A FORMAL NUMBER-TERMED NUMBER SYSTEM BASED ON RECURSION

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ABSTRACT. This is a formal number-termed axiomatization of number theory based on recursion without presupposing or developing general functional concepts or any part of set theory, together with an initial development to indicate its adequacy. The formal inferential system in which it is framed is that of A. P. Morse. The axioms, phrased in the primitive terms '0' and 'II xy $\underline{u}'$ xy m n' and the defined term 'scsr n', are essentially
  (1) II xy $\underline{u}'$ xy m $0 = m$,
  (2) II xy $\underline{u}'xy$ m scsr $n = \underline{u}'n$ II xy $\underline{u}'xy$ mn,
  (3) induction.

Introduction. For full-blown (natural) number theory we require the capability of making recursive definitions. We need to define exponentiation and more generally arbitrary finite summation and multiplication, to introduce the Euler φ function, to state and prove the unique prime power factorization theorem, to justify general commutativity and associativity, to deal with permutations, combinations, partitions, etc.

In order to explore number theory it is usual to presuppose enough set theory to construct ordered pairs, relations and functions and then develop the apparatus needed to make recursive definitions. A modest set theory will suffice, even one whose sets are constrained to be finite [3; Ch. 1, Sec. 6]. However, there are several ways [5, pp. 81–85; 3, Ch. 1, Sec. 6; 7; 2; 8] in which number theory can be axiomatized without presupposing a set-theoretic setting. In each of these the early development, culminating in a point where recursive definitions may be made, is somewhat tortuous. In the system presented here we formally axiomatize recursion directly so that an immediate assault on recursive problems is possible. This is done without either general functional concepts or any part of set theory, in such a way that all terms denote only numbers or meaninglessness. Although we do not have the full facility of the set-theoretic approach with its capacity for term denotation of finite sets and sequences as well as

numbers, we have initially more than in the more narrowly based number theories alluded to which have the same restriction of denotation to numbers. We shall carry a development of the theory far enough to give an idea of the extent of its flexibility within these confines.

Because our system is couched in a formalism allowing for the incorporation of new definitions as needed, it approximates a working mathematical discipline. Moreover, being formal, it is one which might perhaps lend itself to a program of automatic proof checking and theorem proving in number theory. It might do so, we feel, more readily than more narrowly based non-set-theoretic systems and with less involvement with extraneous questions than a set-theoretically based system.

0. **The Formal System.** We adopt the formal and inferential system of A. P. Morse as described in *A Theory of Sets* [6] and develop our number theory within this framework. We adopt neither the logic nor the set theory espoused there. Instead, we accept a slightly different set of axioms for logic, one which nevertheless leads to very nearly the same logical system and is quite standard. We of course abandon set theory altogether.

Without referring to [6], most of our theory should be intuitively decipherable to one moderately familiar with the idea of a formal system. A little later we shall interject some explanatory remarks designed to ease the reader's path over some of the technical features of this formal inferential system.

We assume that we have made appropriate orienting definitions in the spirit of [6] so that, upon introducing some additional terminology, our constants and non variable forms are categorized as follows:

CONSTANTS.
    PRIMITIVE.
        SCHEMATIC. '$\underline{u}$', '$\underline{u}\,''$', $\cdots$, '$\underline{v}$', '$\underline{v}\,''$', $\cdots$, $\cdots$.
        DEFINITIONAL. '$\equiv$'.
    ORIENTING.
        PARENTHETICAL. '(',')'.
        LOGICAL. '$\sim$', '$\to$', '$\wedge$'.
        EQUALITY-THEORETICAL. '=', 'the'.
        NUMBER-THEORETICAL. '0', 'II'.
DEFINED. Each symbol appearing in a definiendum (left side of a definition) and not in the corresponding definiens (right side) is a constant.

FORMS.
  PRIMITIVE.
    SCHEMATIC. '$\underline{u}x$', '$\underline{u}\,'xx\,''$', $\cdots$,'$\underline{v}x$', '$\underline{v}\,'xx\,''$',$\cdots$, $\cdots$.
    DEFINITIONAL. '$(x \equiv y)$'.
  ORIENTING.
    LOGICAL. '$\sim p$', '$(p \to q)$', '$\bigwedge x\,\underline{u}x$'.
    EQUALITY-THEORETICAL. '$(x = y)$', 'the $x\,\underline{u}x$'.
    NUMBER-THEORETICAL. '0', '$\text{II}\,xy\,\underline{u}\,'xy\,m\,n$'.
  DEFINED. Each definiendum is a form.
  STATEMENTAL. The primitive statemental form is '$(x \equiv y)$'. The orienting statemental forms are precisely the orienting logical forms and the orienting equality-theoretical form '$(x = y)$'. A definiendum arising from a definition in which the definiens is statemental is statemental.
  NOMINAL. The orienting nominal forms are precisely the orienting equality-theoretical form 'the $x\,\underline{u}x$' and the orienting number-theoretical forms. A definiendum arising from a definition in which the definiens is nominal is nominal.

The replacing formulas for the free variables of the orienting logical forms and for the schematic expressions of the third orienting logical form and the second orienting equality-theoretical form are normally statemental.

The replacing formulas for the free variables of the schematic, orienting equality-theoretical and orienting number-theoretical forms and for the schematic expression of the second orienting number-theoretical form are normally nominal. (Nominal formulas are traditionally called terms.)

Less technically the orienting constants and terms may here be thought of as being primitive.

Of course, all formulas are constructed iteratively from the forms by free variable and schematic replacement in accordance with rules set forth in [6]. Appropriate modifications could be introduced into the formalism to mechanically force a distinction between statemental and nominal formulas, that is, to make precise the notion of normal replacement and allow only normal replacements for constructing formulas. This capability is not inherent in the formalism as it now stands. We shall not go to this extreme; instead we tolerate (but never actually use) such uninterpretable mixed formulas as

$$\text{'scsr}\,(x = y)\text{'}, \text{'(scsr}\,x \to \text{pcsr}\,y)\text{'}.$$

Given our initial clues and judiciously chosen definitions, it should be

intuitively clear in any given instance of a formula hereinafter actually exhibited which is neither a variable nor a schematic expression whether it is to be thought of as statemental or nominal and which variables and schematic expressions appearing in it are normally to be replaced by statemental formulas and which by nominal formulas.

Many features of [6] adhere to the present development. We comment on some of these in the next seven paragraphs.

A strictly linear formal notation is maintained throughout.

In our definitions and axioms we shall be parenthetically complete and use no other notational shortenings, although we shall adopt a more relaxed notation elsewhere, but one in keeping with many conventional constructions. For instance, outside of definitions and axioms we would feel free to write

$$'((x < 1 \ \wedge \ 1 < y) \rightarrow x < y \rightarrow 2 \cdot x < y + 1)',$$

instead of the parenthetically complete longer

$$'((((x < 1) \wedge (1 < y)) \rightarrow (x < y)) \wedge (((x < 1) \wedge (1 < y))$$
$$\rightarrow ((2 \cdot x) < (y + 1))))',$$

in accordance with the convention that

$$(p \rightarrow q \rightarrow r)$$

means

$$((p \rightarrow q) \wedge (p \rightarrow r)).$$

We have already taken care, and in future definitions we shall continue to take care, to have no form a proper initial segment of another form. Thus in reading formulas from left to right, when we have read a complete formula, we will be sure that we have read far enough and no other reading is possible. With this convention, many punctuation marks and parentheses can be avoided.

We allow formulas in which variables appear neither free nor bound as in

$$'(x = \text{the } x \ \underline{u}x)',$$

and interpret them by locally (in a subformula) changing bound variables. Thus

$$'((x = \text{the } x \ \underline{u}x) \rightarrow (x = \text{the } t \ \underline{u}t))'$$

is a theorem and in the formula immediately to the right of '$\rightarrow$' we check that 'x' is free and 't' is bound.

The rules of inference are: initiation (accepting as a theorem a definition or an axiom); detachment (modus ponens); free variable substitution (substituting for a free variable in a theorem a formula each of whose variables is free in the theorem); schematic substitution (substituting for a schematic expression in a theorem a formula each of whose variables is free in the theorem or occurs explicitly in the schematic expression); local change of bound variable; and universalization (logical closure).

We mention that in order to perform schematic substitution in a formula we require that all occurrences of schematic expressions initiated by the same schematic constant be identical. The actual mechanics of schematic substitution are thus simple, but certain circumlocutions must be gone through in some instances to attain schematic uniformity preparatory to performing schematic substitution. We shall make each definition involving a schematic expression schematically uniform, and thus immediately amenable to schematic substitution, by introducing a circumlocution if necessary. For instance, see Definition 3.0.8. Operations with these circumlocutions will be illustrated presently after adoption of axioms for equality.

In our term

$$\text{`II } xy \, \underline{u} \, {'}xy \, m \, n\text{'}$$

the variables with double occurrences, namely 'x', 'y', are bound and the variables with single occurrences, namely 'm', 'n' are free. This term is analogous to the term

$$\text{`} \int_{D(m,n)} \underline{u} \, {'}xy \, d(x, y)\text{'}$$

in calculus ($D(m, n)$ is the unit disk in E_2 centered at (m, n)). In either one we may substitute 's' for 'y' with impunity to make an inferential step, but it is hazardous to substitute 'x' for 'y'. One can actually get away with substituting 'm' for 'y' in either term but we shall herein avoid this kind of substitution. Also in either term we may safely replace the schematic expression by any formula each of whose free variables is among 'x', 'y', or is free in the inferential formula undergoing substitution.

For the moment, of course, we are not concerned with the meanings of our formulas, only that they be capable of conveying meaning. Their meanings will be determined by the axioms presently to be adopted.

We shall indicate with an asterisk after its number each definition which enters either directly or indirectly into any subsequent axiom.

1. *Logic.*

1.0 DEFINITIONS.

.0* $((p \land q) \equiv \sim (p \to \sim q))$

.1 $((p \lor q) \equiv \sim (\sim p \land \sim q))$

.2* $((p \leftrightarrow q) \equiv ((p \to q) \land (q \to p)))$

.3* $(\bigvee x\, \underline{u}x \equiv \sim \bigwedge x \sim \underline{u}x)$

1.1 AXIOMS OF DEFINITION FOR LOGIC.

.0 $((x \equiv y) \to (x \to y))$

.1 $((x \equiv y) \to (y \to x))$

1.2 AXIOMS FOR LOGIC.

.0 $((p \to q) \to ((q \to r) \to (p \to r)))$

.1 $(p \to (\sim p \to q))$

.2 $((\sim p \to p) \to p)$

.3 $(\bigwedge x\, \underline{u}x \to \underline{u}x)$

.4 $(y \to \bigwedge xy)$

.5 $(\bigwedge x(\underline{u}x \to \underline{v}x) \to (\bigwedge x\, \underline{u}x \to \bigwedge x\, \underline{v}x))$

With regard to Axioms 1.2.3 and 1.2.5, we may replace '$\underline{u}x$' by any formula whatsoever to obtain a theorem. However, if in Axiom 1.2.4 we were to replace 'y' by a formula, then we would not be guaranteed that we had got a theorem unless each variable in the 'y'-replacing formula was different from 'x'.

Parts of subsequent derivations resting on elementary logic will often be omitted. The reader is referred to [6, Ch. 1] and to the treatment of [1], whose syntax and perspective readily adapt to the present system, for an adequate logical supplement.

2. *Equality.*

2.0 DEFINITIONS.

.0* $((x \neq y) \equiv \sim (x = y))$

.1* $(\bigvee! x\, \underline{u}x \equiv \bigvee y \bigwedge x(\underline{u}x \leftrightarrow (x = y)))$

.2* $(\infty \equiv \text{the } x(x \neq x))$

.3 $(\delta\, a\, b\, p \equiv \text{the } z((p \land (z = a)) \lor (\sim p \land (z = b))))$

2.1 AXIOM OF DEFINITION FOR EQUALITY.

$((x \equiv y) \to (x = y))$

2.2 AXIOMS FOR EQUALITY.

.0 $(x = x)$

.1 $((x = y) \to (y = x))$

.2 $(((x = y) \land (y = z)) \to (x = z))$

.3 $((x = y) \to (\underline{u}x \to \underline{u}y))$

.4 $(\bigvee! x\, \underline{u}x \to \underline{u} \text{ the } x\, \underline{u}x)$

.5 $(\sim \bigvee! x\, \underline{u}x \to (\text{the } x\, \underline{u}x = \infty))$

We observe that

$\bigvee! x \, \underline{u}x$ if and only if there is precisely one x such that $\underline{u}x$,
the $x \, \underline{u}x$ is the unique x such that $\underline{u}x$.

We think of ∞ as not a number but as a mathematical object, mean-inglessness, a catch-all for meaningless designations, capable of being written about and denoted by a variable or schematic expression. Thus it turns out, not unexpectedly, that

$$\text{(the } x(x = 0 \wedge x \neq 0) = \infty \wedge$$
$$\text{the } x(0 < x \wedge x < 3) = \infty \wedge$$
$$\text{predecessor } 0 = \infty \wedge$$
$$(x \neq \infty \rightarrow \text{successor } x \neq \infty) \wedge$$
$$\text{successor } \infty = \infty).$$

To use descriptive definitions (those whose definiens is initiated by 'the') is a great convenience. But, when used only when meaningful, they are logically dispensible [5, pp. 405–420]. Once descriptive definitions are admitted, the expression 'the $x(x \neq x)$' becomes a term. We may use it if we like at no additional cost in formality. We have here chosen to do so and feel that it provides a small increment of freedom. It seems to us, when confronted with the question of decid-ing what predecessor zero should be, that it is less entangling to have it be a non-number than a number. A little later we shall comment on the extent to which adopting Axiom 2.2.5 and incorporating '∞' into the number theoretic Axioms, as we shall presently do, actually strengthens the system.

We have a kronecker 'δ' which assigns the value of a to $\delta \, a \, b \, p$ if p and the value b to $\delta \, a \, b \, p$ if $\sim p$.

Since schematic substitution is crucial in our system, we shall try to give the flavor of it by listing a few elementary theorems on equality.

2.3 THEOREMS.
  .0 $(x = y \rightarrow (\underline{u}x \leftrightarrow \underline{u}y))$
  .1 $(\underline{u}y \leftrightarrow \bigvee x(\underline{u}x \wedge x = y))$

PROOF. We use 2.2.0, 2.2.3 and 2.2.1.

$(\underline{u}y \leftrightarrow \underline{u}y \wedge y = y$
$\leftrightarrow \underline{u}y \wedge \bigvee x(y = x)$
$\leftrightarrow \bigvee x(\underline{u}y \wedge y = x)$
$\leftrightarrow \bigvee x(\underline{u}x \wedge y = x)$
$\leftrightarrow \bigvee x(\underline{u}x \wedge x = y))$

.2 $(\underline{u}\,'yy' \leftrightarrow \bigvee x \bigvee x'(\underline{u}\,'xx' \wedge x = y \wedge x' = y'))$
.3 $(x = y \rightarrow (\underline{v}x \leftrightarrow \underline{v}y))$

PROOF. We use 2.3.0, 2.3.1, schematic substitution, local change of bound variable, a succession of free-variable substitutions, 2.2.1 and 2.2.2.

$(x = y \rightarrow (\underline{u}x \leftrightarrow \underline{u}y \leftrightarrow \bigvee x(\underline{u}x \wedge x = y)))$
$(x = y \rightarrow (\underline{v}x \leftrightarrow \bigvee x(\underline{v}x \wedge x = y)))$
$(x = y \rightarrow (\underline{v}x \leftrightarrow \bigvee t(\underline{v}t \wedge t = y)))$
$(y = x \rightarrow (\underline{v}y \leftrightarrow \bigvee t(\underline{v}t \wedge t = x)))$
$(x = y \rightarrow (\underline{v}y \leftrightarrow \bigvee t(\underline{v}t \wedge t = x)))$
$(x = y \rightarrow (\bigvee t(\underline{v}t \wedge t = x) \leftrightarrow \bigvee t(\underline{v}t \wedge t = y)))$
$(x = y \rightarrow (\underline{v}x \leftrightarrow \underline{v}y))$

.4 $(\underline{v}y \leftrightarrow \bigvee x(\underline{v}x \wedge x = y))$

PROOF. Model a proof after that of 2.3.1 using 2.3.3 instead of 2.2.3.

.5 (the $x(x = a) = a$)
.6 $(\bigvee! x\,\underline{v}x \rightarrow \underline{v}$ the $x\,\underline{v}x)$

PROOF. We use 2.2.4, local change of bound variable, 2.3.1, schematic substitution and 2.3.4.

$(\bigvee! x\,\underline{u}x \rightarrow \underline{u}$ the $x\,\underline{u}x$
$\rightarrow \underline{u}$ the $t\,\underline{u}t$
$\rightarrow \bigvee x(\underline{u}x \wedge x = $ the $t\,\underline{u}t)$
$\rightarrow \bigvee x(\underline{u}x \wedge x = $ the $x\,\underline{u}x))$
$(\bigvee! x\,\underline{v}x \rightarrow \bigvee x(\underline{v}x \wedge x = $ the $x\,\underline{v}x)$
$\rightarrow \bigvee x(\underline{v}x \wedge x = $ the $t\,\underline{v}t)$
$\rightarrow \underline{v}$ the $t\,\underline{v}t$
$\rightarrow \underline{v}$ the $x\,\underline{v}x)$

.7 $((\underline{u}x \rightarrow \underline{v}x) \rightarrow $ the $z\,\underline{u}z = $ the $z\,\underline{v}z)$
.8 $(\underline{u}y = $ the $z \bigvee x(z = \underline{u}x \wedge x = y))$
.9 $(\underline{u}\,'yy' = $ the $z \bigvee x \bigvee x'(z = \underline{u}\,'xx' \wedge x = y \wedge x' = y'))$
.10 $(x = y \rightarrow \underline{v}x = \underline{v}y)$
.11 $((p \rightarrow \delta\,a\,b\,p = a) \wedge (\sim p \rightarrow \delta\,a\,b\,p = b))$

We note that through 2.3.1 and 2.3.9, Axioms 2.2.3 and 2.2.4 above and 3.1.3 and 3.1.4 below become amenable to schematic substitution. We shall henceforth accept all such consequences of equality and in proofs omit the uniformizing circumlocutions preparatory to substituting schematically and the simplifications after, going straight to the end result of the schematic substitution.

3. *Number Theory.*

3.0 DEFINITIONS.

.0* $(\text{pcsr } n \equiv \text{II } xy\, x \infty\, n)$

.1* $(\text{scsr } n \equiv \text{the } x(\text{pcsr } x = n))$

.2 $(1 \equiv \text{scsr } 0)$

.3 $(\text{I } y \underline{u}y \, m \, n \equiv \text{II } xy \, \underline{u}y \, m \, n)$

.4 $((a + b) \equiv \text{I } y \text{ scsr } y \, a \, b)$

.5 $(\Sigma \, x < n \, \underline{u}x \equiv \text{II } xy(y + \underline{u}x) \, 0 \, n)$

.6 $((a \cdot b) \equiv \Sigma \, x < a \, b)$

.7 $((a < b) \equiv (((a \neq \infty) \wedge (b \neq \infty)) \wedge \vee c((c \neq 0) \wedge$
$$((a + c = b))))$$

.8 $(\text{least } x \, \underline{u}x \equiv \text{the } t(\vee \, x(\underline{u}x \wedge (x = t)) \wedge \wedge x((x < t) \to \sim \underline{u}x)))$

.9 $(\text{rm } c \, a \equiv \text{least } r \vee b(c = ((b \cdot a) + r)))$

.10 $(2 \equiv \text{scsr } 1)$

.11 $((a \leqq b) \equiv ((a < b) \vee (a = b)))$

.12 $(\Sigma \, x \leqq n \, \underline{u}x \equiv \Sigma \, x < \text{scsr } n \, \underline{u}x)$

.13 $(\text{greatest } x \, \underline{u}x \equiv \text{the } t(\vee \, x(\underline{u}x \wedge (x = t)) \wedge$
$$\wedge x((x < t) \to \sim \underline{u}x))$$

.14 $(\Pi \, x < n \, \underline{u}x \equiv \text{II } xy(y \cdot \underline{u}x) \, 1 \, n)$

.15 $(\Pi \, x \leqq n \, \underline{u}x \equiv \Pi \, x < \text{scsr } n \, \underline{u}x)$

.16 $(\exp a \, b \equiv \Pi \, x < b \, a)$

.17 $((a \mid b) \equiv (((a \neq \infty) \wedge (b \neq \infty)) \wedge \vee c((a \cdot c) = b)))$

.18 $((a \!\not\mid\! b) \equiv \sim (a \mid b))$

.19 $(\text{prime is } p \equiv (((p \neq 1) \wedge (p \neq \infty)) \wedge \wedge a((a \mid p) \to$
$$((a = 1) \wedge (a = p)))))$$

.20 $(\text{P } n \equiv \text{I } y \text{ least } t((y < t) \wedge (\text{prime is } t)) \, 2 \, n)$

.21 $(\text{M } a \equiv \text{greatest } t(\exp \text{P } t \mid a))$

.22 $(\text{N } a \, n \equiv \text{greatest } t(\exp \text{P } n \, t \mid a))$

.23 $(\vartheta y \, \underline{u}y \, m \, n \equiv \text{pcsr pcsr N II } xy(y \cdot \exp \text{P scsr } x \, \delta \text{ scsr scsr } \underline{u}y$
$$1 \, (\underline{u}y \neq \infty)) \exp 2 \, \delta \text{ scsr scsr } m \, 1(m \neq \infty) n \, n)$$

3.1 AXIOMS FOR NUMBER THEORY.

.0 $\vee \, m(m \neq \infty)$

.1 $(\text{II } xy \, \underline{u} \, 'xy \, m \, 0 = m)$

.2 $(\text{II}xy \, \underline{u} \, 'xy \, m \, \infty = \infty)$

.3 $((n \neq \infty) \to (\text{II } xy \, \underline{u} \, 'xy \, m \text{ scsr } n = \underline{u} \, 'n \text{II } xy \, \underline{u} \, 'xy \, m \, n))$

.4 $((\underline{u}0 \wedge \wedge x(((x \neq \infty) \wedge \underline{u}x) \to \underline{u} \text{ scsr } x)) \to$
$$\wedge x((x \neq \infty) \to \underline{u}x))$$

Axiom 3.1.0 is our axiom of existence. There is a meaningful number.

Although

$$\text{'II } xy \, \underline{u} \, 'xy \, m\text{'}$$

is not a formula in our system and hence does not denote anything among our field of mathematical objects, we, so to speak, think of

$$\text{II } xy \underline{u}\,'xy\, m$$

as a function f determined by m and the function expressed by

$$\text{'}\underline{u}\,'xy\text{'}$$

whose value at n is

$$\text{II } xy \underline{u}\,'xy\, m\, n,$$

at 0 is m and at ∞ is ∞.

If we are at $n > 0$ and have previously defined f at 0 to be m and for each t with $0 < t \leqq n$ by means of the expression '$\underline{u}\,'xy$' (perhaps sometimes even defining it to be ∞), then we advance one step and define f at scsr n to be

$$\underline{u}\,'nf(n).$$

We can schematically diagram our would-be function as follows:

$$m = f(0)\,f(1) \qquad y \quad \underline{u}\,'xy \quad f(n) \quad f(\text{scsr } n) = \underline{u}\,'nf(n)$$

$$0 \qquad 1 \qquad\qquad x \quad \text{scsr } x \qquad n \quad \text{scsr } n$$

Here we think of y and $\underline{u}\,'xy$ as being respectively equal to $f(x)$ and $f(\text{scsr } x)$. This recursive process using the inductor 'II' is our principle of definition by recursion and is expressed by Axioms 3.1.1, 3.1.2 and 3.1.3.

The inductor 'I' is weaker than 'II' being incapable of defining f at scsr x if the definition depends upon the input of information in addition to the value of f at x.

If our definition of f at scsr x is to depend upon, in addition to possibly new inputs of information, the information contained in the finite sequence of all previously defined values of f instead of just the one value of f at x then we can take advantage of Gödel numbers to accumulate this suitably packaged information and accomplish the definition by means of the inductor '$\mathscr{I}$'. Once we have got sufficiently far under way by means of 'II', '$\mathscr{I}$' is just as effective as 'II' and expresses removal of a built-in redundancy in 'II' seemingly needed to get things started. It seems unlikely that one could succeed by starting with only '0' and a recursion form with only one bound variable and a one-place schematic expression no matter what axioms phrased in these terms are adopted.

Recursion theorems for 'I' analogous to 3.1.1, .2, .3 for 'II' are obvious.

Analogues for ' $\mathcal{J}$ ' are postponed to 3.25.

Axiom 3.1.4 is ordinary induction.

3.2 THEOREMS.

.0 $(\wedge x \wedge y(x \neq \infty \rightarrow \underline{u}'xy = \underline{v}xy) \rightarrow \text{II } xy \ \underline{u}'xy \ m \ n =$
$$\text{II } xy \ \underline{v}'xy \ m \ n)$$

.1 $(\wedge y(\underline{u}y = \underline{v}y) \rightarrow \text{I } y \ \underline{u}y \ m \ n = \text{I } y \ \underline{v}y \ m \ n)$

.2 $(\wedge y(\underline{u}y = \underline{v}y) \rightarrow \mathcal{J} \ y \ \underline{u}y \ m \ n = \mathcal{J} \ y \ \underline{v}y \ m \ n)$

PROOF OF .0. Clearly by Axioms 3.1.1 and 3.1.2

$$(n = 0 \vee n = \infty \rightarrow \text{II } xy \ \underline{u}'xy \ m \ n = \text{II } xy \ \underline{v}'xy \ m \ n).$$

Suppose

$$(n \neq \infty \wedge \text{II } xy \ \underline{u}'xy \ m \ n = \text{II } xy \ \underline{v}'xy \ m \ n).$$

Then by Axiom 3.1.3

$$(\text{II } xy \ \underline{u}'xy \ m \ \text{scsr } n = \underline{u}'n \ \text{II } xy \ \underline{u}'xy \ m \ n$$
$$= \underline{v}'n \ \text{II } xy \ \underline{v}' \ xy \ m \ n$$
$$= \text{I } xy \ \underline{v}'xy \ m \ \text{scsr } n).$$

So by Axiom 3.1.4, we are done.

3.3 THEOREMS.

.0 $(0 \neq \infty)$

.1 $(\text{pcsr } 0 = \infty)$

.2 $(\text{pcsr } \infty = \infty)$

.3 $(\text{scsr } \infty = \infty)$

PROOFS. For .0 we use equality and the following consequence of Axioms 3.10, .1, .2.

$$\vee m(\text{II } xy\underline{u}'xy \ m \ 0 = m \neq \infty = \text{II } xy\underline{u}'xy \ m \ \infty).$$

By Axioms 3.1.1, .2 we get immediately .1 and .2. As

$$(\vee! x(\text{pcsr } x = \infty) \vee \sim \vee! x(\text{pcsr } x = \infty)),$$

.3 follows from .2 since, by Axioms 2.2.4, .5, in either event

$$(\text{scsr } \infty = \text{the } x(\text{pcsr } x = \infty) = \infty).$$

3.4 THEOREM. $(b \neq 0 \leftrightarrow \vee a(b = \text{scsr } a))$

PROOF OF '→'. The conclusion follows from 3.3.0 and the following consequence of Axiom 3.1.4

$$(b \neq \infty \rightarrow (b \neq 0 \rightarrow \vee a(b = \text{scsr } a))).$$

Proof of '←'. Suppose contrary to the conclusion that

$$\vee\ a(0 = \text{scsr}\ a).$$

Then by 3.3.1, 3.3.0, Axioms 2.2.5, .4 and 3.3.3 we get, contrary to 3.3.0,

$$\vee\ a(0 = \text{scsr}\ a = \text{the}\ x(\text{pcsr}\ x = a)\ \wedge\ \infty$$
$$= \text{pcsr}\ 0 = a\ \wedge\ 0 = \text{scsr}\ \infty = \infty\ ).$$

3.5 Theorem. $(\text{pcsr scsr}\ a = a)$

Proof. We use Axiom 3.1.3.

$$(a = \infty\ \rightarrow a = \text{pcsr}\ \infty = \text{pcsr scsr}\ \infty = \text{pcsr scsr}\ a)$$
$$(a \neq \infty\ \rightarrow \text{pcsr scsr}\ a = \text{II}\ xy\ x\ \infty\ \text{scsr}\ a = a)$$

3.6 Theorem. $(\text{scsr}\ a = \infty\ \rightarrow a = \infty\ )$

Proof. $(\text{scsr}\ a = \infty\ \rightarrow \infty = \text{pcsr}\ \infty = \text{pcsr scsr}\ a = a)$

3.7 Theorem. $(\text{pcsr}\ a = \infty\ \rightarrow a = 0\ \vee\ a = \infty\ ).$

Proof. We use the '→' part of 3.4.

$$(\text{pcsr}\ a = \infty\ \wedge\ a \neq 0 \wedge a \neq \infty\ \rightarrow$$
$$\vee\ c(\infty \neq c = \text{pcsr scsr}\ c = \text{pcsr}\ a = \infty\ )$$

3.8 Theorem. $(a \neq 0 \rightarrow \text{scsr pcsr}\ a = a)$

Proof. $\vee\ c(a = \text{scsr}\ c = \text{scsr pcsr scsr}\ c = \text{scsr pcsr}\ a)$

3.9 Theorems.
.0 $(a + 0 = a)$
.1 $(a + \infty = \infty\ )$
.2 $(0 + a = a)$
.3 $(\infty + a = \infty\ )$
.4 $(a + 1 = \text{scsr}\ a)$
.5 $((a + b) + 1 = a + (b + 1))$

3.10 Theorems.
.0 $(0 \cdot a = 0)$
.1 $(\infty \cdot a = \infty\ )$
.2 $(a \neq \infty\ \rightarrow a \cdot 0 = 0)$
.3 $(a \neq 0 \rightarrow a \cdot \infty = \infty\ )$
.4 $(1 \cdot a = a)$
.5 $(a \cdot 1 = a)$
.6 $(a + 1) \cdot b = (a \cdot b) + b)$

The proofs of 3.9.2, .3 and 3.10.2, .3, .5 are by induction on a. The others are immediate from definitions and recursion axioms. We give three proofs.

PROOF OF 3.9.2. By 3.9.0 we have

$$(0 + 0) = 0.$$

Now suppose

$$(a \neq \infty \wedge 0 + a = a).$$

Then by Axiom 3.1.3

$$
\begin{aligned}
(0 + \text{scsr } a &= \text{I}y \text{ scsr } y \, 0 \text{ scsr } a \\
&= \text{scsr I } y \text{ scsr } y \, 0 \, a \\
&= \text{scsr } (0 + a) \\
&= \text{scsr } a).
\end{aligned}
$$

By induction and 3.9.1 we arrive at the desired result.

PROOF OF 3.9.4. Since the result is immediate from 3.3.3 and 3.9.3 if $(a = \infty)$, suppose otherwise.

$$
\begin{aligned}
(a + 1 &= \text{I}y \text{ scsr } y \, a \, 1 \\
&= \text{I } y \text{ scsr } y \, a \text{ scsr } 0 \\
&= \text{scsr I } y \text{ scsr } y \, a \, 0 \\
&= \text{scsr } a)
\end{aligned}
$$

PROOF OF 3.10.6. If $(a = \infty)$, then by 3.9.3 and 3.10.1 the statement is true. Suppose $(a \neq \infty)$.

$$
\begin{aligned}
((a + 1) \cdot b &= \Sigma \, x < (a + 1) \, b \\
&= \Sigma \, x < \text{scsr } a \, b \\
&= \text{II } xy \, (y + b) \, 0 \text{ scsr } a \\
&= \text{II } xy \, (y + b) \, 0 \, a + b \\
&= \Sigma \, x < a \, b + b \\
&= (a \cdot b) + b)
\end{aligned}
$$

Let us digress for the moment to consider two alternate number systems together with a somewhat curtailed version of our present system based on the definitions we have used so far, except for 3.0.23 in 3.2.2, namely, 3.0.0–3.0.6, and on the next three definitions, 3.0.7, 3.0.8 and 3.0.9.

The first alternate system uses the same formal-inferential, logical and equality-theoretic foundation as our present system but adopts as orienting number-theoretic forms

$$\text{'0', 'pcsr } x\text{', '}(x + y)\text{', '}(x \cdot y)\text{';}$$

adopts as number-theoretic definitions 3.0.1, .2, .7, .8, .9, the formula obtained from the expression

$$\begin{aligned}
\text{'(II } xy \; \underline{u} \, 'xy \; m \; n \equiv \text{ the } z \; &\vee \; d \; \vee \; c(0 < d \wedge d \neq \infty \wedge 0 < c \wedge c \neq \infty \\
&\wedge \; \delta \;\; \text{scsr} \;\; m0 \;\; (m \neq \infty) = \text{rm} \;\; c(1 + d) \wedge z = \delta \\
&\text{pcsr} \quad \text{rm} \quad c(1 + ((1 + n) \cdot d))\infty\,)n \neq \infty) \wedge \wedge \\
&x(x < n \to \delta \; \text{scsr} \, \underline{u} \, 'x \, \text{pcsr} \; \text{rm} \; c(1 + ((1 + x) \cdot d)) \\
&0 \quad (\underline{u}\,'x \quad \text{pcsr} \quad \text{rm} \quad c(1 + ((1 + x) \cdot d)) \neq \infty) \\
&= \text{rm} \; c(1 + ((1 + x)) \cdot d)))))\text{'}
\end{aligned}$$

by appropriately inserting parentheses and establishing schematic uniformity, 3.0.3 and 3.0.5; and finally adopts as number-theoretic axioms 3.3–3.10 and 3.1.4. As indicated in [5, pp. 238–241; 3, Ch. 1, Sec. 7] one can eventually prove (using the Chinese Remainder Theorem) entirely within this first alternate system that 'II $xy \; \underline{u} \, 'xy \; m \; n$' possesses the properties listed in 3.1.1, .2, .3. Whence it is easy to derive by induction as first alternate theorems the formulas obtained from 3.0.0, .4, .6 by replacing '$\equiv$' by '$=$'. It now follows that the normal theorems devoid of '$\equiv$' in the curtailed system and in the first alternate system are exactly the same. Thus, simply axiomatized, recursive '$+$' and '$\cdot$' are enough to establish general recursion.

The second alternate system (arithmetic) uses the same formal-inferential and logical foundation as our present system but adopts as an orienting equality-theoretic form '$(x = y)$'; equality-theoretic axioms 2.2.0, .1, .2, .3; orienting number-theoretic forms

$$\text{'0', '1', '}(x + y)\text{', '}(x \cdot y)\text{';}$$

and number-theoretic axioms 3.9.0, .5, 3.10.4, .6,

$$\text{'}(((x + 1) = (y + 1)) \to (x = y))\text{', '}{\sim}((x + 1) = 0)\text{',}$$
$$\text{'}((\underline{u} \, 0 \;\; \wedge\wedge x(\underline{u}x \to \underline{u}(x + 1))) \to \wedge \; x \; \underline{u}x)\text{'.}$$

Clearly each formula in arithmetic is a formula in the present system. Furthermore, it is not hard to see that each statement which is a theorem in arithmetic transforms, when all quantifications are restricted to (non-∞) numbers, into a theorem in the present system. Concerning the question raised earlier as to the extent to which incorporating '∞' into our system as we have done strengthens it, we

can make this more precise by asking whether a statement of arithmetic which transforms, upon restricting quantifications to numbers, into a theorem of our present system is also a theorem of arithmetic. This seems indeed to be the case. The referee has indicated to the author that the usual model of arithmetic within finite set theory may be extended to a model of ∞-extended arithmetic (a modification of the first alternate system adopting '1' and '∞' as orienting forms instead of 'pcsr x' and abandoning descriptive definitions) by associating with '∞' a specific effective finite set (say $\{\{\varnothing\}\}$) which is not a finite ordinal, and hence to a model within arithmetic, in such a way that it follows that each arithmetical statement whose quantification-restricted transform is provable in the present system is provable arithmetically. Thus the incorporation of '∞' strengthens the system only peripherally in the sense that any additional theorems relate to ∞. In other words the present system is a conservative extension of arithmetic.

Continuing now with a development of our present system, we shall pass over the derivation of further elementary results, referring the reader to [4], and call attention here to only two of these, the well-ordering

3.11 THEOREMS.

.0 $(x \neq \infty \wedge \underline{u}x \rightarrow \text{least } t \underline{u}t \neq \infty)$

.1 $(x \neq \infty \wedge \underline{u}x \wedge \vee y \wedge t(y < t \rightarrow \sim\underline{u}t) \rightarrow \text{greatest } t \underline{u}t \neq \infty)$

Useful recursive properties of '$\leqq$', 'Σ' and 'Π', easily proved by induction, are

3.12 THEOREMS.

.0 $(a \leqq b \wedge \wedge x(a \leqq x \rightarrow \underline{u}'xc = c) \rightarrow$
   $(\Pi xy \underline{u}'xy m a = c \rightarrow \Pi xy \underline{u}'xy m b = c))$

.1 $(\Sigma i < (a + b) \underline{u}i = \Sigma i < a \underline{u}i + \Sigma i < b \underline{u}(a + i))$

.2 $(b \neq \infty \rightarrow \Pi i < (a + b) \underline{u}i = \Pi i < a \underline{u}i. \Pi i < b \underline{u}(a + i))$

The early development of the usual laws of exponents is similar to that for addition and multiplication given in 3.9 and 3.10. We shall skip this, mentioning only the exponential properties of ∞.

3.13 THEOREMS.

.0 $(\exp a \infty = \infty)$

.1 $(\exp \infty 0 = 1)$

.2 $(a \neq 0 \rightarrow \exp \infty a = \infty)$

Induction on a is used to prove .2.

Although the infinitude of primes can be stated by saying that for each number there is a next greater prime and easily proved after Euclid, we are able with 'P' to say a trifle more, namely, that 'P' con-

secutively enumerates infinitely many and all primes. We formally
state this, omitting the easy proof based upon the infinitude of primes.
We additionally give a useful immediate corollary based on the ele-
mentary fact that each number greater than 1 has a greatest prime
divisor.

3.14   THEOREMS.
.0 $(n \neq \infty \to$ prime is $P\,n \,\wedge n < P\,n < P$ scsr $n =$
$$\text{least } t(P\,n < t \wedge \text{prime is } t))$$
.1   (prime is $q \to \vee! \, n(P\,n = q))$
.2 $(1 < a \to M\,a \neq \infty \,\wedge\, PM\,a \mid a \,\wedge\, \wedge j(M\,a < j \to Pj \!\not| \, a))$

The three following lemmas are based on the elementary fact that a
prime dividing the product of two numbers divides one of them and on
the elementary cancellation law of mulitplication and are fairly easy
to prove by induction on n.

3.15   THEOREMS.
.0 $(0 < k \wedge$ prime is $p \wedge$ prime is $q \to$
$$(\exp pk \mid \exp q\,n \to p = q \,\wedge k \leqq n))$$
.1 $(n \neq \infty \,\wedge\, \wedge i \wedge j(i \leqq$ scsr $n \wedge j \leqq$ scsr $n \wedge i \neq j \to$ prime
is $\underline{u}i \wedge \underline{u}i \neq \underline{u}j \wedge 0 < \underline{v}i) \to \wedge j \wedge r(j \leqq n \wedge \underline{v}j < r \to$
$\exp \underline{u}j\,\underline{v}j \mid \Pi \, i \leqq n \exp \underline{u}i\,\underline{v}i \wedge \exp \underline{u}jr\,r \!\not| \, \Pi \, i \leqq n \exp \underline{u}i\,\underline{v}i)$
$\exp \underline{u}$ scsr $n\,\underline{v}$ scsr $n \!\not| \, \Pi \, i \leqq n \exp \underline{u}i \,\underline{v}i)$
.2   (prime is $p \wedge \exp p\,n \mid a \wedge c \mid a \wedge p \!\not| \, c \to (c \cdot \exp p\,n) \mid a)$

Ordered retrieval of information stored in the Gödel number of a
finite sequence of numbers and in the Gödel number of a finite se-
quence of numbers and ∞'s is given by the next three theorems. With
regard to the third, one might say that each $\underline{v}i$ which is not ∞ is stored
at the i-th numbered address in the Gödel number packaged as
scsr scsr $\underline{v}i$ and each $\underline{v}i$ which is ∞ is stored at the i-th numbered
address in the Gödel number packaged as 1. The stored information
is retrieved by means of 'pcsr' and 'N' and the end is signaled by 'M'.

3.16   THEOREMS
.0 $(j \leqq n \neq \infty \,\wedge\, \wedge i(i \leqq n \to \underline{u}i \neq \infty) \to$
$$N\Pi \, i \leqq \exp P\,i\,\underline{u}i\,j = \underline{u}j)$$
.1 $(n \neq \infty \,\wedge\, \wedge i(i \leqq n \to \underline{u}i \neq \infty) \wedge \underline{u}n \neq 0 \to$
$$M\,\Pi\, i \leqq n \exp P\,i\,\underline{u}i = n)$$
.2 $(j \leqq n \neq \infty \to$
pcsr pcsr $N\,\Pi\, i \leqq n \exp P\,i\,\delta$ scsr scsr $\underline{u}i\, 1(\underline{u}i \neq \infty)j = \underline{u}j$
$\wedge M\,\Pi\, i \leqq n \exp P\,i\,\delta$ scsr scsr $\underline{u}i\, 1(\underline{u}i \neq \infty) = n)$

The proofs are immediate from 3.15.1 and 3.14.

The utility, if not necessity, of Gödel number representation of finite sequences is demonstrated by the proof of the

3.17 THEOREM. (Pigeon Hole Principle).

$$(h < k \,\wedge\, \bigwedge i (i \leqq k \to \underline{u}i \leqq h) \to \bigvee i \bigvee j (i \leqq k \,\wedge\, j \leqq k \,\wedge$$
$$\wedge\, i \neq j \,\wedge\, \underline{u}i = \underline{u}j))$$

PROOF. Suppose contrarily that

$$(\mathrm{h} < k \,\wedge\, \bigwedge i (i \leqq k \to \underline{u}i \leqq h) \,\wedge\, \bigwedge i \bigwedge j (i \leqq k \,\wedge\, j \leqq k \,\wedge$$
$$i \neq j \to \underline{u}i \neq \underline{u}j)).$$

Let

$$(a = \Pi\, i \leqq k \exp \mathrm{P}\, i \operatorname{scsr} \underline{u}i \,\wedge$$
$$n = \operatorname{least} t \bigvee b \bigvee s (\mathrm{M}b = t \,\wedge\, s < t \,\wedge\, \bigwedge i (i \leqq t \to \operatorname{pcsr} \mathrm{N} bi \leqq s) \,\wedge$$
$$\bigwedge i \bigwedge j (i \leqq t \,\wedge\, j \leqq t \,\wedge\, i \neq j \to \operatorname{pcsr} \mathrm{N}\, b\, i \neq \operatorname{pcsr} \mathrm{N}\, b\, j))).$$

By 3.16.0, .1

$$(\mathrm{M}\, a = k \,\wedge\, h < k \,\wedge\, \bigwedge i (i \leqq k \to \operatorname{pcsr} \mathrm{N}\, a\, i = \underline{u}i \leqq h) \,\wedge$$
$$\bigwedge i \bigwedge j (i \leqq k \,\wedge\, j \leqq k \,\wedge\, i \neq j \to \operatorname{pcsr} \mathrm{N}\, a\, i = \underline{u}i \neq \underline{u}j = \operatorname{pcsr} \mathrm{N}\, a\, j)),$$

so

$$(n \neq \infty \,\wedge\, n \neq 0).$$

Let b and s be such that

$$(\mathrm{M}\, b = n \,\wedge\, s < n \,\wedge\, \bigwedge i (i \leqq n \to \operatorname{pcsr} \mathrm{N}\, b\, i \leqq s) \,\wedge$$
$$\bigwedge i \bigwedge j (i \leqq n \,\wedge\, j \leqq n \,\wedge\, i \neq j \to \operatorname{pcsr} \mathrm{N}\, b\, i \neq \operatorname{pcsr} \mathrm{N}\, b\, j)).$$

Clearly, since $(1 \leqq n)$,

$$(s = 0 \to \operatorname{pcsr} \mathrm{N}\, b\, 0 = 0 = \operatorname{pcsr} \mathrm{N}\, b\, 1 \neq \operatorname{pcsr} \mathrm{N}\, b\, 0).$$

So

$$(s \neq 0).$$

Now let

$$(\bigwedge i (\underline{v}'' \, bsi = \delta \operatorname{pcsr} \mathrm{N}\, b\, i \operatorname{pcsr} \mathrm{N}\, b\, n(pcrs\, \mathrm{N}\, b\, i \neq s)) \,\wedge$$
$$\underline{v}'bs = \Pi\, i \leqq \operatorname{pcsr} n \exp \mathrm{P}\, i \operatorname{scsr} \underline{v}'' bsi)$$

and check, contrary to the minimal nature of n, that

$$(M\underline{v}'bs = \text{pcsr } n \wedge \text{pcsr } s < \text{pcsr } n \wedge$$

$$\bigwedge i(i \leqq \text{pcsr } n \rightarrow \text{pcsr } N \underline{v}'bsi = \underline{v}''bsi$$
$$= \delta \text{ pcsr } N \, b \, i \text{ pcsr } N \, b \, n(\text{pcsr } N \, b \, i \neq s)$$
$$\leqq \text{pcsr } s) \wedge$$
$$\bigwedge i \wedge j(i \leqq \text{pcsr } n \wedge j \leqq \text{pcsr } n \wedge i \neq j \rightarrow$$
$$\text{pcsr } N \underline{v}'bsi = \underline{v}''bsi = \delta \text{ pcsr } N \, b \, i \text{ pcsr } N \, b \, n(\text{pcsr } N \, b \, i \neq s)$$
$$\neq \delta \text{ pscr } N \, b \, j \text{ pcsr } N \, b \, n(\text{pcsr } N \, b \, j \neq s)$$
$$= \underline{v}''bsj = \text{pcsr } N \underline{v}'bsj)).$$

We next state and prove in some detail the

3.18 THEOREM. (Unique Prime Power Factorization).
$$(a \neq 0 \wedge a \neq 1 \wedge a \neq \infty \rightarrow$$
.0 (Existence) $a = \Pi \, i \leqq M \, a \exp P \, i \, N \, a \, i \wedge$
.1 (Uniqueness) $(a = \Pi \, i \leqq M \, a \exp P \, i \, \underline{u}i \rightarrow$
$$\wedge \, i(i \leqq M \, a \rightarrow \underline{u}i = N \, a \, i)))$$

PROOF OF .0. Let

$$(q = \text{least } t(t \mid a \wedge t \neq 1) \wedge$$
$$d = \text{the } t \, (P \, t = q) \wedge$$
$$k = \text{greatest } t(\Pi \, i \leqq t \exp P \, i \, N \, a \, i \mid a \wedge N \, a \, t \neq 0)).$$

Check that

$$(\text{prime is } q \wedge q \mid a \wedge P \, d = q \wedge \bigwedge t(t < d \rightarrow P \, t \nmid a \rightarrow N \, a \, t = 0))$$

and hence by way of 3.12.0 and 3.12.2 that

$$(\Pi \, i \leqq d \exp P \, i \, N \, a \, i = \exp P \, d \, N \, a \, d \mid a \wedge N \, a \, d \neq 0 \wedge N \, a \, a = 0 \wedge$$

$$(0) \qquad k \neq \infty \wedge \Pi \, i \leqq k \exp P \, i \, N \, a \, i \mid a \wedge N \, a \, k \neq 0 \wedge$$

$$(1) \qquad \wedge \, j(k < j \rightarrow (\Pi \, i \leqq j \exp P \, i \, N \, a \, i \mid a \rightarrow N \, a \, j = 0))).$$

If

$$(\Pi \, i \leqq k \exp P \, i \, N \, a \, i \neq a),$$

then there is b such that

$$(2) \qquad (\Pi \, i \leqq k \exp P \, i \, N \, a \, i \cdot b = a \wedge b \neq 1).$$

Let

$$(\underline{v}b = \text{least } t(t \mid b \wedge t \neq 1) \wedge \underline{w}b = \text{the } t \, (P \, t = \underline{v}b))$$

Then

(3) $\quad$ (prime is $\underline{v}b \wedge \underline{v}b \mid b \wedge \mathrm{P}\,\underline{w}b = \underline{v}b \wedge \mathrm{N}\,a\,\underline{w}b \neq 0\ \wedge$
$$\wedge\ t(t < \underline{w}b \to \mathrm{P}\,t\!\not|\,b)).$$

Check from 3.15.1, (3) and (2) that

$(\underline{w}b \leqq k\ \to\ (\exp \mathrm{P}\,\underline{w}b\,\mathrm{N}\,a\,\underline{w}b) \mid \Pi\,i \leqq k\,\exp \mathrm{P}\,i\,\mathrm{N}\,ai\ \wedge\ \mathrm{P}\,\underline{w}b \mid b\ \wedge$

$\Pi\,i \leqq k\,\exp \mathrm{P}\,i\,\mathrm{N}\,ai \cdot b = a$
$\to (\exp \mathrm{P}\,\underline{w}b\,\mathrm{N}\,a\,\underline{w}b \cdot \mathrm{P}\,\underline{w}b) \mid a$
$\to \exp \mathrm{P}\,\underline{w}b\,(\mathrm{N}\,a\,\underline{w}b + 1) \mid a$
$\to \mathrm{N}\,a\,\underline{w}b + 1 < \mathrm{N}\,a\,\underline{w}b).$

Thus

$$(k < \underline{w}b).$$

Since by (0), 3.15.1 and (3)

$(\Pi\,i \leqq k\,\exp \mathrm{P}\,i\,\mathrm{N}\,a\,i \mid a \wedge \mathrm{P}\,\underline{w}b \not| \Pi\,i \leqq k\,\exp \mathrm{P}\,i\,\mathrm{N}\,a\,i\ \wedge$
$\exp \mathrm{P}\,\underline{w}b\,\mathrm{N}\,a\,\underline{w}b \not| \Pi\,i \leqq k\,\exp \mathrm{P}\,i\,\mathrm{N}\,a\,i \wedge \mathrm{P}\,\underline{w}b \mid a),$

we infer from 3.15.2, (3) and 3.15.5 that

$((\Pi\,i \leqq k\,\exp \mathrm{P}\,i\,\mathrm{N}\,a\,i \cdot \mathrm{P}\,\underline{w}b) \mid a \wedge \wedge\ t(k < t < \underline{w}b \to$
$\mathrm{P}\,t\!\not|\,a \to \mathrm{N}\,a\,t = 0 \to \exp \bar{\mathrm{P}}\,t\,\mathrm{N}\,a\,t = 1))$

and hence by 3.12.0, 3.12.2, the definition of $\mathrm{N}\,a\,\underline{w}b$ and again 3.15.2 that

$(\Pi\,i \leqq \underline{w}b\,\exp \mathrm{P}\,i\,\mathrm{N}\,a\,i = (\Pi\,i \leqq k\,\exp \mathrm{P}\,i\,\mathrm{N}\,a\,i$
$\cdot \exp \mathrm{P}\,\underline{w}b\,\mathrm{N}\,a\,\underline{w}b) \mid a).$

We are led by way of (1) and (3) to the contradiction

$$(\mathrm{N}\,a\,\underline{w}b = 0 \wedge \mathrm{N}\,a\,\underline{w}b \neq 0).$$

So

$$(\Pi\,i \leqq k\,\exp \mathrm{P}\,i\,\mathrm{N}\,a\,i = a).$$

Finally, by (0) and 3.16.1, we conclude that $(k = \mathrm{M}\,a)$.

PROOF OF .1 Check by way of 3.12.2, 3.13.0 and 3.10.0–.3 that

$$\wedge\ i(i \leqq \mathrm{M}\,a \to \underline{u}i \neq \infty)$$

and then use 3.16.0.

To state general commutativity for addition is easy.

3.19 THEOREM. $(\bigwedge x(x \leqq n \to \bigvee! \ y(y \leqq n \wedge \underline{v}y = x)) \to$
$$\Sigma x \leqq n \underline{u}x = \Sigma x \leqq n \underline{u} \underline{v}x)$$

To state general associativity for addition is somewhat harder. It can be done using the inductors 'I' and 'II' and Gödel numbers of finite sequences of numbers which determine the binary tree structures of acceptable distributions of parentheses and Gödel numbers of finite sequences of summands. As somewhat of a tour de force we give a formal version of general associativity for addition.

3.20 DEFINITIONS.

.0 $(K \, an \equiv$
$I y \Pi j < M \, y$
$\exp P \, j \, \delta \, N \, y \, j$
$\qquad \delta \, \text{pcsr} \, N \, y \, j \, N \, y \, \text{scsr} \, j (\, j = \text{least} \, t(N \, y \, t = N \, y \, \text{scsr} \, t))$
$\qquad (\, j < \text{least} \, t(N \, y \, t = N \, y \, \text{scsr} \, t)) a \, n)$

.1 $(S \, i \leqq n \, a \, \underline{u} i \equiv$
$N \, II \, xy \, \Pi j < M \, y$
$\exp P \, j \, \delta \, N \, yj$
$\qquad \delta \, (N \, yj + N \, y \, \text{scsr} \, j) \, N \, y \, \text{scsr} \, j$
$\qquad (\, j = \text{least} \, t(N \, K \, axt = N \, K \, ax \, \text{scsr} \, t)$
$\qquad (\, j < \text{least} \, t(N \, K \, axt = N \, K \, ax \, \text{scsr} \, t))$
$\qquad \Pi \, i \leqq n \, \exp P \, i \, \underline{u} i \, n \, 0)$

3.21 THEOREM. $(K \, a \, M \, a = 1 \to \Sigma \, i \leqq M \, a \, \underline{u} i = S \, i \leqq M \, a \, a \, \underline{u} i).$

The proofs of 3.19 and 3.21 are perhaps best handled simultaneously by induction on n with $M \, a = n$.

For example one can check that if

$$a = 2^2 \cdot 3^3 \cdot 5^5 \cdot 7^5 \cdot 11^4 \cdot 13^1,$$

then

$$(M \, a = 5 \wedge K \, a \, M \, a = 1 \wedge S \, i \leqq M \, a \, a \, \underline{u} i$$
$$= ((\underline{u}0 + (\underline{u}1 + ((\underline{u}2 + \underline{u}3) + \underline{u}4))) + \underline{u}5)).$$

The mechanism for reiterating an infinite sequence is built into the system. Thus the m-th order iterate of the sequence expressed by '$\underline{u}y$' evaluated at n could be defined by

$$(\text{Reit} \, y \, \underline{u}y \, m \, n \equiv I \, y \, \underline{u}y \, n \, m).$$

Arbitrary rereiteration, though more difficult, can be done by representing a sufficiently large initial segment of an infinite sequence by

the Gödel number of this segment.

3.22 DEFINITIONS.

.0 $(\text{reit}''\, a\, n\, k \equiv \Pi\, i \leqq k\, \exp P\, i$
$$\delta\, \text{scsr}\, I\, y\, \text{pcsr}\, N\, a\, y\, n\, i\, 0(I\, y\, \text{pcsr}\, N\, a\, y\, n\, i \neq \infty))$$

.1 $(\text{rereit}''\, x\, \underline{u}x\, k\, a\, j \equiv \Pi\, xy\, \text{reit}''\, y\, \underline{u}x\, k\, a\, j)$

.2 $(\text{rereit}'\, x\, \underline{u}x\, k\, \underline{v}x\, j \equiv \text{rereit}''\, x\, \underline{u}x\, k\, \Pi\, x \leqq k\, \exp P\, x\, \delta\, \text{scsr}\, \underline{v}x\, 0$
$$(\underline{v}x \neq \infty)j)$$

.3 $(\text{Rereit}\, x\, \underline{u}x\, \underline{v}x\, jm \equiv \text{pcsr}\, N\, \text{rereit}'\, x\, \underline{u}x\, \text{least}\, t$
$$(\text{pcsr}\, N\, \text{rereit}'\, x\, \underline{u}x\, t\, \underline{v}x\, j\, m \neq \infty)\underline{v}x\, j\, m)$$

3.23 THEOREM. $(j \neq \infty\, \wedge\, m \neq \infty\, \wedge\, \wedge\, x(\underline{u}x \neq \infty\, \wedge\, \underline{v}x \neq \infty) \to$
Rereit $x\, \underline{u}x\, \underline{v}x\, 0\, m = \underline{v}m\, \wedge$

Rereit $x\, \underline{u}x\, \underline{v}x\, \text{scsr}\, j\, m = I\, y\, \text{Rereit}\, x\, \underline{u}x\, \underline{v}x\, jy\, \underline{u}\, j\, m \neq \infty)$

The proof is by induction on j.

For example one can check that if

$(m \neq \infty\, \wedge\, n \neq \infty\, \wedge\, \underline{u}x = \delta\, n\, \delta\, 0\, 1\, (x = 1)\, (x = 0) \wedge \underline{v}x = \text{scsr}\, x),$

so that

$(m \neq \infty\, \wedge\, \underline{u}0 = n \neq \infty\, \wedge\, \underline{u}1 = 0\, \wedge\, \underline{u}2 = 1\, \wedge\, \underline{u}3 = 1\, \wedge\, \underline{v}x = \text{scsr}\, x),$

then

$$(\text{Rereit}\, x\, \underline{u}x\, \underline{v}x\, 0\, m = \text{scsr}\, m\, \wedge$$
$$\text{Rereit}\, x\, \underline{u}x\, \underline{v}x\, 1\, m = n + m\, \wedge$$
$$\text{Rereit}\, x\, \underline{u}x\, \underline{v}x\, 2\, m = n \cdot m \wedge$$
$$\text{Rereit}\, x\, \underline{u}x\, \underline{v}x\, 3\, m = \exp n\, m)$$

and

$$\text{Rereit}\, x\, \underline{u}x\, \underline{v}x\, 4\, m = \left. n^{\displaystyle\left(n^{\left(n^{\,n}\right)}\right)} \right. \quad (m \text{ occurrences of 'n' understood}).$$

In 3.25 below are stated the recursion properties for ' $\mathscr{I}$ ' corresponding to 3.1.1–.3 for 'II'. In preparation we give

3.24 THEOREMS. $(0 < m\, \wedge\, n \neq \infty\, \wedge\, \wedge\, y(1 < y \to 0\, \underline{v}y) \to$

.0 $\Pi\, xy(y \cdot \exp P\, \text{scsr}\, x\, \underline{v}y)\, \exp 2\, mn =$
$\Pi\, i \leqq n\, \exp P\, i\, N\, \Pi\, xy(y \cdot \exp P\, \text{scsr}\, x\, \underline{v}y)\, \exp 2\, m\, i\, i =$
$\Pi\, i \leqq n\, \exp P\, i\, N\, \Pi\, xy(y \cdot \exp P\, \text{scsr}\, x\, \underline{v}y)\, \exp 2\, m\, n\, i\, \wedge$

.1 $\,$ N II $xy(y \cdot$ exp P scsr $x\,\underline{v}y)$ exp 2 M scsr n scsr $n =$
$\quad$ $\underline{v}$ II $xy(y \cdot$ exp P scsr $x\,\underline{v}y)$ exp 2 $m\ n \wedge$
.2 $\,$ II $xy(y \cdot$ exp P scsr $x\,\underline{v}y)$ exp 2 $m\ n =$
$\quad$ $\Pi\ i \leqq n$ exp P $i\,\delta\,\underline{v}$ II $xy(y \cdot$ exp P scsr $x\,\underline{v}y)$ exp 2 m pcsr $i\ m$
$$(i \neq 0) \wedge$$
.3 $\,$ $\wedge\ i(i \leqq n \rightarrow$ P $i\ |$ II $xy(y \cdot$ exp P scsr $x\,\underline{v}y)$ exp 2 $m\ n))$

The proofs of .0 and .1 are done simultaneously by induction on n using 3.16.0. From these .2 immediately follows. Then .3 follows from .2 by induction on n.

3.25 Theorems. (Recursion for ' g ').
$\quad$ .0 $\,(\mathit{g}\ y\ \underline{u}y\ m\ 0 = m)$
$\quad$ .1 $\,(\mathit{g}\ y\ \underline{u}y\ m\infty = \infty)$
$\quad$ .2 $\,(n \neq \infty \rightarrow$
$\qquad$ $\mathit{g}\ y\ \underline{u}y\ m$ scsr n
$\qquad$ $= \underline{u}$ II $xy(y \cdot$ exp P scsr $x\ \delta$ scsr scsr $\underline{u}y$ 1 $(\underline{u}y \neq \infty$)) exp 2 δ
$\qquad\qquad\qquad\qquad\qquad\qquad$ scsr scsr m 1$(m \neq \infty$)n
$\qquad$ $= \underline{u}\ \Pi\ i \leqq n$ exp P i N II $xy(y \cdot$ exp P scsr x
$\qquad\quad$ δ scsr scrs $\underline{u}y$ 1$(\underline{u}y \neq \infty$)) exp 2 δ scsr scsr m 1$(m \neq \infty$)ii
$\qquad$ $= \underline{u}\ \Pi\ i \leqq n$ exp P$i\ \delta$scsr scsr $\mathit{g}\ y\ \underline{u}y\ mi$ 1 $(\ y\underline{u}y\ mi \neq \infty$))

The proofs of .0 and .1 are trivial and .2 follows from 3.24 and 3.16.0.

We are ready to show the redundance in 'II' previously alluded to, that is, that recursive definition with the inductor 'II' can be effected with the inductor ' g '. In an earlier manner of speaking, the number of items, counting multiplicities, stored in z, the Gödel number of the finite sequence of previously defined values, is M z, the address of the previously stored item is M z and the latest stored item is pcsr pcsr N z M z packaged in z as N z M z.

3.26 Theorem. (II $xy\ \underline{u}'xy\ m\ n = \mathit{g}\ z\ \underline{u}'$ M z pcsr pcsr N z M zmn)

The proof is by induction on n aided by 3.16.2 and 3.25.

Finally we give an example of a sequence defined recursively using the inductor ' g '. It is rendered informally as

$$A_0 = 2,$$

$$A_{n+1} = \left\{ \begin{array}{l} \infty,\ \text{if } n \text{ is not prime,} \\[4pt] \text{the least prime } t \text{ such that P } n \cdot \displaystyle\sum_{i < n} A_i < t, \\[2pt] \qquad\qquad\qquad A_i \text{ is prime} \\[4pt] \qquad \text{if } n \text{ is prime and } n + 2 \text{ is not prime,} \\ \text{the number of primes less than } n \text{ each} \\ \text{of which is a member of a prime pair,} \\ \quad \text{if } n \text{ is the least prime of a prime pair} \end{array} \right\}, \ \text{for } n \geqq 0.$$

Formally this becomes

3.27 DEFINITION.
   (A $n \equiv$ $\mathit{d}z\delta\delta$
   $\Sigma i < M z\delta10((\text{prime is } i \wedge \text{prime is } (i + 2))\vee$
                                       $(\text{prime is pcsr pscr } i \wedge \text{prime is } i))$

   least $t(\text{prime is } t \wedge$
   $((\text{PM } z \cdot \Sigma i < M z\delta \text{ pcsr pcsr N}zi \text{ 0 prime is pcsr pcsr N } zi) < t))$

   prime is $(M z + 2)$

   ∞ prime is $M z 2 n)$

Since we insist that A n be undefined (that is, ∞) for n a positive non-prime, and since the statement that there are infinitely many prime pairs implies that A n takes on all numbered values, it could confuse the record if we were to use a *number* instead of ∞ to indicate the un-defined state. As things stand, if A n is a number we are convinced that n is 0 or is prime.

By taking enough pains one could, using the inductor ' d ', con-structively enumerate, after a fashion, the finite sequences of numbers as well as count the permutations and combinations of any collection of n numbers taken r at a time. Similarly, one could enumerate and total the partitions of a number. To achieve these ends one would represent finite sequences of numbers by Gödel numbers as we have done and represent finite sequences of finite sequences by Gödel numbers of finite sequences of Gödel numbers of finite sequences. And so on. We shall stop here, feeling that our development has been carried far enough to provide a basis for a fair appraisal of this approach to number theory.

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