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STRUCTURE INHERENT IN A FREE GROUPOID

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ABSTRACT. It has often been of interest in mathematics to consider the problem of embedding a certain algebraic system in other more restricted systems, i.e., in those satisfying additional properties. In the present paper, an opposite problem is considered, namely, the possibility of finding certain more restricted systems within a given one. A new characterization of a free groupoid, in terms of a successor mapping, is also given.

1. **Introduction.** Let S be any non-empty collection of symbols not containing the element $*$. A free groupoid $\langle G, \bullet \rangle$, with free basis S, may be obtained in the following manner:

(G-1) G is the collection of exactly those strings of symbols from $S \cup \{ * \}$ which satisfy the condition that, reading from left to right in any string, the number of elements of S encountered never exceeds the number of $*$'s until the final symbol is reached and in the entire string the number of elements of S exceeds the number of $*$'s by one.

(G-2) For $a, b \in G$, the operation $\bullet$, with domain $G \times G$ and range $G - S$, is defined by

$$a \bullet b = * ab;$$

that is, the $\bullet$ product of a and b is obtained by concatenating the symbol $*$, the symbols of a, and the symbols of b, in left-to-right order.

Clearly, for $a, b \in G$, $a \bullet b$ is a member of G since condition (G-1) is satisfied.

The equivalence of the above definition of a free groupoid to the more usual one, stated in terms of homomorphisms, is demonstrated in [2]. A useful necessary and sufficient condition [1, p. 6] that a groupoid $\langle G, \bullet \rangle$ be free with free basis $S \subset G$ is: each $a \in S$ is prime in G, whereas if $a \in G - S$, say $a = g_1 g_2 \cdots g_n$, where $g_i \in S \cup \{*\}$ for $i = 1, 2, \cdots, n$, then $a = b \bullet c$ for exactly one ordered pair of elements b, c of G—namely, $b = g_2 g_3 \cdots g_m$ and $c = g_{m+1} \cdots g_n$, where m is the least positive (even) integer such that the number of occurrences of $*$ in the string $g_1 g_2 \cdots g_m$ is $m/2$.

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2. A New Characterization of the Free Groupoid. For simplicity of exposition, in the following we consider only the free groupoid $\langle G, \bullet \rangle$ with a single generator, s, although nearly all results translate easily to the arbitrary free groupoid. For our discussion, it is useful first to make the following definition.

DEFINITION 1. Let λ be the mapping, from G to the set of positive integers, defined by

(i) $\lambda(s) = 1$, and

(ii) if $g = *g'g''$, $g', g'' \in G$, then $\lambda(g) = \lambda(g') + \lambda(g'')$. $\lambda(g)$ is called the *length* of g.

It is easily shown [2, p. 17] that λ is a well-defined mapping whose domain is indeed G and whose range is the entire set of positive integers. In each string $g \in G$ the symbol s occurs $\lambda(g)$ times, a fact which is a corollary to Theorem 1, itself readily provable by induction on $\lambda(g)$.

THEOREM 1. *If $g = g_1 g_2 \cdots g_k \in G$, with each $g_i \in \{*, s\}$, then $k = 2\lambda(g) - 1$.*

The following concept of successor proves useful in obtaining a new characterization (Theorem 8) of a free groupoid.

DEFINITION 2. The element $*ss$ of G will be called the *successor* of s in G and if $g = *g'g''$, for $g', g'' \in G$, then a *successor* of g in G is any element of G of the form $*h'g''$ or $*g'h''$ where h' is a successor of g' in G and h'' is a successor of g'' in G.

Theorems 2, 3 and 4 are easily established by mathematical induction on $\lambda(g)$.

THEOREM 2. *Each g in G has a successor in G.*

THEOREM 3. *For $g \in G$, if g' is a successor of g in G then $\lambda(g') = \lambda(g) + 1$.*

THEOREM 4. *If $g = g_1 g_2 \cdots g_k \in G$, with each $g_i \in \{*, s\}$, then, for each $j \in \{1, 2, \cdots, k\}$ such that $g_j = s$, the element $g' = g_1 \cdots g_{j-1} * s g_j g_{j+1} \cdots g_k$ is a successor of g in G.*

As a direct consequence of Theorems 1 and 4 we obtain this next result.

THEOREM 5. *If an element g of G has $\lambda(g) = n$, then g has n distinct successors.*

If Z_g denotes the set of successors of g in G, then a straightforward proof by contraposition yields the following.

THEOREM 6. *For $g, h \in G$, if $g \neq h$ then $Z_g \neq Z_h$.*

The proof depends on the fact that every set Z_g contains exactly one element of the form $g_1 g_2 \cdots g_{k+1}$, where $k = \lambda(g)$, where each $g_i \in \{*, s\}$, and where $g_{k-1} = *$, $g_k = s$, $g_{k+1} = s$ (which implies that $g = g_1 g_2 \cdots g_{k-2} s$).

The successor concept will now be used to obtain an alternate characterization of a free groupoid. For that purpose, let A be the union of a countably infinite sequence $\{A_i\}$ of nonempty, pairwise disjoint sets subject to the following conditions:

(Z-1) A_1 contains a single element.

(Z-2) There exists an injective mapping z, with domain A and range contained in $P(A)$, the power set of A, such that if $a \in A_k$ then $z(a)$ contains k elements and $\bigcup_{a \in A_k} z(a) = A_{k+1}$ for each k.

z will be called a *successor mapping* and, for each $a \in A$, $z(a)$ will be called the *successor set of a*. It is an immediate consequence of (Z-1) and (Z-2) that A_i is a finite set, for each positive integer i. In fact, the addition of conditions (Z-3) to (Z-6) below guarantees [2, p. 35] that each A_i contains exactly $1/i\binom{2i-2}{i-1}$ elements.

In the free groupoid $\langle G, \bullet \rangle$, if we define $z' : G \rightarrow P(G)$ by $z'(g) = Z_g$ for each g in G and if $G_i = \{g \in G : \lambda(g) = i\}$, then z' is a successor mapping of A and $\{G_i\}$ is a countable collection of nonempty, pairwise disjoint sets for which (Z-1) and (Z-2) are satisfied relative to z'.

If in addition to (Z-1) and (Z-2) there exists a relation R, with domain $A \times A$ and range $A - A_1$, compatible with the successor mapping as prescribed by the conditions below, then it can be shown that R defines a single-valued mapping @ of $A \times A$ into A. In fact, $\langle A, @ \rangle$ is (Theorem 8) a free groupoid with free basis A_1. Writing $a @ b = c$ to mean $((a, b), c) \in R$, and defining $z(a) @ b = \{a' @ b : a' \in z(a)\}$, $a @ z(b) = \{a @ b' : b' \in z(b)\}$, the conditions are, for a, b, c, d in A:

(Z-3) $c = a @ b$ if and only if $z(a) @ b \cup a @ z(b) = z(c)$.

(Z-4) If $z(a) @ b \cap c @ z(d) \neq \emptyset$ then $c \in z(a)$ and $b \in z(d)$.

(Z-5) If $a \in A_j$ then $z(a) @ b$ and $b @ z(a)$ each contain j distinct elements, for any b in A.

(Z-6) *If* $a, b \in A_j$ with $a \neq b$, then $z(a) @ c \neq z(b) @ d$ and $c @ z(a) \neq d @ z(b)$ for any $c, d \in A_i$, for any positive integer i.

On the basis of the previously stated results for successors, we may deduce the following proposition.

THEOREM 7. *With z' as defined above, the operation $\bullet$ in the free* groupoid $\langle G, \bullet \rangle$ *satisfies the compatibility conditions* (Z-3) to (Z-6).

Theorem 8. *The system* $\langle A, @ \rangle$, *where* A *and* $@$ *are subject to conditions* (Z-1) to (Z-6), *is a free groupoid with free basis* A_1.

Proof. We first observe that it is sufficient to show that $@$ is an injective mapping from $A \times A$ onto $A - A_1$.

For $a, b, c, d \in A$, if $a @ b = c @ d$, then $z(a) @ b \cup a @ z(d) = z(c) @ d \cup c @ z(d)$. Suppose $a \in A_i$, $b \in A_j$. If $z(a) @ b \cap c @ z(d) \neq \emptyset$, then $c \in z(a)$ and $b \in z(d)$, hence $c \in A_{i-1}$, $d \in A_{j-1}$. Thus $z(c) @ d$ contains $i + 1$ elements, at least one of which must lie in $a @ z(b)$ since $z(a) @ b$ contains only i elements. But if $z(c) @ d \cap a @ z(b) \neq \emptyset$, then $a \in z(c)$ and $d \in z(b)$, which is a contradiction. Thus we conclude that $z(a) @ b \cap c @ z(d) = \emptyset$, and, by similar reasoning, $a @ z(b) \cap z(c) @ d = \emptyset$. Hence $z(a) @ b = z(c) @ d$ and $a @ z(b) = c @ z(d)$ from which it follows that $c \in A_i$, $d \in A_j$. Condition (Z-6) then implies that $a = c$ and $b = d$ and the injectivity of $@$ is established.

For $a \in A_i$, $b \in A_j$, $z(a) @ b$ contains i distinct elements and $a @ z(b)$ contains j distinct elements. If $z(a) @ b \cap a @ z(b) \neq \emptyset$, from (Z-4) it follows that $a \in z(a)$ and $b \in z(b)$, both of which are impossible. Thus $z(a) @ b \cup a @ z(b) = z(a @ b)$ contains $i + j$ elements and hence $a @ b \in A_{i+j}$. The range of $@$ is therefore $A - A_1$, and the theorem is established. ∎

3. **Structure Inherent in the Free Groupoid** $\langle G, \bullet \rangle$. In addition to the notion of successor, it is possible to define within G operations and relations resulting in additional algebraic structure. For example, we may define binary operations $\bullet_I$ and $\bullet_F$ in G in the manner described below. For the ordered pair g, g' of elements of G, $\bullet_I gg'$ and $\bullet_F gg'$ will denote those elements of G obtained by applying the operations $\bullet_I$ and $\bullet_F$ respectively. This prefix notation will be advantageous primarily because it is parenthesis-free; for example, instead of $(g \bullet_I g') \bullet_I g''$ and $g \bullet_F (g' \bullet_F g'')$, as in the more usual infix notation, we have $\bullet_I \bullet_I gg'g''$ and $\bullet_F g \bullet_F g'g''$, for g, g', g'' in G.

(I) For $g, g' \in G$, $\bullet_I gg'$ is that element of G obtained by substituting the string g' for the initial occurrence of s in g, reading from left to right.

(F) For $g, g' \in G$, $\bullet_F gg'$ is that element of G obtained by substituting the string g' for the final occurrence of s in g, reading from left to right.

Theorem 9. *Each of the algebraic systems* $\langle G, \bullet_I \rangle$ *and* $\langle G, \bullet_F \rangle$ *is a semigroup with identity.*

Proof. The result will be established for $\langle G, \bullet_I \rangle$ with a similar proof applying in the case of $\langle G, \bullet_F \rangle$. The condition for membership in G, stated in (G-1), is seen to be satisfied for $\bullet_I gg'$ whenever it is satisfied for g and g'. Hence $\bullet_I$ is indeed an operation in G. Further, the element s of G is seen to serve as an identity for $\bullet_I$. It remains to show that for $a, b, c \in G$, $\bullet_I \bullet_I abc = \bullet_I a \bullet_I bc$. This will be proved by mathematical induction on $\lambda(a)$. If $\lambda(a) = 1$, then $a = s$ and $\bullet_I \bullet_I abc = \bullet_I bc = \bullet_I a \bullet_I bc$. If $\lambda(a) > 1$ then there exist unique a', a'' in G such that $a = *a'a''$ and $\lambda(a') < \lambda(a)$, $\lambda(a'') < \lambda(a)$. Thus we have $\bullet_I a \bullet_I bc = \bullet_I *a'a'' \bullet_I bc$ and, by definition of $\bullet_I$, this latter is equivalent to $*\bullet_I a' \bullet_I bca''$. By the induction hypothesis, $\bullet_I a' \bullet_I bc = \bullet_I \bullet_I a'bc$, so that $*\bullet_I a' \bullet_I bca'' = *\bullet_I \bullet_I a'bca''$. Again utilizing the definition of $\bullet_I$, we may transform the right hand side into $\bullet_I \bullet_I *a'a''bc = \bullet_I \bullet_I abc$, and the proof is complete. ■

Although $\bullet_I$ and $\bullet_F$ are not commutative in G, there are infinite subsets of G in which this property also holds. For each positive integer k, let s^k denote that element of G consisting of $k - 1$ repetitions of $*$ followed by k repetitions of s, and let s_k denote that element of G consisting of $k - 1$ repetitions of $*s$ followed by a final s. If we define $G' = \bigcup \{s^k\}$ and $G'' = \bigcup \{s_k\}$, the union being taken over all positive integers, then the following result is apparent.

Theorem 10. *Each of the algebraic systems $\langle G', \bullet_I \rangle$ and $\langle G'', \bullet_F \rangle$ is a commutative semigroup with identity s.*

The mapping which associates with each element g of $G'(G'')$ the non-negative integer $\lambda(g) - 1$ is an isomorphism between $\langle G', \bullet_I \rangle(\langle G'', \bullet_F \rangle)$ and the additive semigroup of non-negative integers, and thus we may also make this next assertion.

Theorem 11. *The commutative semigroups $\langle G', \bullet_I \rangle$ and $\langle G'', \bullet_F \rangle$ are each freely generated by $*ss$.*

In addition, the element $*ss$ plays the role of generator in the set G, in a sense made precise by the following.

Theorem 12. *For $g \in G$, either $g = s$ or there exists a positive integer k such that $g = g_1 g_2 \cdots g_k$ with each $g_i \in \{\bullet_I, \bullet_F, *ss\}$.*

Proof. By induction on $\lambda(g)$. If $\lambda(g) \leqq 2$ then $g = s$ or $g = *ss$, and the theorem holds. If $\lambda(g) > 2$, then there exist unique g', g'' in G with $\lambda(g') < \lambda(g)$ and $\lambda(g'') < \lambda(g)$, such that $g = *g'g''$. If $\lambda(g') = 1$, then $\lambda(g'') > 1$ and $g = \bullet_F * ssg''$; if $\lambda(g'') = 1$, then $\lambda(g') > 1$ and $g = \bullet_I *ssg'$. When $\lambda(g') > 1$ and $\lambda(g'') > 1$, we have $g = \bullet_F \bullet_I *ssg'g''$.

In each case the inductive hypothesis applied to g' and/or g'' establishes the desired result. ■

Using the notion of successor, defined previously, one may define a partial order in G, i.e., a reflexive, antisymmetric and transitive relation in G.

DEFINITION 3. For $g, g' \in G$ we will say that g *precedes* g', denoted $g \leqq g'$, if and only if there exist a positive integer k and $g_1, g_2, \cdots, g_k \in G$ such that $g_1 = g$, $g_k = g'$, and g_i is a successor of g_{i-1} for each $i = 2, 3, \cdots, k$.

For completeness the following obvious results are stated as theorems.

THEOREM 13. *The set G is partially ordered relative to $\leqq$, and has s as its least element.*

THEOREM 14. *The sets G' and G'' are each linearly ordered by $\leqq$.*

THEOREM 15. *For g, g', g'' in G, if $g = \bullet_I g'g''$ or $g = \bullet_F g'g''$, then $g' \leqq g$ and $g'' \leqq g$.*

After observing that, for $g, g_1, g_2, g', g_1', g_2'$ in G with $g = *g_1g_2$ and $g' = *g_1'g_2'$, $g \leqq g'$ if and only if $g_1 \leqq g_1'$ and $g_2 \leqq g_2'$, the following may be proved easily by mathematical induction on $\lambda(g) + \lambda(g')$.

THEOREM 16. *Every pair g, g' has an infimum,* $\inf(g, g')$, *and a supremum,* $\sup(g, g')$, *in G.*

Theorem 16 asserts that the system $\langle G, \leqq \rangle$ is a lattice and we may further state the following.

THEOREM 17. *The lattice $\langle G, \leqq \rangle$ is distributive.*

PROOF. What the theorem asserts is that, for g, g', g'' in G, the formulas

$$\inf(g, \sup(g', g'')) = \sup(\inf(g, g'), \inf(g, g'')),$$

$$\sup(g, \inf(g', g'')) = \inf(\sup(g, g'), \sup(g, g'')),$$

hold and their verification rests on the following well-known result:

> A lattice is distributive if and only if it does not contain a sublattice isomorphic to either of the 5-element lattices of Figure 1. In each of these lattices the ordering is defined by: $p \leqq q$ if and only if $p = q$ or there exists a downward path from q to p.

The assumption that a sublattice isomorphic to either of the lattices of Figure 1 can occur in G quickly leads to contradiction of aforementioned properties of G, and the desired conclusion is thus established. ■

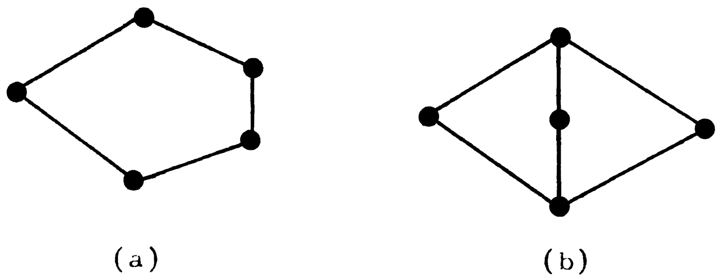

FIGURE 1

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