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NUMERICAL APPROXIMATION FOR 2m TH ORDER DIFFERENTIAL SYSTEMS VIA SPLINES

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1. **Introduction.** In [1] an approximation theory for elliptic forms on Hilbert spaces was given. The principal results were concerned with inequalities involving the signature $s(\sigma)$ and the nullity $n(\sigma)$ of the form $J(x;\sigma)$ defined on $\mathcal{A}(\sigma)$, where σ is a member of the metric space (Σ, ρ) and $\mathcal{A}(\sigma)$ denotes a closed subspace of a Hilbert space $\mathcal{A}$. These results were later applied to second order differential systems ([2]).

In this paper we consider elliptic forms whose associated Euler equations are self adjoint, $2m$ *th* order ordinary differential equations in p dependent variables. It is shown that the inequalities hold for the approximation of arcs (whose component functions $x_\alpha(t)$ satisfy $x_\alpha \in C^{m-1}$, $x_\alpha{}^{(m-1)}$ is absolutely continuous, and $x_\alpha{}^{(m)} \in L^2$) by $2m$ *th* order splines. Thus the approximation is by finite dimensional problems. The indices $s(\sigma)$ and $n(\sigma)$ are shown to be given by the number of negative and zeros eigenvalues of a symmetric matrix. A description is given for the application of these procedures to the numerical approximation of eigenvalue problems in this setting.

Splines are used in this paper in two ways. On the abstract level, their well known approximation properties simplify the "proofs" needed to show that the approximating hypothesis of [1] is satisfied. On the applied level it is shown that splines are the right approximating elements. It is interesting to observe that the "chain is complete", if we view splines as solutions to fixed end point problems in the calculus of variations.

It is clear that ideas and results of this paper may be applied to a wide variety of problems; for example to eigenvalue or focal point problems associated with linear self adjoint systems of ordinary or partial differential equations. In addition these results can be related to problems in optimal control theory as well as the calculus of variations. In a later work the results of this paper will be applied to numerical solution of oscillation points for $2m$ *th* order differential systems.

Section 2 contains the ideas from [1] which are needed in this paper. In Section 3 we define the fundamental quadratic form and Hilbert space, and the approximating forms and spaces. Section 4

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contains the necessary properties of splines, proofs that the approximating hypothesis hold, and finally the fundamental inequalities (Theorems 12 and 13). Section 5 shows that the approximating indices $s(\sigma)$ and $n(\sigma)$ may be obtained as the number of negative and zero eigenvalues of a real, symmetric, sparse matrix. Finally Section 6 shows that general compact eigenvalue problem can be "solved" by our approximation methods. Theorem 16 shows that the k th eigenvalue is a continuous function of the approximation parameter σ.

2. **Preliminaries.** We now state the approximation hypothesis given in [1]. These hypotheses are contained in conditions (1) and (2). In this paper $\mathcal{A}$ will denote a Hilbert space with inner product (x, y) and norm $\|x\| = (x, x)^{1/2}$. Strong convergence will be denoted by $x_q \Rightarrow x_0$ and weak convergence by $x_q \to x_0$. The bilinear forms $Q(x, y)$ in this paper are assumed to be bounded and symmetric. The associated quadratic form is given by $Q(x) = Q(x, x)$.

Let Σ be a metric space with metric ρ. A sequence $\{\sigma_r\}$ in Σ converges to σ_0 in Σ, written $\sigma_r \to \sigma_0$, if $\lim_{r\to\infty}\rho(\sigma_r, \sigma_0) = 0$. For each σ in Σ let $\mathcal{A}(\sigma)$ be a closed subspace of $\mathcal{A}$ such that

(1a) if $\sigma_r \to \sigma_0$, x_r in $\mathcal{A}(\sigma_r)$, $x_r \to y_0$ then y_0 is in $\mathcal{A}(\sigma_0)$;

(1b) if x_0 is in $\mathcal{A}(\sigma_0)$ and $\epsilon > 0$ there exists $\delta > 0$ such that whenever $\rho(\sigma, \sigma_0) < \delta$ there exists x_σ in $\mathcal{A}(\sigma)$ satisfying $\|x_0 - x_\sigma\| < \epsilon$.

For each σ in Σ let $J(x;\sigma)$ be a quadratic form defined on $\mathcal{A}(\sigma)$ with $J(x, y; \sigma)$ the associated bilinear form. For $r = 0, 1, 2, \cdots$ let x_r be in $\mathcal{A}(\sigma_r)$, y_r in $\mathcal{A}(\sigma_r)$ such that: if $x_r \to x_0$, $y_r \Rightarrow y_0$ and $\sigma_r \to \sigma_0$ then

(2a) $\lim_{r\to\infty} J(x_r, y_r; \sigma_r) = J(x_0, y_0; \sigma_0)$;

(2b) $\lim_{r\to\infty}\inf J(x_r; \sigma_r) \geqq J(x_0; \sigma_0)$; and

(2c) $\lim_{r\to\infty} J(x_r; \sigma_r) = J(x_0; \sigma_0)$ implies $x_r \Rightarrow x_0$.

The form $J(x)$ is *elliptic* on $\mathcal{A}$ if conditions (2b) and (2c) hold with $J(x)$ replacing $J(x;\sigma)$ and $\mathcal{A}$ replacing $\mathcal{A}(\sigma)$. The *signature* (index) of $Q(x)$ on a subspace $\mathcal{B}$ of $\mathcal{A}$ is the dimension of a maximal, linear subclass $\mathcal{C}$ of $\mathcal{B}$ such that $x \neq 0$ in $\mathcal{C}$ implies $Q(x) < 0$.

The *nullity* of $Q(x)$ on $\mathcal{B}$ is the dimension of the space $\mathcal{B}_0 = \{x \text{ in } \mathcal{B} \mid Q(x, y) = 0 \text{ for all } y \text{ in } \mathcal{B}\}$. In this paper we denote the index and nullity of $J(x;\sigma)$ on $\mathcal{A}(\sigma)$ by $s(\sigma)$ and $n(\sigma)$ respectively. Let $m(\sigma) = s(\sigma) + n(\sigma)$.

Theorems 1 to 4 have been given in [1].

THEOREM 1. *Assume conditions* (1a), (2b), *and* (2c) *hold. Then for any σ_0 in Σ there exists $\delta > 0$ such that $\rho(\sigma_0, \sigma) < \delta$ implies*

$$s(\sigma) + n(\sigma) \leqq s(\sigma_0) + n(\sigma_0). \tag{3}$$

THEOREM 2. *Assume conditions* (1b) *and* (2a) *hold. Then for any* σ_0 *in* Σ *there exists* $\delta > 0$ *such that* $\rho(\sigma_0, \sigma) < \delta$ *implies*

$$s(\sigma_0) \leqq s(\sigma). \tag{4}$$

Combining Theorems 1 and 2 we obtain

THEOREM 3. *Assume conditions* (1) *and* (2) *hold. Then for any* σ_0 *in* Σ *there exists* $\delta > 0$ *such that* $\rho(\sigma,\sigma_0) < \delta$ *implies*

$$s(\sigma_0) \leqq s(\sigma) \leqq s(\sigma) + n(\sigma) \leqq s(\sigma_0) + n(\sigma_0). \tag{5}$$

COROLLARY 4. *Assume* $\delta > 0$ *has been chosen such that* $\rho(\sigma,\sigma_0) < \delta$ *implies inequality* (5) *holds. Then if* $\rho(\sigma,\sigma_0) < \delta$ *we have*

(6a) $n(\sigma) \leqq n(\sigma_0)$,

(6b) $n(\sigma) = n(\sigma_0)$ implies $s(\sigma) = s(\sigma_0)$ and $m(\sigma) = m(\sigma_0)$, and

(7) $n(\sigma_0) = 0$ implies $s(\sigma) = s(\sigma_0)$ and $n(\sigma) = 0$.

3. **The Forms $J(x;\sigma)$ and the Spaces $\mathcal{A}(\sigma)$.** In this paper A will denote the totality of arcs x in $(t, x_1, \cdots, x_p)$ space defined by a set of p real valued functions $x : x_\alpha(t)$, $(0 \leqq t \leqq 1;\ \alpha = 1, \cdots, p)$ such that $x_\alpha(t)$ is of class C^{m-1}; $x_\alpha{}^{(m-1)}(t)$ is absolutely continuous; $x_\alpha{}^{(m)}(t)$ is square integrable. In the remainder of this section α denotes a parameter with values $1, 2, \cdots, p$; q a parameter with values $0, \cdots, m-1$; superscripts denote the order of differentiation; and repeated indices are assume summed. The inner product is given by

$$(x, y) = x_\alpha{}^{(q)}(0)y_\alpha{}^{(q)}(0) + \int_0^1 x_\alpha{}^{(m)}(t)y_\alpha{}^{(m)}(t)\, dt, \quad (m \text{ not summed})$$

with corresponding norm given by $\|x\|^2 = (x, x)$.

Let Σ denote the set of real numbers $\sigma = 1/n$ $(n = 1, 2, 3, \cdots)$ and zero. The metric on Σ is the absolute value function. Let $\mathcal{A}(0) = \mathcal{A}$. To construct $\mathcal{A}(\sigma)$ for $\sigma = 1/n$ define the partition $\pi(\sigma) = \{k/n \mid k = 0, 1, \cdots, n\}$. The space $\mathcal{A}(\sigma)$ is the space of spline functions with knots at $\pi(\sigma)$, which shall be described in Theorem 5. The space is a $p(n+1)$ dimensional space.

The fundamental (real) bilinear form is given by

$$J(x, y) = H(x, y) + \int_0^1 R^{ij}_{\alpha\beta}(t)x_\alpha{}^{(i)}(t)y_\beta{}^{(j)}(t)\, dt \tag{9}$$

where

$$H(x, y) = A^{k\ell}_{\alpha\beta}x_\alpha{}^{(k)}(0)y_\beta{}^{(\ell)}(0) + B^{k\ell}_{\alpha\beta}[x_\alpha{}^{(k)}(0)y_\beta{}^{(\ell)}(1) + x_\alpha{}^{(k)}(1)y_\beta{}^{(\ell)}(0)] + C^{k\ell}_{\alpha\beta}\, x_\alpha{}^{(k)}(1)y_\beta{}^{(\ell)}(1),$$

$A^{k\ell}_{\alpha\beta} = A^{\ell k}_{\beta\alpha}$, $C^{k\ell}_{\alpha\beta} = C^{\ell k}_{\beta\alpha}$ and $B^{k\ell}_{\alpha\beta}$ are constant matrices; $R^{ij}_{\alpha\beta}(t) = R^{ji}_{\beta\alpha}(t)$ are (for purposes of simplicity) continuous functions on $0 \leqq t \leqq 1$; and the inequality

$$R^{mm}_{\alpha\beta}(t)\phi_\alpha\phi_\beta \geqq h\phi_\alpha\phi_\beta \tag{10}$$

holds almost everywhere on $0 \leqq t \leqq 1$, for every $\phi = (\phi_1, \cdots, \phi_p)$ in E^p, and some $h > 0$. In the above $\alpha, \beta = 1, \cdots, p$; $k, \ell = 0, \cdots, m-1$; $i, j = 0, \cdots, m$.

The fundamental quadratic form is

$$J(x; 0) = J(x) = H(x, x) + \int_0^1 R^{ij}_{\alpha\beta}(t)x_\alpha{}^{(i)}(t)x_\beta{}^{(j)}(t)\, dt. \tag{11}$$

For $\sigma = 1/n (n = 1, 2, 3, \cdots)$ we now define the quadratic form $J(x; \sigma)$ for x in $\mathcal{A}(\sigma)$. Thus let $R^{ij}_{\alpha\beta\sigma}(t) = R^{ij}_{\alpha\beta}(k/n)$ if $t \in [k/n, (k+1)/n)$ and $R^{ij}_{\alpha\beta\sigma}(1) = R^{ij}_{\alpha\beta}((n-1)/n)$ for $\alpha, \beta = 1, \cdots, p$; $i, j = 0, \cdots, m$. Finally set

$$J(x; \sigma) = H(x, x) + \int_0^1 R^{ij}_{\alpha\beta\sigma}(t)x_\alpha{}^{(i)}(t)x_\beta{}^{(j)}(t)\, dt \tag{12}$$

where $x = [x_1(t), \cdots, x_p(t)]$, $x(t) \in \mathcal{A}(\sigma)$.

4. **Splines and Inequalities.** In this section we show that conditions (1) and (2) hold for the spaces $\mathcal{A}(\sigma)$ and forms $J(x; \sigma)$ defined in Section 3. We first state the necessary results from the theory of Splines which we need.

By a *spline function* of degree $2m - 1$ (or order $2m$), having knots at $\pi(1/n)$, we mean a function $S(t)$ in $C^{2m-2}(-\infty, \infty)$ with the property $S(t) \in P_{2m-1}$ (a polynomial of degree at most $2m - 1$) in each of the intervals $(-\infty, 0)$, $(0, 1/n)$, $\cdots$, $((n-1)/n, 1)$, $(1, \infty)$. Let $m \leqq n + 1$ and denote by $\Sigma_{2m}(n)$, those spline functions of degree $2m - 1$ which reduce to an element of P_{m-1} in each of the intervals $(-\infty, 0)$ and $(1, \infty)$. The last condition implies $S^{(v)}(0) = S^{(v)}(1) = 0$ for $v = m, \cdots, 2m-2$. Theorems 5 to 7 are given in [6].

THEOREM 5. *If $y_0, \cdots, y_n$ are real numbers there exists a unique $S(t) \in \Sigma_{2m}(n)$ such that $S(k/n) = y_k$ $(k = 0, \cdots, n)$.*

THEOREM 6. *Let $f(t) \in \mathcal{A}$ (with $p = 1$), and suppose $S(t)$ is the unique element of $\Sigma_{2m}(n)$ such that $S(k/n) = f(k/n)$, $(k = 0, \cdots, n)$.*
(a) *If $s(t) \in \Sigma_{2m}(n)$ then*

$$\int_0^1 [s^{(m)}(t) - f^{(m)}(t)]^2\, dt \geqq \int_0^1 [S^{(m)}(t) - f^{(m)}(t)]^2\, dt,$$

with equality if and only if $s(t) - S(t) \in P_{m-1}$.

(b) $$\int_0^1 (f^{(m)}(t))^2\,dt \geqq \int_0^1 (S^{(m)}(t))^2\,dt,$$

with equality if and only if $f(t) = S(t)$ *in* $[0, 1]$.

THEOREM 7. *Let* $f(t)$ *and* $S_n(t) \in \Sigma_{2m}(n)$ *satisfy (for each* n *such that* $m \leqq n + 1$*) the hypothesis in Theorem* 6. *Then*

(a) $$\lim_{n\to\infty} \int_0^1 [S_n^{(m)}(t) - f^{(m)}(t)]^2\,dt = 0,$$

(b) *For each* $v = 0, 1, \cdots, m - 1$

$$\lim_{n\to\infty} S_n^{(v)}(t) = f^{(v)}(t) \text{ uniformly on } [0, 1], \text{ and}$$

(c) $$\lim_{n\to\infty} \int_0^1 (S_n^{(m)}(t))^2\,dt = \int_0^1 (f^{(m)}(t))^2\,dt.$$

The following result which characterizes weak and strong convergence in $\mathcal{A} = \mathcal{A}(0)$ is found in [5]. Let $\alpha = 1, \cdots, p$; $k = 0, \cdots, m - 1$, then:

THEOREM 8. *The relation* $x_q = [x_{q1}(t), x_{q2}(t), \cdots, x_{qp}(t)]$ *converges strongly to* $x_0 = [x_{01}(t), x_{02}(t), \cdots, x_{0p}(t)]$, *denoted by* $x_q \Rightarrow x_0$, *holds if and only if, for each* α *and* k, $x_{q\alpha}^{(k)}(0) \to x_{0\alpha}^{(k)}(0)$ *and* $x_{q\alpha}^{(m)}(t) \to x_{0\alpha}^{(m)}(t)$ *in the mean of order two. Similarly* x_q *converges weakly to* x_0, *denoted by* $x_q \to x_0$, *holds if and only if, for each* α *and* k, $x_{q\alpha}^{(k)}(0) \to x_{0\alpha}^{(k)}(0)$ *and* $x_{q\alpha}^{(m)}(t) \to x_{0q}^{(m)}(t)$ *weakly in the class of Lebesgue summable square functions. In either case for each* α *and* k, $x_{q\alpha}^{(k)}(t) \to x_{0\alpha}^{(k)}(t)$ *uniformly on* $0 \leqq t \leqq 1$.

We now show that conditions (1) and (2) hold in light of the theorems on splines. Let x_0 in $\mathcal{A}(0) = \mathcal{A}$ be given. For $\sigma = 1/n$; $n = 1, 2, 3, \cdots$ let $x_{\sigma j}(t)$ be the unique element of $\Sigma_{2m}(n)$ such that $x_{\sigma j}(t) = x_{0j}(t)$ for $t \in \pi(\sigma)$ and $j = 1, \cdots, p$, described in Theorem 6. Let $x_\sigma(t) = [x_{\sigma 1}(t), x_{\sigma 2}(t), \cdots, x_{\sigma p}(t)]$. Condition (1b) now holds from Theorem 8.

THEOREM 9. *Assume for each* $\sigma = 1/n$ $(n = 1, 2, 3, \cdots)$ *that* x_σ *is the arc constructed above which agrees with the arc* x_0 *in* $\mathcal{A}(0) = \mathcal{A}$ *at the points* $\pi(\sigma)$. *Then* $x_\sigma \Rightarrow x_0$. *Thus condition* (1b) *holds.*

Since

$$\|x_\sigma - x_0\|^2 = [x_{\sigma\alpha}^{(q)}(0) - x_{0\alpha}^{(q)}(0)]\,[x_{\sigma\alpha}^{(q)}(0) - x_{0\alpha}^{(q)}(0)] +$$

$$\int_0^1 [x_{\sigma\alpha}^{(m)}(t) - x_{0\alpha}^{(m)}(t)]\,[x_{\sigma\alpha}^{(m)}(t) - x_{0\alpha}^{(m)}(t)]\,dt,$$

(where $\alpha = 1, \cdots, p$; $q = 0, \cdots, m-1$; α and q summed; m not summed) the result follows from parts (a) and (b) of Theorem 7.

THEOREM 10. *Condition* (1a) *holds.*

Since $\mathcal{A}(\sigma)$ is a subspace of $\mathcal{A} = \mathcal{A}(0)$ for each $\sigma = 1/n$, the result follows from the weak completeness of Hilbert spaces.

THEOREM 11. *If we define* $J(x; 0) = J(x)$ *then* $J(x; \sigma)$ *defined on* $\mathcal{A}(\sigma)$ *and given by* (12) *satisfies condition* (2).

For (2a) assume x_r, y_r in $\mathcal{A}(\sigma_r)$, $x_r \to x_0$, $y_0 \Rightarrow y_0$ and $\sigma_r = 1/r \to 0$. Let $J(x, y; \sigma)$ be the bilinear form associated with $J(x; \sigma)$, then $|J(x_r, y_r; \sigma_r) - J(x_0, y_0)| \leqq |J(x_r, y_r; \sigma_r) - J(x_r, y_r)| + |J(x_r, y_r) - J(x_0, y_0)|$. The second difference becomes arbitrarily small as $J(x, y)$ is an elliptic form on $\mathcal{A}$. The first difference is bounded by

$$\int_0^1 [R^{ij}_{\alpha\beta}(t) - R^{ij}_{\alpha\beta\sigma}(t)]\, \overset{(i)}{x}_{\sigma\alpha}(t) \overset{(j)}{y}_{\sigma\beta}(t)\, dt \leqq M_1 \psi(\sigma) \|x_r\|\, \|y_r\| \leqq M_2 \psi(\sigma),$$

where $\psi(\sigma) = 2 \sup \{|R^{ij}_{\alpha\beta}(t) - R^{ij}_{\alpha\beta\sigma}(t)|\}$, and the supremum is taken for t in $[0, 1]$; $i, j = 0, \cdots, m$; $\alpha, \beta = 1, \cdots, p$. Thus the first difference tends to zero as $\sigma \to 0$ by the continuity of $R^{ij}_{\alpha\beta}(t)$ and the fact that both weak and strong convergence imply boundedness.

For (2b) assume $x_r \to x_0$ and note that $J(x_r; \sigma_r) - J(x_0) = J(x_r; \sigma_r) - J(x_r) + J(x_r) - J(x_0)$. As above $|J(x_r; \sigma_r) - J(x_r)| \leqq M_3 \psi(\sigma_r)$ and can be made arbitrarily small. The result now follows as $J(x)$ is elliptic and hence weakly lower semicontinuous.

For (2c) suppose $x_r \to x_0$ and $J(x_r; \sigma_r) \to J(x_0)$. We note that

$$|J(x_r; \sigma_r) - J(x_0)| \geqq |\,|J(x_r; \sigma_r) - J(x_r)| - |J(x_0) - J(x_r)|\,|.$$

As above $|J(x_r; \sigma_r) - J(x_r)| \to 0$ so that $J(x_r) \to J(x_0)$. But $J(x)$ is elliptic so that $x_r \Rightarrow x_0$. This completes the proof.

THEOREM 12. *Let* $J(x)$ *be given by* (11). *For* $\sigma = 1/n$ $(n = 1, 2, \cdots)$ *let* $J(x; \sigma)$ *be defined on* $\mathcal{A}(\sigma)$ *and given by* (12). *Let* $s(\sigma)$ *and* $n(\sigma)$ *be the index and nullity of* $J(x; \sigma)$ *on* $\mathcal{A}(\sigma)$ *and* $s(0)$ *and* $n(0)$ *be the index and nullity of* $J(x)$ *on* $\mathcal{A}$. *Then there exists* $\delta > 0$ *such that whenever* $|\sigma| < \delta$

$$s(0) \leqq s(\sigma) \leqq s(\sigma) + n(\sigma) \leqq s(0) + n(0). \tag{13}$$

This result follows by Theorems 3, 9, 10 and 11.

In many types of problems, such as eigenvalue problems, focal point problems, or normal oscillation problems the nullity $n(0) = 0$ except at a "finite number of points". In this case we have

COROLLARY 13. *Assume the hypothesis and notation of Theorem* 12 *and that* $n(0) = 0$. *Then there exists a* $\delta > 0$ *such that whenever* $|\sigma| < \delta$ *we have*

$$s(\sigma) = s(0) \quad \text{and} \quad n(\sigma) = 0. \tag{14}$$

5. **The Finite Dimensional Problem.** In this section we will show that the indices $s(\sigma)$ and $n(\sigma)$ are the number of negative and zero eigenvalues of a real symmetric matrix. In a later paper we will apply the methods of this section to find oscillation points for $2mth$ order differential equations.

Let $\alpha, \beta = 1, \cdots, p$; $i, j = 0, \cdots, m-1$; $k, \ell = 0, \cdots, n$; and $\epsilon = (\alpha - 1)(n + 1) + (k + 1)$; $\eta = (\beta - 1)(n + 1) + (\ell + 1)$. Repeated indices are summed unless otherwise indicated.

Let $z = [z_1(t), \cdots, z_p(t)]$ be a fixed vector in $\mathcal{A}$. We now construct an approximate vector $x = [x_1(t), \cdots, x_p(t)]$ in $\mathcal{A}(\sigma)$. Assume as above the αth component function $x_\alpha(t)$ is given by $\xi_{\alpha k} y_k(t)$ where $y_k(t)$ is a basis element of the spline space $\Sigma_{2m}(n)$ described in Theorem 5. We note that $x_\alpha^{(i)}(0) = \xi_{\alpha k} y_k^{(i)}(0) \to z_\alpha^{(i)}(0)$ and $x_\alpha^{(i)}(1) = \xi_{\alpha k} y_k^{(i)}(1) \to z_\alpha^{(i)}(1)$.

From (12) we have

$$\begin{aligned} J(x; \sigma) &= H(x) + \int_0^1 R^{ij}_{\alpha\beta\sigma}(t) x_\alpha^{(i)}(t) x_\beta^{(j)}(t)\, dt \\ &= A^{ij}_{\alpha\beta} \xi_{\alpha k} y_k^{(i)}(0) \xi_{\beta \ell} y_\ell^{(j)}(0) + 2B^{ij}_{\alpha\beta} \xi_{\alpha k} y_k^{(i)}(0) \xi_{\beta\ell} y_\ell^{(j)}(1) \\ &\quad + C^{ij}_{\alpha\beta} \xi_{\alpha k} y_k^{(i)}(1) \xi_{\beta\ell} y_\ell^{(j)}(1) + \int_0^1 R^{ij}_{\alpha\beta\sigma} \xi_{\alpha k} \xi_{\beta\ell} y_k^{(i)}(t) y_\ell^{(j)}(t)\, dt \\ &= \chi^{k\ell}_{\alpha\beta} \xi_{\alpha k} \xi_{\beta\ell}, \end{aligned}$$

where

$$\begin{aligned} \chi^{k\ell}_{\alpha\beta} = {} & A^{ij}_{\alpha\beta} y_k^{(i)}(0) y_\ell^{(j)}(0) + 2B^{ij}_{\alpha\beta} y_k^{(i)}(0) y_\ell^{(j)}(1) \\ & + C^{ij}_{\alpha\beta}\, y_k^{(i)} y_\ell^{(j)}(1) \\ & + \int_0^1 R^{ij}_{\alpha\beta\sigma} y_k^{(i)}(t) y_\ell^{(j)}(t)\, dt. \end{aligned}$$

If we set $\Gamma_\epsilon = \xi_{\alpha k}$, $\Gamma_\eta = \xi_{\beta\ell}$, and $d_{\epsilon\eta} = \chi^{k\ell}_{\alpha\beta}$ we have

$$J(x; \sigma) = d_{\epsilon\eta}(\sigma) \Gamma_\epsilon \Gamma_\eta\ , \tag{15}$$

for $\epsilon, \eta = 1, \cdots, p(n + 1)$.

We note that the matrix $(d_{\epsilon\eta}(\sigma))$ is symmetric. For $p = 1$ and $m = 1$ we obtain a tridiagonal matrix for zero boundary data. For the general problem with zero boundary data we note that a different class of interpolating splines have support on at most $2m$ intervals. Hence our matrix will appear in diagonal form, each diagonal of length at most $4m - 1$, and "separated" from the next diagonal by length n. Thus the matrix is sparse (a preponderance of zeros) and existing computor techniques may be used to find the number of negative and zero eigenvalues of this real symmetric matrix.

THEOREM 14. *The indices $s(\sigma)$ and $n(\sigma)$ are respectively the number of negative and zero eigenvalues of the $p(n + 1) \times p(n + 1)$ matrix $(d_{\epsilon\eta}(\sigma))$.*

6. **The Associated Eigenvalue Problem.** We will briefly indicate how the above results may be applied to compact eigenvalue problems. Let $J(x)$ be given by (11). The most general compact form in our setting is the form

$$K(x) = \bar{H}(x) + \int_0^1 \bar{R}^{ij}(t)x^{(i)}(t)x^{(j)}(t)\,dt\,, \tag{16}$$

where

$$\bar{H}(x) = \bar{A}^{k\ell}_{\alpha\beta}x_\alpha{}^{(k)}(0)y_\beta{}^{(\ell)}(0) + 2\bar{B}^{k\ell}_{\alpha\beta}x_\beta{}^k(0)x_\beta{}^\ell(1) + \bar{C}^{k\ell}_{\alpha\beta}x_\alpha{}^k(1)x_\beta{}^\ell(1).$$

The barred matrices described in (16) satisfy exactly the same conditions as the unbarred matrices for $J(x)$ in (9) and (11) except that in equation (16), $0 \leqq i + j < 2m$. That is except for the $x_\alpha{}^{(m)}y_\beta{}^{(m)}$ term, $J(x)$ given in (11) is a compact quadratic form. We also assume that $K(x) \leqq 0$, x in $\mathcal{A}$ implies $J(x) > 0$. References [1] and [3] explain in detail the relationship between our problem and the eigenvalue problem for linear, compact, self-adjoint operators.

Set $\bar{R}^{ij}_{\alpha\beta\sigma}(t) = \bar{R}^{ij}_{\alpha\beta}(k/n)$ if t is in $[k/n, (k + 1)/n)$ and $\bar{R}^{ij}_{\alpha\beta\sigma}(1) = \bar{R}^{ij}_{\alpha\beta}((n - 1)/n)$ where $0 \leqq i + j < 2m$. Then we may define

$$K(x;\sigma) = \bar{H}(x) + \int_0^1 \bar{R}^{ij}_{\alpha\beta\sigma}(t)x_\alpha{}^{(i)}(t)x_\beta{}^{(j)}(t)\,dt.$$

Let $M = E^1 \times \Sigma$ be the metric space with metric d given by $d(\mu_1, \mu_2) = |\lambda_2 - \lambda_1| + |\sigma_2 - \sigma_1|$ where $\mu_i = (\lambda_i, \sigma_i)$ in M. Let $K(x; 0) = K(x)$. For each real λ define

$$L(x;\mu) = J(x;\sigma) - \lambda K(x;\sigma)\,, \tag{17}$$

on the space $\mathcal{A}(\mu) = \mathcal{A}(\sigma)$ where $\mu = (\lambda, \sigma)$.

We note that the results given above in the J, σ notation hold in the L, μ notation (see Reference [1]). For example, if we define $s(\mu) = s(\lambda, \sigma)$ and $n(\mu) = n(\lambda, \sigma)$ to be the index and nullity of $L(x; \mu)$ on $\mathcal{A}(\mu) = \mathcal{A}(\sigma)$, then Theorem 12 becomes

THEOREM 15. *Let λ_0 in E^1 be given. Let $M = E^1 \times \Sigma$, $\mu = (\lambda, \sigma)$ in M, $L(x; \mu)$ be defined on $\mathcal{A}(\mu) = \mathcal{A}(\sigma)$ and given by* (17). *Let $\mu_0 = (\lambda_0, 0)$ in M and $s(\mu), n(\mu)$ be the index and nullity of $L(x; \mu)$ on $\mathcal{A}(\mu) = \mathcal{A}(\sigma)$. Then there exists a $\delta > 0$ such that whenever $|\sigma| < \delta$*

$$s(\mu_0) \leqq s(\mu) \leqq s(\mu) + n(\mu) \leqq s(\mu_0) + n(\mu_0). \tag{18}$$

The result follows by an extension of Theorem 3 (see Reference [1]).
Proceeding analogously to equation (15) we obtain

$$L(x; \mu) = L(x; \lambda, \sigma) = e_{\epsilon\eta}(\lambda, \sigma)\Gamma_\epsilon\Gamma_\eta \tag{19}$$

where $\epsilon, \eta = 1, \cdots, p(n+1)$.

We note that the matrix $(e_{\epsilon\eta}(\lambda, \sigma))$ is a linear function of λ, and hence is easily computed for each λ and fixed σ. It is also symmetric and sometimes sparse and has the properties described for $(d_{\epsilon\eta}(\sigma))$.

The following definition gives the relationship between our indices and the definition of eigenvalues. It is equivalent to the usual definition for linear, compact, self adjoint operators on a Hilbert space.

Let σ_0 in Σ be given. A real number λ_0 is an *eigenvalue* (characteristic value) of $J(x; \sigma_0)$ relative to $K(x; \sigma_0)$ on $\mathcal{A}(\sigma_0)$ if $n(\lambda_0, \sigma_0) \neq 0$. The number $n(\lambda_0, \sigma_0)$ is its *multiplicity*. An eigenvalue λ_0 will be counted the number of times equal to its multiplicity. If λ_0 is an eigenvalue and $x_0 \neq 0$ in $\mathcal{A}(\sigma_0)$ such that $J(x_0, y; \sigma_0) = \lambda_0 K(x_0, y; \sigma_0)$ for all y in $\mathcal{A}(\sigma_0)$ then x_0 is an *eigenvector* corresponding to λ_0.

Setting $\sigma_0 = 0$, Theorem 15 relates the eigenvalues for the finite dimension σ-problem to the eigenvalue of $J(x)$ relative to the compact form $K(x)$ on $\mathcal{A}$. Continuity of the nth eigenvalue follows by Theorem 15 since $n(\lambda, 0) = 0$ except at the eigenvalues of $J(x)$ relative to $K(x)$. It is a well known result, that the set $\Lambda = \{\lambda \text{ in } E^1 \mid n(\lambda, 0) \neq 0\}$ had no finite cluster point. Let λ^* be such that $L(x; \lambda^*, 0)$ is positive definite. Let $\lambda_k(\sigma)$ $(k = 0, \pm 1, \pm 2, \cdots)$ denote the $(k+1)$ *st* eigenvalue greater than λ^* if $k \geqq 0$ and the ℓth eigenvalue $(\ell = -k)$ less than λ^* if $k < 0$. Then

COROLLARY 16. *If the* kth *eigenvalue $\lambda_k(\sigma)$ $(k = 0, \pm 1, \pm 2, \cdots)$ exists for $\sigma = 0$ it exists in a neighborhood of $\sigma = 0$ and is a continuous function of σ.*

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