set where two g_i's are equal is closed, and the set where two g_i's are not equal is open. Each $M_{\mathcal{P}}$, being defined by a finite number of statements that pairs of $\{g_i\}$'s are equal, together with a finite number of statements that pairs of $\{g_i\}$'s are unequal, is the finite intersection of sets, each of which is either open or closed. Thus, each $M_{\mathcal{P}}$ is the intersection of an open set and a closed set.

We will show, first, that if any $M_{\mathcal{P}}$ has a non-empty interior, all the $\{g_i\}$'s are C^∞ on that interior; and second, that the union of the interiors of the $M_{\mathcal{P}}$'s is dense in M. Given an $M_{\mathcal{P}}$ with the interior of $M_{\mathcal{P}} \neq \emptyset$, let the corresponding partition be $\mathcal{P} = (P_1) \cdots (P_j)$. Then, on $M_{\mathcal{P}}$, we may express the polynomial $P_t(G^{-1}G')$ as

$$P_t(G^{-1}G') = (t - g_{i_1})^{p_1}(t - g_{i_2})^{p_2} \cdots (t - g_{i_j})^{p_j},$$

where for all ℓ, $\ell = 1, \cdots, j$, (i_ℓ) is any element of the subset P_ℓ, and (p_ℓ) is the number of elements in the subset P_ℓ. By the definition of the $\{M_{\mathcal{P}}\}$'s, the powers (p_ℓ) in the above expression are maximal, and all the $\{g_{i_\ell}\}$ are distinct.

To show that each g_{i_ℓ} is C^∞ on the interior of $M_{\mathcal{P}}$, we let x be a point of the interior. We then take $Q_\ell(t, x)$ to be the partial derivative of $P_t(G^{-1}G')$ with respect to t $(p_\ell - 1)$ times, at the point x. Thus,

$$Q_\ell(t, x) = \frac{\partial^{p_\ell - 1} P_t(G^{-1}G')}{(\partial t)^{p_\ell - 1}}(x),$$

on $M_{\mathcal{P}}$. If $p_\ell = 1$, we set $Q_\ell(t, x) = P_t(G^{-1}G')(x)$. Since $P_t(G^{-1}G')(x)$ is C^∞ in t and x, so is $Q_\ell(t, x)$. (Note that, if we considered the case of a C^k manifold, we would have to assume here that $k \geqq n$, to conclude that the $\{g_i\}$'s are even C^1.) By standard arguments on derivatives of polynomials, $Q_\ell(g_{i_\ell}(x), x) = 0$, and $\partial Q_\ell/\partial t(g_{i_\ell}(x), x) \neq 0$, since the power of the factor $(t - g_{i_\ell}(x))$, p_ℓ, is maximal. So, by the implicit function theorem, there are open sets $A \subset M$ containing x, and $B \subset R$ containing $g_{i_\ell}(x)$, such that for each point $y \in A$, there is a unique $h(y) \in B$, such that $Q_\ell(h(y), y) = 0$. Moreover, $h(y)$ is C^∞. But the function g_{i_ℓ} also satisfies the equation $Q_\ell(g_{i_\ell}(y), y) = 0$ in a neighborhood of x. Hence, by the uniqueness of $h(y)$, the functions $g_{i_\ell}(y)$ and $h(y)$ must be equal near x. Thus, g_{i_ℓ} is C^∞ on a neighborhood of x. Since x was an arbitrary point of the interior of $M_{\mathcal{P}}$, g_{i_ℓ} is C^∞ on the interior of $M_{\mathcal{P}}$.

The only task remaining is to show that the union of the interiors of the $M_{\mathcal{P}}$ is dense in M. We denote the interior of a set A by int A and its boundary by (∂A); moreover, we let A' be the complement of a set in M. Since any set $A \subset \bar{A} = (\text{int } A) \cup (\partial A)$, the whole manifold $M = \bigcup(M_{\wp}) = [\bigcup(\text{int } M_{\wp})] \cup [\bigcup \partial(M_{\wp})]$, where all unions are indexed by $\{\wp\}$. Thus $[\bigcup(\text{int } M_{\wp})] \supset [\bigcup \partial(M_{\wp})]'$, and it is sufficient to show that $[\bigcup \partial(M_{\wp})]'$ is dense in M. This is equivalent to showing that $[\bigcup \partial(M_{\wp})]$ has empty interior. As noted earlier, each $M_{\wp} = (O_p \cap C_p)$, where O_p is open and C_p is closed. Also $\partial(O_p \cap C_p) \subset (\partial O_p) \cup (\partial C_p)$; this follows from a formula in ([7], page 62): Given sets X and Y,

$$(\partial X) \cup (\partial Y) = \partial(X \cup Y) \cup \partial(X \cap Y) \cup (\partial X \cap \partial Y).$$

Thus, $\bigcup_{\wp} (\partial M_{\wp}) \subset \bigcup_{\wp} [\partial O_{\wp} \cup \partial C_{\wp}]$. Clearly, any subset of a set with empty interior has empty interior, so we have reduced the problem to showing that $\bigcup_{\wp} [\partial O_{\wp} \cup \partial C_{\wp}]$ has empty interior. To show this, we employ two elementary lemmas:

LEMMA 1.2(a). *If a set is either closed or open, its boundary is closed and has empty interior.*

PROOF. Given any set C,

$$(*) \qquad \partial C = (\overline{C'}) \cap (\bar{C}).$$

(See [6], page 46.) Thus, the boundary of any set is closed. Now, if the set C is closed, $(*)$ gives $\partial C = (\overline{C'}) \cap (C)$. Thus, the boundary of C is a subset of C. If it had any interior points, these would be interior points of C. Since $\partial C = \bar{C}\text{-int } C$, this is impossible. If the set C is open, its complement C', is closed. By $(*)$, the boundary of a set is the boundary of its complement; so, as before, the boundary of C has no interior points.

LEMMA 1.2(b). *The finite union of closed sets with empty interior has empty interior.*

PROOF. By induction, it suffices to prove this for two sets, A and B, each of which is closed and has no interior points. If $X \subset (A \cup B)$ is open, $X \cap B'$ is open and $\subset A$. Since A has empty interior, $X \cap B' = \emptyset$. Thus, $X \subset B$; since B has empty interior, $X = \emptyset$. Taking $X = \text{int}(A \cup B)$, we conclude that $X = \emptyset$; hence, $A \cup B$ is closed and has empty interior.

By Lemmas 1.2(a) and 1.2(b), the above union, $\bigcup_{\wp}(\partial O_{\wp} \cup \partial C_{\wp})$, has empty interior, and the union of the interiors of the $\{M_{\wp}\}$ is dense, as was to be shown.

Our last "smoothness" theorem shows that on this open, dense subset of M the eigenvectors $\{e_i\}$ corresponding to the eigenvalues $\{g_i\}$ are also C^∞, provided the $\{e_i\}$ are chosen correctly. In fact, we can even choose the $\{e_i\}$ to be orthonormal with respect to G:

THEOREM 3. *Given any point p in $\bigcup_{\wp}(\operatorname{int} M_{\wp})$, we can find a neighborhood of p, N_p, and C^∞ basis vector fields on N_p, $\{e_1, \cdots, e_n\}$, which are eigenvectors for the linear transformation $G^{-1}G'(q)$ at each point $q \in N_p$ and which are also orthonormal for G on N_p.*

We note that this set of vector fields $\{e_1, \cdots, e_n\}$ may also be thought of as the vector fields which simultaneously orthonormalize G and orthogonalize G'.

PROOF. The point p is in the interior of one of the $M_{\wp}$. If we call this interior I, each g_i has constant multiplicity, say m_i, on I. For this reason, we restrict ourselves from now on to the subset I. Now, suppose we can find m_i C^∞ vector fields which are a basis for the eigenspace of g_i at each point q in N_p. We can then orthonormalize these vector fields with respect to G by using the Gram-Schmidt process. The resulting vector fields will still be a basis for the eigenspace of g_i at each point q in N_p. Moreover, the Gram-Schmidt process is C^∞, so these new vector fields will also be C^∞. We claim that the collection of n vector fields obtained as above from all the eigenvalues $\{g_i\}$ is the collection $\{e_1, \cdots, e_n\}$ described in the theorem. Since the eigenvectors coming from different eigenvalues of $(G^{-1}G')$ are orthogonal in the metric G, by a standard algebraic theorem ([2], p. 314), the collection of vector fields is orthonormal for G and so, *a fortiori*, they are linearly independent. Thus, they have all the properties required for $\{e_1, \cdots, e_n\}$.

We will find these m_i vector fields by giving an explicit expression for a set of m_i linearly independent eigenvectors of a linear transformation corresponding to an eigenvalue of multiplicity m_i. This expression will give the components of the eigenvectors as C^∞ functions of g_i and the entries of $(G^{-1}G')$. So, if we start with C^∞ basis vector fields in a neighborhood N_p of p such that the entries of $G^{-1}G'$ are C^∞, as we can always do, the m_i vector fields obtained by this process will be C^∞.

If we let $L = G^{-1}G'(p)$, we must solve the vector equation $L(X) = g_i \cdot X$, or, $(L - g_iI)X = 0$. By choosing *some* basis at each point q near p, which diagonalizes L at q but does not necessarily vary smoothly with q, we find that $L - g_iI$ is a singular matrix of rank $(n - m_i)$ at every point q near p. Now, pick a new basis which is C^∞ in a neighborhood of p. Then, if

$$M = \left\| \begin{matrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \cdots & m_{nn} \end{matrix} \right\|$$

is the matrix of $L - g_i I$ with respect to this basis, M is singular, of rank $(n - m_i)$. We must find an explicit expression for m_i linearly independent vectors $X_1, \cdots, X_{m_i}$, which span the kernel of M. We let $s = m_i$ and $r = n - m_i$, to simplify notation. Since the $\{g_i\}$'s do not change multiplicity in a neighborhood of p, any $(r+1) \times (r+1)$ minor of M has zero determinant in this neighborhood. Moreover, at the point p, there exists at least one $(r \times r)$ minor, D, with non-zero determinant. By the continuity of the determinant, $\det D \neq 0$ in a neighborhood of p. We may permute the basis near p so that this minor is in the lower right hand corner of M. Thus,

$$D = \left\| \begin{matrix} m_{(s+1)(s+1)} & \cdots & m_{(s+1)n} \\ \vdots & & \vdots \\ m_{n(s+1)} & \cdots & m_{nn} \end{matrix} \right\| .$$

For $j = 1, \cdots, s$, we define X_j (which we write as a row vector, though it is actually a column vector) by the expression

$$X_j = (\underbrace{0, \cdots, d, \cdots, 0}_{s}, \underbrace{x(j_1), \cdots, x(j_r)}_{r}),$$

where $d = \det \mathrm{D}$ is the j-th coordinate of X_j, and $x(j_k)$, $k = 1, \cdots, r$, is defined as follows. Let

$$M_\ell{}^j = \left\| \begin{array}{c|c} m_{\ell j} & m_{\ell (s+1)} \cdots m_{\ell n} \\ \hline m_{(s+1)j} & \\ \vdots & D \\ m_{nj} & \end{array} \right\| .$$

This is an $(r + 1) \times (r + 1)$ minor of M, so its determinant $= 0$. Let $(M_\ell{}^j)_{1,v} =$ this matrix with the first row and v-th column deleted. Then, define $x(j_k) = (-1)^k \det (M_1{}^j)_{1,k+1}$.

If we now let M act to the left on the column vector X_j, we find that the t-th coordinate of $M(X_j) =$

$$(d \cdot m_{tj} + (x(j_1)) \cdot m_{t(s+1)} + \cdots + (x(j_r)) \cdot m_{tn}).$$

But, this is merely the expansion of the determinant of $M_t{}^j$ by minors of the first row of $M_t{}^j$, since $(M_1{}^j)_{1,v} = (M_t{}^j)_{1,v}$, for all v and t. Thus, $\det M_\ell{}^j = 0$, $\forall j$ and ℓ, implies that $M(X_j) = 0$, $\forall j$. The X_j's are clearly linearly independent on the set where $\det D \neq 0$. Moreover, the entries of each X_j are polynomial functions of the entries of $G^{-1}G'$ and of the

$\{g_i\}$'s; since both the entries of $G^{-1}G'$ and the $\{g_i\}$'s are C^∞ on a neighborhood of p, each X_j is clearly C^∞ in this same neighborhood. So, the $\{X_j\}$'s are linearly independent on one neighborhood of p and C^∞ on a (perhaps larger) neighborhood of p. We may take N_p to be the smaller of these two neighborhoods. Clearly, on N_p, the $\{X_j\}$'s have all the properties required above.

We also note that the above construction does not work for $m_i = n$. But in this case, *any* basis vector field which orthonormalizes G will orthogonalize G', since G' is conformal to G (i.e., $G' = f \cdot G$ for some function f).

We note that the above construction is entirely local; in fact, we cannot even pick the eigenvectors to be continuous on all of M, much less C^∞, whenever M is compact and has non-zero Euler characteristic, since any continuous vector field on such a manifold vanishes at some point.

In summary, even though the $\{g_i\}$'s and $\{e_i\}$'s are not C^∞ everywhere, the situation is still fairly good, since the $\{g_i\}$'s are C^∞ on an open, dense subset of M. This result is often sufficient for applications in differential geometry; one just proves a local result, assuming that the $\{g_i\}$'s are C^∞, and then uses a limiting argument to extend the result to all of M. This technique is used most frequently in the case where the $\{g_i\}$'s are the principal curvatures, but other applications are also known; for instance, G and G' can be two Riemannian metrics on M. All of the above results hold as well for this case, which has also been extensively studied ([3], [4]).

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Columbia University and The University of Minnesota