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ON THE EQUIVALENCE OF OSCILLATION AND THE EXISTENCE OF INFINITELY MANY CONJUGATE POINTS

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ABSTRACT. Consider the linear nth order ordinary differential equation $y^{(n)} + \sum_{i=0}^{n-1} p_i(t)y^{(i)} = 0$ where the coefficients are continuous on a half-line. Sufficient conditions are given under which oscillation is equivalent to the existence of infinitely many conjugate points.

Introduction. Consider the linear differential equation

$$L(y) \equiv y^{(n)} + \sum_{k=0}^{n-1} p_k(t)\, y^{(k)} = 0, \tag{1.1}$$

where $p_k(t), k = 0, 1, \cdots, n-1$, is continuous on an interval $[a, \infty)$, $a > -\infty$. Equation (1.1) is said to be oscillatory provided it has an oscillatory solution, i.e., a nontrivial solution with arbitrarily large zeros. Otherwise, equation (1.1) is said to be nonoscillatory. For $n = 2$ it is well known that nonoscillation of (1.1) on $[a, \infty)$ implies (1.1) is disconjugate on some interval $[b, \infty)$ for $a \leqq b$. Whether or not a similar statement is true for $n > 2$ was resolved in the negative by Dolan [1], and Gustafson [2]. A. Levin [4] also gives an example of a nonoscillatory third order equation for which, given arbitrary numbers t_0 and r, there exists a nontrivial solution with at least r zeros on (t_0, ∞). In [2], for each $n > 2$ there is exhibited a linear equation of order n which is nonoscillatory yet $\eta_p(a)$ exists for each integer $p \geqq 1$. ($\eta_p(a)$ is defined in § 2 below.) For a class of fourth-order self-adjoint linear differential equations, Leighton and Nehari [6] show that oscillation is equivalent to the existence of $\eta_p(a)$ for each integer $p \geqq 1$ and used this equivalence to give sufficient conditions on the coefficients for nonoscillation. In addition, they are able to characterize for each $p \geqq 1$ the extremal solutions defining $\eta_p(a)$. Recently Ridenhour [10] has given conditions on equation (1.1) in terms of boundary-value functions under which he is able to obtain a characterization of the extremal solution for $\eta_p(a)$ analogous to the characterization given by Leighton and Nehari in [6]. The purpose

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of this paper is to give conditions on equation (1.1) under which oscillation is equivalent to the existence of $\eta_p(a)$ for each integer $p \geqq 1$. The conditions given here, although more restrictive than those in [10], are equivalent to those in [6] for $n = 4$. We shall make extensive use of several theorems given in [10].

2. **Definitions and Preliminaries.** Let $k, i_1, i_2, \cdots, i_k$ denote positive integers for which $k \geqq 2$ and $\sum_{j=1}^{k} i_j \geqq n$. A nontrivial solution $u(t)$ of equation (1.1) is said to have an $i_1 - i_2 - \cdots - i_k$ distribution of zeros on an interval I provided there exists points $t_1 < t_2 < \cdots < t_k$ for which $m[u(t), t_j] \geqq i_j$ for $j = 1, 2, \cdots, k$. Here $m[u(t), t_j]$ denotes the exact multiplicity of the zero of $u(t)$ at $t = t_j$. The boundary value function $r_{i_1 i_2 \cdots i_k}(t)$, $t \geqq a$, is defined as the infimum of the set of numbers $b > t$ for which there exists a nontrivial solution of (1.1) with an $i_1 - i_2 - \cdots - i_k$ distribution of zeros on the interval $[t, b]$. Here we take the infimum of the empty set to be ∞. If $r_{i_1 i_2 \cdots i_k}(t) = \infty$ for all $t \geqq a$, we write $r_{i_1 i_2 \cdots i_k} = \infty$. To simplify the notation we adopt the following notation used in [5]:

$$S_{(n-1)p}(t) \equiv r_{i_1 i_2 \cdots i_k}(t),$$

where $k = n - 1$, $i_j = 1$, $j \neq p$, and $i_p = 2$.

For $k \geqq 1$ and $t \in [a, \infty)$ we define the kth conjugate point $\eta_k(t)$ (in the sense of Sherman [12] and Gustafson [2]) as the infimum of the set of numbers b for which there exists a nontrivial solution of (1.1) with $n + k - 1$ zeros on the interval $[t, b]$. Throughout this paper zeros of a solution are counted according to their multiplicities. An extremal solution defining $\eta_k(t)$ is a nontrivial solution of (1.1) having $n + k - 1$ zeros on $[t, \eta_k(t)]$. It should be pointed out that the notion of the conjugate point arising from the calculus of variations is different from the one defined here.

The boundary-value functions and conjugate points defined above have been studied by a variety of authors. Some ([9], [11]) have given various ordering properties of boundary-value functions, while others ([8], [13]) have studied the relationships between $\eta_1(t)$ and boundary-value functions. Of particular interest Peterson [8] has shown that

$$\eta_1(t) = \min_{k \neq j}\{S_{(n-1)k}(t), S_{(n-1)j}(t)\}.$$

It follows that if $\eta_1(t) < \infty$, there exists at most one integer p, $1 \leqq p \leqq n - 1$, for which $S_{(n-1)p} = \infty$. For $n = 4$ the conclusion of a basic lemma (Lemma 2.3) in [6] may be interpreted as $S_{32} = \infty$. Hanan [3] has studied third order equations for which $S_{21} = \infty$ or $S_{22} = \infty$. The results presented here generalize some of the theorems in [6] and [3]. In [5] the author has studied oscillatory properties of a class of nth-order equations for which $S_{(n-1)p} = \infty$ for some integer p. In another vein Nehari [7] has shown that if $\eta_1 = \infty$ for (1.1) (i.e., $L(y)$ is disconjugate on (a, ∞)) and $q(t) \neq 0$ on (a, ∞), then for the equation $L(y) = q(t)y$, $r_{ik} = \infty$ whenever $i + k > n$.

The following theorems are due to Ridenhour [10] and will be referred to often in the remaining portions of this paper. In order to simplify the notation on these theorems and those of § 3 we shall introduce the following definition.

DEFINITION. Let p be a fixed integer, $1 \leqq p \leqq n - 1$. A distribution of zeros $i_1 - i_2 - \cdots - i_k$, $k \geqq 2$, for a nontrivial solution of (1.1) is called a p-distribution provided $\sum_{j=1}^{k} i_j = n$, and there exists an integer s, $1 \leqq s \leqq k$, such that $\sum_{j=1}^{s} i_j = p$. Using the definition the following theorems are easy interpretations of the indicated theorems in [10].

THEOREM 2.1 (Theorem 2.3 [10]). *For equation* (1.1) *suppose p is an integer, $1 \leqq p \leqq n - 1$, for which $S_{(n-1)p} = r_{(p+1)(n-p)} = r_{p(n-p+1)} = \infty$. Suppose further that $y(t)$ is a nontrivial solution of* (1.1) *with an $i_1 - i_2 - \cdots - i_k$ distribution of zeros, $\sum_{j=1}^{k} i_j = n$, which is not a p-distribution. Let $s \equiv \min \{r \mid \sum_{j=1}^{r} i_j \geqq p\}$. Then $\sum_{j=1}^{s} i_j = p + 1$ and $i_s = 2$.*

THEOREM 2.2 (Lemma 4.2 [10]). *For equation* (1.1), *suppose p is an integer, $2 \leqq p \leqq n - 2$, for which $S_{(n-1)p} = r_{(p+1)(n-p)} = r_{p(n-p+1)} = \infty$. Suppose further that $y(t)$ is a nontrivial solution of* (1.1) *with an $i_1 - i_2 - \cdots - i_k$ distribution of zeros for which $i_j \geqq 2$ for each $j = 1, 2, \cdots, k$. Then,*

(i) *if $\sum_{j=1}^{k} i_j = n$, the distribution is a p-distribution;*

(ii) *if $\sum_{j=1}^{k} i_j > n$, then $k = 3$, $i_1 = p$, $i_2 = 2$, $i_3 = n - p$ and $y(t)$* has no other zeros on $[a, \infty)$.

THEOREM 2.3 (Theorems 4.1, 4.2 [10]). *For equation* (1.1) *suppose p is an integer, $1 \leqq p \leqq n - 1$, for which $S_{(n-1)p} = r_{(p+1)(n-p)} = r_{p(n-p+1)} = \infty$. Suppose further that $y_k(t)$ is an extremal solution for $\eta_k(b)$ where $b > a$. Then*

(i) *$m[y_k(t), b] = p, m[y_k(t), \eta_k(b)] = n - p$;*

(ii) *$y_k(t)$ has exactly $(k - 1)$ zeros on $(b, \eta_k(b))$;*

(iii) *if $y_k(c) = 0$ for $c \in (\eta_k(b), \infty)$, then $k = 1$, $p = n - 2$, $m[y_k(t), c] = 2$;*

(iv) *$y_k(t)$ is unique to within constant multiples.*

The following well known lemma has been used by various authors. Its proof may be found in [12] and is stated here for easy reference.

LEMMA 2.1. *Let $b < c$ be real numbers and suppose $u(t)$ and $v(t)$ are nontrivial solutions of equation* (1.1) *for which* $m[u(t), b] > m[v(t), b] \geqq 0$, $m[u(t), c] > m[v(t), c] \geqq 0$, *and* $v(t) \neq 0$ *for* $t \in (b, c)$. *Then there exists a nontrivial linear combination of $u(t)$ and $v(t)$ with a double zero on (b, c).*

3. **Main Results.** Consider the following equation together with the conditions for some fixed integer p, $1 \leqq p \leqq n - 1$,

$$(3.1)\qquad \begin{gathered} L(y) \equiv y^{(n)} + \sum_{k=0}^{n-1} p_k(t)\, y^{(k)} = 0, \\ S_{(n-1)p} = r_{(p+1)(n-p)} = r_{p(n-p+1)} = r_{p2(n-p)} = \infty . \end{gathered}$$

In this section we shall show that (3.1) is oscillatory if and only if, for $b > a$, $\eta_k(b)$ exists for each integer $k \geqq 1$. Using different techniques than those presented here Ridenhour [10] obtained this result for the special case $p = n - 1$. If $p = n - 1$ or $p = 1$, then the condition $r_{p2(n-p)} = \infty$ is redundant and may be omitted. For $n = 3$ or $n = 4$, this same condition is redundant for any choice of $p = 1, 2, 3$. We begin with the following fundamental theorem.

THEOREM 3.1. *For the system* (3.1) *if $u(t)$ is a nontrivial solution with an $i_1 - i_2 - \cdots - i_k$ distribution of zeros, $\sum_{j=1}^{k} i_j = n$, then $i_1 - i_2 - \cdots - i_k$ is a p-distribution of zeros.*

Let $t_1 < t_2 < \cdots < t_k$ denote points for which $m[u(t), t_j] \geqq i_j$ for $j = 1, 2, \cdots, k$. Suppose first that $2 \leqq p \leqq n - 2$. If $m[u(t), t_j] \geqq 2$ for each $j = 1, 2, \cdots, k$, then by the hypothesis and Theorem 2.2 there exists an integer σ for which

$$\sum_{j=1}^{k} m[u(t), t_j] = n \text{ and } \sum_{j=1}^{\sigma} m[u(t), t_j] = p.$$

It follows that $m[u(t), t_j] = i_j$ for each $j = 1, 2, \cdots, k$, and the distribution is therefore a p-distribution.

Assume then that $m[u(t), t_{q_1}] = i_{q_1} = 1$ for some integer q_1, $1 \leqq q_1 \leqq k$. We shall show that if the given distribution is not a p-distribution, then there exists a nontrivial solution with a zero distribution which is not a p-distribution and has $(k + 1)$ distinct zeros in this distribution. Repeating this argument $(n - k - 1)$ times would yield a nontrivial solution having a zero distribution which is not a p-distribution of zeros at $(n - 1)$ distinct points, contradicting $S_{(n-1)p} = \infty$.

To this end, suppose $i_1 - i_2 - \cdots - i_k$ is not a p-distribution of zeros and let $s = \min \{r \mid \sum_{j=1}^{r} i_j \geqq p\}$. By Theorem 2.1, we have

$$\sum_{j=1}^{s} i_j = p + 1 \text{ and } i_s = 2. \tag{3.3}$$

If $i_j = 1, j \neq s$, it follows that $s = p$, contradicting $S_{(n-1)p} = \infty$. We may then assume there exists an integer $q \neq s$ for which $i_q \geqq 2$, and consider a nontrivial solution $v(t)$ satisfying the following $(n-1)$ conditions:

$$v^{(j)}(t_r) = 0, j = 0, 1, \cdots, i_r - 1, r \neq q, q_1;$$
$$v^{(j)}(t_q) = 0, j = 0, 1, \cdots, i_q - 2;$$
$$v''(t_s) = 0.$$

Observe that $m[u(t), t_s] = 2$. Otherwise $u(t)$ has a zero distribution on the set $\{t_j\}_{j \neq q_1}$ which is not a p-distribution and obviously contradicts Theorem 2.1. Similarly, $v^{(i_q-1)}(t_q) \neq 0$ since $m[v(t), t_s] \geqq 3$. Consequently, no nontrivial linear combination of $u(t)$ and $v(t)$ is identically zero. If a constant c is chosen sufficiently small, it follows that either $u(t) - cv(t)$ or $u(t) + cv(t)$ has zeros near $t = t_{q_1}$ and $t = t_q$. Hence this linear combination has $(k+1)$ distinct zeros, and this zero distribution is not a p-distribution by (3.3). As indicated above, repeating this argument leads to the conclusion that $S_{(n-1)p}(a) < \infty$. This contradiction of the hypothesis completes this portion of the proof of the theorem.

Suppose now that $p = 1$. We shall show that if $i_1 - i_2 - \cdots - i_k$ is not a p-distribution, then there exists a nontrivial solution of (3.1) which has at least $(k+1)$ distinct zeros and a zero distribution which is not a 1-distribution. As before this will contradict $S_{(n-1)1} = \infty$.

By hypothesis it clearly follows that there exists an integer $q > 1$ for which $i_q \geqq 2$. If there also exists an integer q_1 for which $m[u(t), t_{q_1}] = 1$ we may proceed in a way analogous to that in the appropriate above portion for $2 \leqq p \leqq n-2$. We assume then $m[u(t), t_j] \geqq 2$ for all $j \geqq 2$. There are several cases to be considered. Suppose first that there exists $m \neq q, m > 1$, for which $i_m \geqq 2$. Consider then a nontrivial solution $v(t)$ of (3.1) defined by the following $(n-1)$ conditions

$$v^{(j)}(t_r) = 0, j = 0, 1, \cdots, i_{r-1}, r \neq q, \bar{q};$$
$$v^{(j)}(t_q) = 0, j = 0, 1, \cdots, i_q - 2;$$
$$v^{(j)}(t_m) = 0, j = 0, 1, \cdots, i_m - 2;$$
$$v''(t_1) = 0.$$

It follows from Theorem 2.1 that $u''(t_1)v^{(i_q-1)}(t_q)v^{(i_m-1)}(t_m) \neq 0.$

Consequently, if a constant c is chosen sufficiently small, then either $u(t) - cv(t)$ or $u(t) + cv(t)$ has $(k + 2)$ distinct zeros and a zero distribution which is not a 1-distribution.

Now suppose $k = 2$. By hypothesis $r_{2(n-1)} = \infty$, and hence $m[u(t), t_2] = n - 2 \geqq 2$. Choose a nontrivial solution $v(t)$ of (3.1) satisfying the following $(n - 1)$ conditions:

$$v^{(j)}(t_1) = 0, j = 0, 1, 2;$$

$$v^{(j)}(t_2) = 0, j = 0, 1, \cdots, n - 5, \text{ if } n \geqq 5.$$

Then as before a constant c may be chosen so small that either $u(t) - cv(t)$ or $u(t) + cv(t)$ has $(k + 2)$ distinct zeros and a double zero at $t = t_1$.

Finally, suppose $i_j = 1, j \neq 1, q, k > 2$, and $m[u(t), t_j] \geqq 2$ for all $j = 1, 2, \cdots, k$. The technique is similar to that above. The choice of the solution $v(t)$ differs according to $m[u(t), t_q]$. If $m[u(t), t_q] - i_q$ is even, choose $v(t)$ satisfying

$$v^{(j)}(t_1) = 0, j = 0, 1, 2;$$

$$v(t_j) = 0, j \neq q,$$

$$v^{(j)}(t_q) = 0, j = 0, 1, \cdots, i_q - 3, \text{ if } i_q \geqq 3.$$

If $m[u(t), t_q] - i_q$ is odd, choose $v(t)$ satisfying

$$v^{(j)}(t_1) = 0, j = 0, 1, 2,$$

$$v(t_j) = 0, j \neq q_1 \text{ for some } q_1 \neq 1, q,$$

$$v^{(j)}(t_q) = 0, j = 0, 1, \cdots, i_q - 2.$$

In either case, it follows that $m[u(t), t_q] - m[v(t), t_q]$ is an even positive integer, and hence a constant c may be chosen so small that either $u(t) - cv(t)$ or $u(t) + cv(t)$ has n zeros distributed over $(k + 1)$ distinct points and a double zero at $t = t_1$. A similar argument may be given for the case $p = n - 1$, concluding the proof of the theorem.

The following corollaries are immediate consequences of Theorem 3.1. Two nontrivial solutions of (3.1) are essentially different provided they are not constant multiples of each other.

COROLLARY 3.1. *If $a < \alpha < \beta$, any nontrivial solution of* (3.1) *with $(p - 1)$ zeros on $[a, \alpha]$ and $(n - p - 1)$ zeros on $[\beta, \infty)$ has only simple zeros on (α, β).*

Corollary 3.2. *If $a < \alpha < \beta$ and $u(t)$ and $v(t)$ are essentially different nontrivial solutions of* (3.1) *with $(p - 1)$ zeros in common on $[a, \alpha]$ and $(n - p - 1)$ zeros in common on $[\beta, \infty)$, then $W[u(t)\, v(t)] \equiv u'(t)v(t) - v'(t)u(t) \neq 0$ for $t \in (\alpha, \beta)$.*

Since $W[u(t), v(t)] \neq 0$ implies $u(t)$ and $v(t)$ form a fundamental set for a second-order linear ordinary differential equation, Corollary 3.2 may be considered as a separation theorem, i.e., the zeros of $u(t)$ and $v(t)$ separate each other on (α, β). Consequently, the number of zeros of $u(t)$ on (α, β) differs from the number of zeros of $v(t)$ on (α, β) by at most one.

In [5] it is assumed for equation (1.1) that no two essentially different solutions have $(n-1)$ zeros in common. The assumption plays a crucial role in the proofs of many of the theorems in [5]. The following theorem shows that the results of [5] are applicable to the system (3.1).

THEOREM 3.2. *For the system* (3.1) *no two essentially different solutions have* $(n-1)$ *zeros in common.*

Suppose to the contrary that $u(t)$ and $v(t)$ are essentially different solutions of (3.1) with $(n-1)$ zeros in common located as distinct points $t_1 < t_2 < \cdots < t_k$. Clearly $k \geqq 2$, and for $i = 1, 2, \cdots, k$ let

$$m_i \equiv \min \{m[u(t), t_i], m[v(t), t_i]\} \leqq n-2.$$

Define $s = \min \{r \mid \sum_{i=1}^{r} m_i \geqq p\}$, and consider the nontrivial solution $w(t)$ of (3.1) given by

$$w(t) = u^{(m_s)}(t_s)v(t) - v^{(m_s)}(t_s)u(t).$$

It follows that $w(t)$ has a distribution of zeros which is not a p-distribution. This contradication of Theorem 3.1 completes the proof of the theorem.

The following two lemmas indicate to a certain extent how the zeros of an extremal solution defining $\eta_k(b)$ for (3.1) are distributed on $(b, \eta_k(b))$, $b > a$. Let $b > a$ be a real number and $\{z_i(b; t)\}_{i=0}^{n-1}$ the fundamental set of solutions of (3.1) for which $z_i^{(j)}(b; b) = \delta_{ij}$ (the Kronecker delta), $j = 0, 1, 2, \cdots, n-1$. By Theorem 2.3 any extremal solution of (3.1) for $\eta_k(b)$, $k \geqq 1$, is a linear combination of the solutions $z_i(b; t)$, $i = p, \cdots, n-1$. In the remainder of this paper $y_k(t)$ shall denote the extremal solution of (3.1) for $\eta_k(b)$ for which $y_k(t) = \sum_{j=p}^{n-1} c_{kj} z_j(b; t)$

(3.4) $$\sum_{j=p}^{n-1} c_{kj}^2 = 1 \text{ and } c_{kp} > 0.$$

LEMMA 3.1. *For the system* (3.1) *if* $b > a$ *and* $\eta_k(b) < \infty$ *and* $1 < q < k$, *then* $y_k(t)$ *has at least* q *distinct zeros on* $(b, \eta_q(b))$.

To prove the lemma consider a set of nontrivial solutions $\{u_i(t)\}_{i=0}^{n-p}$ satisfying the following $(n-1)$ conditions for $i = 1, 2, \cdots, n-p-1$:

$$
\begin{aligned}
&u_i^{(j)}(b) = 0, j = 0, 1, \cdots, p-2, \text{ if } p \geqq 2; \\
(3.5) \qquad &u_i^{(j)}(\eta_q(b)) = 0, j = 0, 1, \cdots, n-p-i-1; \\
&u_i^{(j)}(\eta_k(b)) = 0, j = 0, 1, \cdots, i-1.
\end{aligned}
$$

Let $u_0(t) \equiv y_q(t)$ and $u_{n-p}(t) \equiv y_k(t)$. Note that for $i = 1, 2, \cdots, n-p$, $u_i(t)$ and $u_{i-1}(t)$ have $(p-1)$ zeros in common at $t = b$ and $n-p-1$ zeros in common on the set $\{\eta_q(b), \eta_k(b)\}$. Let $\alpha_{01} < \alpha_{02} < \cdots < \alpha_{0(q-1)}$ denote the simple zeros of $u_0(t)$ on $(b, \eta_q(b))$. Since $u_1(t)$ and $u_0(t)$ are essentially different, by Lemma 2.1 and Corollary 3.2, $u_1(t)$ must have a simple zero on (b, α_{01}) and on each interval $(\alpha_{0j}, \alpha_{0(j+1)})$, $j = 1, 2, \cdots, q-1$, where $\alpha_{0q} \equiv \eta_q(b)$. Hence $u_1(t)$ has at least q simple zeros on $(b, \eta_q(b))$. Suppose $u_i(t)$ for some i, $1 \leqq i < n-p$, has q simple zeros on $(b, \eta_q(b))$ at points $\alpha_{i1} < \alpha_{i2} < \cdots < \alpha_{iq}$, where $u_i(t) \neq 0$ on $(\alpha_{iq}, \eta_q(b))$. If $u_i(t)$ and $u_{i+1}(t)$ are essentially different, it follows from Corollary 3.2 and Lemma 2.1 that $u_{i+1}(t)$ vanishes on $(\alpha_{ij}, \alpha_{i(j+1)})$, $j = 1, 2, \cdots, q-1$, and on $(\alpha_{i(q-1)}, \eta_q(b))$. Consequently, $u_{i+1}(t)$ has at least q zeros on $(b, \eta_q(b))$. Hence $u_i(t)$ has at least q zeros on $(b, \eta_q(b))$ for each $i = 0, 1, \cdots, n-p$, and the proof is complete by choosing $i = n-p$.

The method of proof used in Lemma 3.1 actually yields a somewhat stronger result than that stated. Instead of using the extremal solution $y_k(t)$, a similar proof may be constructed using any nontrivial solution $u(t)$ of (3.1) having at least $p-1$ zeros at $t = b$ and $(n-p)$ zeros on $(\eta_q(b), \infty)$. Furthermore, if $u(t)$ and $y_q(t)$, $q \geqq 1$, are essentially different solutions and a set of solution, $\{u_i(t)\}_{i=0}^{n-p}$ similar to that used in the proof of Lemma 3.1 is introduced, then for some $i = 1, 2, \cdots, n-p$, $u_i(t)$ and $u_{i-1}(t)$ are essentially different. It follows that $u(t)$ has at least q zeros on $(b, \eta_q(b))$. In [5] it is shown that if (3.1) is oscillatory, there exists oscillatory solutions with a zero at $t = b$ of multiplicities $p-1$ and p. Clearly, these oscillatory solutions must have q distinct zeros on $(b, \eta_q(b))$, $q \geqq 1$.

LEMMA 3.2. *For equation* (3.1) *if* $b > a$, $1 \leqq q \leqq k + p - n$, *and* $\eta_k(b) < \infty$, *then* $y_k(t)$ *has at most* $n - p + q - 1$ *zeros on* $(b, \eta_q(b))$.

To prove the lemma suppose to the contrary that $y_k(t)$ has at least $n-p+q$ distinct zeros (recall the zeros of $y_k(t)$ are simple on $(b, \eta_k(b))$) on $(b, \eta_q(b))$. Consider a set $\{v_i(t)\}_{i=0}^{n-p}$ of nontrivial solutions of (3.1) given by $v_i(t) \equiv u_{n-p-i}(t)$, $i = 0, 1, \cdots, n-p$, where $u_j(t)$, $j = 0, 1, 2, \cdots, n-p$, is defined by (3.5). Corollary 3.2 implies that for $i = 1, \cdots, n - p$ the number of zeros of $v_i(t)$ differs from the number of zeros of $v_{i-1}(t)$ by at most one. Hence $y_q(t) \equiv v_{n-p}(t)$ has at least q zeros on $(b, \eta_q(b))$ contradicting Theorem 2.3. This concludes the proof of the lemma.

We are now prepared to prove the primary theorem of the paper.

THEOREM 3.3. *For the system* (3.1) *the equation is oscillatory provided for some $b > a$, $\eta_k(b)$ exists for each integer $k \geqq 1$.*

We need only show that if $b > a$ and $\eta_k(b)$ exists for $k \geqq 1$, then there is an oscillatory solution of the equation. To this end recall the definition (3.4) of $y_k(t)$. Since each member of the sequence of vectors $\{(c_{kp}, c_{k(p+1)}, \cdots, c_{k(n-1)})\}_{k=1}^{\infty}$ lies on the unit ball in a finite dimensional vector space, there exists a subsequence of vectors which converge to a vector $(c_p, c_{p+1}, \cdots, c_{n-1})$ on the unit ball. Let $\{y_{k_i}(t)\}_{i=1}^{\infty}$ denote the corresponding subsequence of solutions, and define $y(t) = \sum_{j=p}^{n-1} c_j z_i(b; t)$. Clearly, $y(t) \not\equiv 0$ and the zeros of $y(t)$ are the accumulation points of the zeros of $\{y_{k_i}(t)\}_{i=1}^{\infty}$.

We now show that $y(t)$ has arbitrarily large zeros on (b, ∞). If $q > 1$ denotes an integer, then, for each integer $k > n - p + q + 1 \equiv \alpha(q)$, $y_k(t)$ has at least $n - p + q$ zeros on $(b, \eta_{\alpha(q)}(b))$ by Lemma 3.1. On the other hand, Lemma 3.2 yields that $y_k(t)$ has at most $n - p + q - 1$ zeros on $(b, \eta_q(b))$. Therefore $y_k(t)$ vanishes at least once on $(\eta_q(b), \eta_{\alpha(q)}(b))$ for every $b > n - p + q + 1$. It follows that $y(t)$ vanishes on $[\eta_q(b), \eta_{\alpha(q)}(b)]$ for each integer $q > 1$, and the proof is complete.

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