

STOCHASTICALLY PERTURBED DYNAMICAL SYSTEMS

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1. Introduction. A variety of natural phenomena have been modeled by stochastic differential equations containing one or more small parameters. Such models allow one to study the effects of random perturbations of small magnitude. When the parameters are set equal to 0, the stochastic effects disappear from the problem; the resulting ordinary differential equations describe the evolution of the unperturbed dynamical system. Examples can be found in the study of wave propagation in random media [17], of noise in electronic systems [19] or in a control system [6]. Yet another class of examples, which we mention in § 6, arises from population genetics models.

For simplicity, we consider only one small positive parameter ϵ . The state space of the system being modeled is taken as n -dimensional R^n . The state process $\xi^\epsilon = (\xi_1^\epsilon, \dots, \xi_n^\epsilon)$ is assumed to be a Markov diffusion, with generator \mathcal{L}^ϵ the partial differential operator

$$(1.1) \quad \mathcal{L}^\epsilon \psi = \epsilon \operatorname{tr} \alpha \psi_{xx} + \psi_x \cdot f.$$

Here ψ denotes any function of class C^2 (continuous second order partial derivatives); the functions α, f are respectively matrix and vector valued:

$$\begin{aligned} \alpha(x) &= (\alpha_{ij}(x)), \quad i, j = 1, \dots, n, \\ f(x) &= (f_1(x), \dots, f_n(x)). \end{aligned}$$

The vector $\psi_x = (\psi_{x_1}, \dots, \psi_{x_n})$ is the gradient of ψ , and

$$\operatorname{tr} \alpha \psi_{xx} = \sum_{i,j=1}^n \alpha_{ij} \psi_{x_i x_j}.$$

In fact, we shall take ξ^ϵ as the solution of a stochastic differential equation of Ito type (§ 2).

A variety of questions can be asked about the perturbed process ξ^ϵ . Which of these has a reasonable answer in a particular instance

¹This research was supported in part by the Air Force Office of Scientific Research under Grant No. AF-AFOSR 71-2078, and in part by the National Science Foundation under Grant No. GP 15132.

²Presented at the Conference on Stochastic Differential Equations, University of Alberta, July 1972.

Received by the Editors June 22, 1973.

often depends on the structure of the unperturbed system and on the way randomness enters. One question which one can always ask is the following: given a real valued function Φ on R^n , estimate the expected value $E\Phi[\xi^\epsilon(t)]$ for fixed $t > 0$ and small $\epsilon > 0$. For instance, if $\Phi(x) = |x|^m$, then $E|\xi^\epsilon(t)|^m$ is the m th absolute moment.

In § 3 a good approximation to $E\Phi[\xi^\epsilon(t)]$ is obtained in the form of an asymptotic expansion in powers of ϵ . This result is a corollary of a theorem about expanding solutions of the Cauchy problem for the nonlinear parabolic equation (4.1 $^\epsilon$), whose second order coefficients contain the small positive parameter ϵ .

If Φ is conserved in the unperturbed system, then it is interesting to estimate $E\Phi[\xi^\epsilon(t^\epsilon)]$, where $t^\epsilon = O(\epsilon^{-1})$ as $\epsilon \rightarrow 0$. Some rough estimates for the growth of this quantity are obtained in § 3 as an easy consequence of the Ito stochastic differential rule. Similar estimates were found previously by Carrier [1] using other methods. For systems of the type of a randomly perturbed harmonic oscillator occurring in wave propagation problems, a more exact description of the behavior of $\xi^\epsilon(t^\epsilon)$ is known [17].

In some problems, one is interested in the process ξ^ϵ only up to the time τ^ϵ when $\xi^\epsilon(t)$ exits from a given region B . One may then wish to estimate, for instance, $E\Phi[\xi^\epsilon(\tau^\epsilon)]$. This question is considered in § 5. We obtain an asymptotic expansion in powers of ϵ . As is well-known in the theory of singular perturbations, such expansions cannot be expected to hold in all of B . Let us set $\varphi^\epsilon(x) = E_x\Phi[\xi^\epsilon(\tau^\epsilon)]$, where the subscript x indicates the initial state $\xi^\epsilon(0) = x$. Then φ satisfies for $\epsilon > 0$ the elliptic second order partial differential equation $\mathcal{L}^\epsilon\varphi^\epsilon = 0$ in B , with boundary data $\varphi^\epsilon = \Phi$ on ∂B . For $\epsilon = 0$, the operator \mathcal{L}^0 is of first order. Some (or all) of the boundary data may be lost. The expansion in § 5 is valid in portions of B from which trajectories of the unperturbed dynamical system lead nontangentially to ∂B . It corresponds to the "regular expansion" obtained in the theory of singular perturbations by non-probabilistic methods [4]. It would seem interesting to study probabilistically boundary layer expansions, or other asymptotic properties of φ^ϵ , outside regions where a regular expansion holds. However, we have not done so.

The equation $\mathcal{L}^\epsilon\varphi^\epsilon = 0$ is linear. The probabilistic method to get regular expansions can also be used when φ^ϵ satisfies some nonlinear elliptic equation. When the nonlinearity involves only φ^ϵ (see (5.1 $^\epsilon$), or the corresponding parabolic equation (4.1 $^\epsilon$)) the expansion problem is scarcely more difficult than for linear equations. The only additional step is to prove convergence of φ^ϵ to φ^0 as $\epsilon \rightarrow 0$ in regions where the regular expansion holds. If the nonlinearity involves the gradient φ_x^ϵ , then one has to prove convergence of φ_x^ϵ to φ_x^0 as $\epsilon \rightarrow 0$. This is

substantially more difficult. In certain cases it was done in [6], by writing the equation as the dynamic programming equation of a stochastic control problem.

2. **The system equations.** Let the Markov process ξ^ϵ satisfy the stochastic differential equations

$$(2.1^\epsilon) \quad d\xi^\epsilon = f[\xi^\epsilon(t)] dt + (2\epsilon)^{1/2}\sigma[\xi^\epsilon(t)] dw, \quad t \geq 0$$

where w is a brownian motion of some finite dimension. As initial data, let

$$(2.2) \quad \xi^\epsilon(0) = x,$$

where x is a given vector. Occasionally, we write $\xi^\epsilon(t; x)$ to indicate dependence on the initial state x . More often, this dependence is shown by writing probabilities and expectations as $P_x(\dots)$, $E_x(\dots)$. For $\epsilon = 0$, we get the unperturbed system equations for the (non-random) function ξ^0 .

$$(2.1^0) \quad d\xi^0 = f[\xi^0(t)] dt, \quad t \geq 0,$$

with the same initial data $\xi^0(0) = x$.

Let us assume that the functions f, σ (respectively vector, matrix valued) are of class C^∞ . Moreover, we assume that there exist a positive constant M and a positive C^∞ function V such that:

$$(2.3) \quad |\sigma(x)| \leq M(1 + |x|)$$

$$(2.4a) \quad \mathcal{L}^\epsilon V \leq M(1 + V), \quad 0 \leq \epsilon \leq 1$$

$$(2.4b) \quad (1 + |x|)|V_x| \leq M(1 + V)$$

$$(2.4c) \quad V(x) \rightarrow \infty \quad \text{as} \quad |x| \rightarrow \infty.$$

In the formula (1.1) for α , take $\alpha = \sigma\sigma^T$. Then $f, 2\epsilon\alpha$ are respectively the local drift and local covariance coefficients for the Markov process ξ^ϵ . These assumptions imply that ξ^ϵ is defined for all $t \geq 0$. Moreover, for any $l = 1, 2, \dots$, there is a bound on $E_x V^l[\xi^\epsilon(t)]$ depending only on l , a bound for $V(x)$, and t . See Appendix (A.2).

We shall use the following estimate, which shows that $\xi^\epsilon(t)$ is near $\xi^0(t)$ with probability very nearly 1. A proof is given in the Appendix.

LEMMA 2.1. *Let $\|\xi^\epsilon - \xi^0\|_t = \max_{0 \leq s \leq t} |\xi^\epsilon(s) - \xi^0(s)|$. Then given $a > 0$ a compact set $K \subset R^n$, and $t_1 > 0$, there exist $\lambda > 0$, and $\epsilon_1 > 0$ such that*

$$P_x(\|\xi^\epsilon - \xi^0\|_t > a) \leq 2n \exp(-a\epsilon^{-1/2\lambda})$$

for all $x \in K, 0 \leq t \leq t_1$, and $0 < \epsilon \leq \epsilon_1$.

We also need the following probabilistic representation for solutions $\psi(t, x)$ of a linear partial differential equation

$$(2.5) \quad -\psi_t + \mathcal{L}^\epsilon \psi + A(t, x)\psi + g(t, x) = 0.$$

LEMMA 2.2. *Let $G \subset R^n$ be an open set, and ψ a solution of (2.5) in the cylinder $Q = (0, t) \times G$, such that ψ is continuous and bounded in the closure \bar{Q} while the partial derivatives $\psi_t, \psi_{x_i x_j}, i, j = 1, \dots, n$, are continuous in Q . Assume that A, g are continuous and bounded on Q . Let τ denote the minimum of t and the exit time of $\xi^\epsilon(s)$ from G . Then*

$$(2.6) \quad \begin{aligned} \psi(t, x) = & E_x \int_0^\tau D(s)g(t-s, \xi^\epsilon(s)) ds \\ & + E_x [D(\tau)\psi(t-\tau, \xi^\epsilon(\tau))], \end{aligned}$$

$$D(s) = \exp \int_0^s A(t-u, \xi^\epsilon(u)) du.$$

This result follows rather easily from the Ito stochastic differential rule (Appendix). An important case is $A = 0$; then $D(s) = 1$. If, in addition, $g = 0$, then ψ is a solution of the backward equation $-\psi_t + \mathcal{L}^\epsilon \psi = 0$ for the diffusion process ξ^ϵ . This special case is of interest in § 3.

When g and ψ are autonomous and $A \equiv 0$, Lemma 2.2 has the following corollary, which will be useful in § 5.

LEMMA 2.3. *Let $G \subset R^n$ be an open set, and τ the exit time of $\xi^\epsilon(s)$ from G starting at $x \in G$. Assume that $E_x \tau < \infty$, and that g is continuous and bounded on G . Let ψ be a C^2 solution of*

$$(2.7) \quad \mathcal{L}^\epsilon \psi + g(x) = 0 \text{ in } G,$$

such that ψ is continuous and bounded in \bar{G} . Then

$$(2.8) \quad \psi(x) = E_x \int_0^\tau g[\xi^\epsilon(s)] ds + E_x \psi[\xi^\epsilon(\tau)].$$

To get Lemma 2.3, introduce the bounded stopping time $\tau_t = \min(t, \tau)$, and use Lemma 2.2. Then let $t \rightarrow \infty$.

3. Estimates for $E_x \Phi[\xi^\epsilon(t)]$. Given t , we are interested in various statements which can be made about the random variables $\xi^\epsilon(t)$ for small $\epsilon > 0$. These could be answered in principle from knowing the distribution of $\xi^\epsilon(t)$, which by Lemma 2.1 is mostly concentrated near

the point $\xi^0(t)$. Instead of trying to find this distribution, we estimate integrals of various functions Φ with respect to it. Thus we are concerned with expectations of the form $E_x \Phi[\xi^\epsilon(t)]$. We begin with some estimates for this quantity which can be obtained by elementary arguments from the Ito stochastic differential rule. Later in the section we discuss expansions in powers of ϵ .

We assume that Φ is C^∞ , and that for some positive constants M, ℓ

$$(3.1a) \quad |\Phi| \leq M(1 + V^\ell),$$

$$(3.1b) \quad (1 + |x|)|\Phi_x| \leq M(1 + V^\ell),$$

$$(3.1c) \quad (1 + |x|)^2|\Phi_{xx}| \leq M(1 + V^\ell).$$

These assumptions hold, for instance, if Φ is a polynomial and V a positive polynomial.

By the Ito stochastic differential rule,

$$d\Phi[\xi^\epsilon(s)] = \mathcal{L}^\epsilon \Phi ds + (2\epsilon)^{1/2} \Phi_x \sigma dw.$$

By taking $E_x \int_0^t$ and recalling that $\xi^\epsilon(0) = x$, we get

$$(3.2) \quad E_x \Phi[\xi^\epsilon(t)] = \Phi(x) + E_x \int_0^t \mathcal{L}^\epsilon \Phi[\xi^\epsilon(s)] ds.$$

By (2.3), (3.1a, c) and Appendix (A.2) the expectations exist in (3.2). By (3.1b),

$$|\Phi_x \sigma|^2 \leq M_1(1 + V^{2\ell})$$

for some M_1 . Since $E_x V^{2\ell}[\xi^\epsilon(s)]$ is bounded for $0 \leq s \leq t$ (by Appendix (A.2) with 2ℓ in place of ℓ there), $E_x |\Phi_x \sigma|^2$ is also bounded. This implies

$$E_x \int_0^t \Phi_x[\xi^\epsilon(s)] \sigma[\xi^\epsilon(s)] dw = 0$$

as required in (3.2).

In differentiated form (3.2) becomes, upon interchanging expectation with time integration,

$$(3.2') \quad \frac{d}{dt} E_x \Phi[\xi^\epsilon(t)] = E_x \mathcal{L}^\epsilon \Phi[\xi^\epsilon(t)].$$

The left side is the time derivative of the quantity we wish to compute. Unfortunately, the right side involves the expectation of $\mathcal{L}^\epsilon \Phi$ rather than Φ . Nevertheless, useful information can often be obtained from (3.2').

EXAMPLE 1. Consider a linear system

$$d\xi = A\xi dt + (2\epsilon)^{1/2}\sigma dw, \quad \sigma = \text{constant.}$$

The process $\xi (= \xi^\epsilon)$ is gaussian. The distribution of $\xi(t)$ is determined by the means and covariances

$$\mu_i(t) = E_x \xi_i(t), \quad v_{ij}(t) = E_x (\xi_i(t) - \mu_i(t))(\xi_j(t) - \mu_j(t)).$$

These can be computed using (3.2'), taking first $\Phi(x) = x_i$, then $\Phi(x) = x_i x_j$, $i, j = 1, \dots, n$. One gets for the mean vector $\mu(t) = (\mu_1(t), \dots, \mu_n(t))$ and covariance matrix $v(t) = (v_{ij}(t))$ the differential equations

$$\begin{aligned} \frac{d\mu}{dt} &= A\mu, \quad \mu(0) = x, \\ \frac{dv}{dt} &= Av + vA^T + 2\epsilon\alpha, \quad v(0) = 0. \end{aligned}$$

Conservative systems. Let us next discuss the special case when $\Phi \geq 0$ and $\Phi_x \cdot f = 0$. Then $\Phi[\xi^0(t)]$ is constant on each trajectory of the unperturbed system (2.1⁰). In many cases, $\Phi(x)$ can be regarded as the "energy" associated with the state x . Thus, we wish to estimate the average energy for the perturbed system

$$e(t) = E_x \Phi[\xi^\epsilon(t)].$$

By (1.1) we now have $\mathcal{L}^\epsilon \Phi = \epsilon \operatorname{tr} \alpha \Phi_{xx}$. Since ϵ appears as a coefficient on the right side of (3.2'), it is reasonable to anticipate that $e(t)$ will differ appreciably from $e(0)$ for times t of order ϵ^{-1} . An interesting question is the rate of growth of $e(t)$ as a function of ϵt . Some easy estimates of this type were discussed in [7], and applied to two examples of Carrier [1]. We shall now carry that discussion a bit further.

EXAMPLE 2. Let $n = 2$,

$$d\xi_1 = \xi_2 dt, \quad d\xi_2 = -\xi_1 dt + (2\epsilon)^{1/2}\xi_1 dw.$$

This is a randomly perturbed harmonic oscillator. Let $\Phi(x_1, x_2) = (1/2)(x_1^2 + x_2^2)$. Then

$$\operatorname{tr} \alpha \Phi_{xx} = x_1^2 \Phi_{x_2 x_2} = x_1^2.$$

Since the right side is no more than $2\Phi(x_1, x_2)$, we have from (3.2')

$$\frac{de}{dt} \leq 2\epsilon e,$$

$$e(t) \leq e_0 \exp(2\epsilon t)$$

where e_0 is the energy of the unperturbed system. This estimate is not sharp. A more careful analysis shows that the mean energy grows like $(1/2)\exp(\epsilon t)$. Consider the mean squared displacement $\mu(t) = E[\xi_1(t)^2]$. Carrier [1] showed that $\mu(t) \sim (1/2)\exp(\epsilon t)$ as $\epsilon t \rightarrow \infty$. Since

$$(*) \quad e(t) = e_0 + \epsilon \int_0^t \mu(s) ds,$$

we have also $e(t) \sim (1/2)\exp(\epsilon t)$ as $\epsilon t \rightarrow \infty$. The following (heuristic) argument also explains why the correct exponent is ϵt rather than $2\epsilon t$. When $\epsilon = 0$

$$\int_{t_1}^{t_1+2\pi} \xi_1^0(s)^2 ds = \int_{t_1}^{t_1+2\pi} \Phi[\xi^0(s)] ds,$$

since the unperturbed system has as 2π -periodic orbits the circles $\Phi = \text{constant}$. For small $\epsilon > 0$ we should then have

$$(**) \quad \int_{t_1}^{t_1+2\pi} \mu(s) ds \approx \int_{t_1}^{t_1+2\pi} e(s) ds.$$

One then replaces $\mu(s)$ by $e(s)$ in $(*)$, after writing the integral for $t = 2m\pi$ with m large as the sum of integrals over periods. (The error in the approximation $(**)$ could be estimated with the help of the expansion (3.5) for the mean energy and mean square displacement, conditioned on the state $\xi(t_1)$. However, we shall not carry out these estimates here.)

Considerably more is known about the oscillator problem in Example 2. By introducing as new variables the amplitude r and phase shift ϕ and a new time scale $t' = \epsilon t$, it can be shown that (in the new time scale) the (r, ϕ) process is closely approximated by a 2-dimensional diffusion whose generator can be explicitly calculated. See [17]–[19].

For our next class of examples we find an increasing function u , such that $u(e) \geq 0$ for $e \geq 0$ and

$$(3.3) \quad E_x \text{tr } \alpha \Phi_{xx} \leq u[e(t)].$$

When such a function u exists, from (3.2') e satisfies the differential inequality

$$\frac{de}{dt} \leq \epsilon u(e).$$

If $F'(e) = 1/u(e)$ and e_0 is the energy of the unperturbed system, we then have

$$(3.4) \quad e(t) \leq F^{-1}(F(e_0) + \epsilon t).$$

EXAMPLE 3. Let us put a nonlinearity in Example 2, as follows:

$$\begin{aligned} d\xi_1 &= \xi_2 dt, \quad d\xi_2 = -(\xi_1 + G'(\xi_1)) dt + (2\epsilon)^{1/2} \xi_1 dw, \\ G(x_1) &= \Gamma(x_1^2), \quad \Gamma(0) = \Gamma'(0) = 0, \quad \Gamma'' > 0. \end{aligned}$$

Carrier considered in [1] the case $G'(x_1) = x_1^3$, $\Gamma(y) = (1/4)y^2$. We now take as energy function

$$\Phi(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + G(x_1).$$

Again $\text{tr } \alpha \Phi_{xx} = x_1^2$. Take $u = \Gamma^{-1}$, the inverse of Γ . Then u is a concave function, and $x_1^2 = u[G(x_1)]$. Hence, writing $\xi_1(t) = \xi_1$,

$$E_x(\xi_1)^2 = E_x u[G(\xi_1)] \leq u[E_x G(\xi_1)].$$

Since u is increasing and $G \leq \Phi$, we have

$$E_x \text{tr } \alpha \Phi_{xx} = E_x(\xi_1)^2 \leq u[e(t)],$$

as required in (3.3). Thus the estimate (3.4) holds. In particular, if $G(x_1) = cx_1^{2m}$, $m \geq 2$, then for suitable constant c_1

$$\begin{aligned} F^{-1}(r) &= c_1 r^{m/(m-1)}, \\ e(t) &\leq c_1 (\epsilon t)^{m/(m-1)} \end{aligned}$$

for ϵt large. The nonlinear term $-G'(\xi_1) dt$ in the system equations imposes a much slower growth rate on the mean energy than in Example 2. It would be interesting to see whether this estimate might be improved, perhaps by using an averaging technique over periods of the unperturbed system. However, we have not succeeded in doing so.

Expansions in powers of ϵ , for fixed t . We were concerned above with upper bounds for $E_x \Phi[\xi^\epsilon(t)]$, particularly for $t = t(\epsilon)$ of order ϵ^{-1} . We now turn to the problem of making much sharper estimates for this quantity when t is fixed. Let us write

$$\varphi^\epsilon(t, x) = E_x \Phi[\xi^\epsilon(t)], \quad \varphi^0(t, x) = \Phi[\xi^0(t)].$$

We wish to expand φ^ϵ in powers of ϵ :

$$(3.5) \quad \varphi^\epsilon = \varphi^0 + \epsilon \theta_1 + \epsilon^2 \theta_2 + \dots + \epsilon^k \theta_k + o(\epsilon^k).$$

In § 4 it will be shown that such an expansion is valid for any k (finite). The coefficients $\theta_1, \theta_2, \dots$ satisfy

$$(3.6) \quad \theta_k(t, x) = \int_0^t \text{tr } \alpha(\theta_{k-1})_{xx} ds, \quad k = 1, 2, \dots,$$

where the integrand is evaluated at $(t - s, \xi^0(s))$ and $\theta_0 = \varphi^0$. One can derive formally equations (3.6) in the following way. Under perhaps more stringent growth conditions than (3.1) on $\Phi(x)$ for $|x|$ large, φ^ϵ must satisfy the partial differential equation

$$(3.7) \quad -\varphi_t^\epsilon + \mathcal{L}^\epsilon \varphi^\epsilon = 0, \quad t \geq 0$$

with $\varphi^\epsilon(0, x) = \Phi(x)$. By formally differentiating (3.7) repeatedly with respect to ϵ and setting $\epsilon = 0$, one gets the first order partial differential equations

$$(3.8) \quad -(\theta_k)_t + \mathcal{L}^0 \theta_k + \text{tr } \alpha(\theta_{k-1})_{xx} = 0$$

with $\theta_k(0, x) = 0$ for $k = 1, 2, \dots$. The characteristic equations for (3.8) are just (2.1⁰), and thus we get formally (3.6).

To compute $\theta_1, \theta_2, \dots$ for given initial state x , additional ordinary differential equations besides (2.1⁰) are needed. Let us explicitly indicate the dependence of ξ^0 on the initial state, by writing $\xi^0 = \xi^0(t, x)$. Then

$$\varphi^0(t, x) = \Phi[\xi^0(t, x)].$$

To get θ_1 , one needs the matrix of second derivatives φ_{xx}^0 . Partial derivatives $\varphi_{x_i}^0, \varphi_{x_i x_j}^0, \dots$ of any order can be found from the derivatives of Φ and from $\xi_x^0, \xi_{xx}^0, \dots$. The latter satisfy familiar ordinary differential equations obtained from (2.1⁰) by differentiation with respect to initial data. To get θ_2 one needs $(\theta_1)_{xx}$. By differentiating (3.6) with $k = 1$ with respect to x_i and x_j , one gets $(\theta_1)_{x_i x_j}$ in terms of third and fourth order derivatives of φ^0 ; and so on. Unfortunately, the number of differential equations needed increases rapidly with k .

In the 1-dimensional case, there is a closed form expression provided $f[\xi^0(s)] \neq 0, 0 \leq s \leq t$. In fact,

$$\theta_k(t, x) = H_k[\xi^0(t, x), x],$$

where

$$H_0(y, x) = \Phi(y),$$

$$H_1(y, x) = \frac{1}{[f(x)]^2} \int_x^y \alpha(z) \{ \Phi_{yy}(z) f(z) + \Phi_y(z) [f'(z) - f'(x)] \} dz$$

$$H_{k+1}(y, x) = \frac{1}{f(x)} \int_x^y \alpha(z) \left\{ H_{ky} \frac{f(z)}{f(x)} + 2H_{kyx} + H_{kxx} f(x) + H_{ky} \left[\frac{f'(z) - f'(x)}{f(x)} \right] \right\} dz.$$

This can be shown by direct calculation.

4. **Results for fixed t (finite).** Let us consider a second order partial differential equation of the form

$$(4.1^\epsilon) \quad -\varphi_t^\epsilon + \mathcal{L}^\epsilon \varphi^\epsilon + F(x, \varphi^\epsilon) = 0, \quad t \geq 0,$$

with the initial data

$$(4.2) \quad \varphi^\epsilon(0, x) = \Phi(x).$$

When $F = 0$, this becomes (3.7). For $\epsilon = 0$, we have the first order equation

$$(4.1^0) \quad -\varphi_t^0 + \mathcal{L}^0 \varphi^0 + F(x, \varphi^0), \quad t \geq 0,$$

with the same initial data (4.2). The solution φ^0 can then be expressed by the method of characteristics.

Let us suppose that φ^ϵ is a bounded solution of (4.1 $^\epsilon$)–(4.2) with continuous partial derivatives $\varphi_t^\epsilon, \varphi_x^\epsilon, \varphi_{i,x}^\epsilon, i, j = 1, \dots, n$. We wish to expand φ^ϵ in powers of ϵ .

The coefficients in (3.5) must have the property that $k! \theta_k$ is the k th derivative of φ^ϵ with respect to ϵ when $\epsilon = 0$. By formally differentiating (4.1 $^\epsilon$) repeatedly with respect to ϵ , we get

$$(4.3^0) \quad -(\theta_k)_t + \mathcal{L}^0 \theta_k + F_\varphi \theta_k + \Gamma_k^0 + \text{tr} \alpha(\theta_{k-1})_{xx} = 0, \\ \theta_0 = \varphi^0, \Gamma_1^0 = 0, \Gamma_2^0 = \frac{1}{2} \theta_1^2 F_{\varphi\varphi}, \dots$$

In general, Γ_k^0 is a polynomial in $\theta_1, \dots, \theta_{k-1}$ of degree k , with coefficients partial derivatives $F_{\varphi\varphi}, F_{\varphi\varphi\varphi}, \dots$ evaluated at $(x, \varphi^0(x))$. If $F = A(x)\varphi + g(x)$ is linear in φ , then all $\Gamma_k^0 = 0$.

THEOREM 4.1. *Assume that $F_\varphi, F(x, 0)$, and Φ are bounded. Then the expansion (3.5) holds uniformly for (t, x) in any compact set.*

PROOF. Let

$$\begin{aligned} \theta_1^\epsilon &= \epsilon^{-1}(\varphi^\epsilon - \varphi^0), \dots, \theta_k^\epsilon = \epsilon^{-1}(\theta_{k-1}^\epsilon - \theta_{k-1}) \\ &= \epsilon^{-k}(\varphi^\epsilon - \varphi^0 - \epsilon \theta_1 \dots - \epsilon^{k-1} \theta_{k-1}). \end{aligned}$$

We must show that, for each k , $\theta_k^\epsilon(t, x) \rightarrow \theta_k(t, x)$ as $\epsilon \rightarrow 0$, uniformly on any compact set. The functions θ_k^ϵ satisfy linear partial differential equations (by direct calculation)

$$(4.3^\epsilon) \quad -(\theta_k^\epsilon)_t + \mathcal{L}^\epsilon \theta_k^\epsilon + A^\epsilon \theta_k^\epsilon + \Gamma_k^\epsilon + \text{tr } \alpha(\theta_{k-1})_{xx} = 0,$$

$$(4.4a) \quad A^\epsilon = \int_0^1 F_\varphi(x, \varphi^0 + \lambda(\varphi^\epsilon - \varphi^0)) d\lambda,$$

$$(4.4b) \quad \Gamma_1^\epsilon = 0, \Gamma_2^\epsilon = \theta_1 \theta_1^\epsilon \int_0^1 \int_0^1 F_{\varphi\varphi}(x, \varphi^0 + \lambda\mu(\varphi^\epsilon - \varphi^0)) d\lambda d\mu, \dots$$

For $k > 2$, Γ_k^ϵ is a polynomial of degree k in θ_ℓ , θ_m^ϵ , $\ell, m \leq k - 1$, with coefficients involving $F_{\varphi\varphi}, F_{\varphi\varphi\varphi}, \dots$ of orders up to k .

We first apply Lemma 2.2 to φ^ϵ , with $A = 0$, $g = F(x, \varphi^\epsilon)$, $G = R^n$, to get

$$(4.5) \quad \begin{aligned} \varphi^\epsilon(t, x) &= E_x \int_0^t F[\xi^\epsilon(s), \varphi^\epsilon(t - s, \xi^\epsilon(s))] ds \\ &+ E_x \Phi[\xi^\epsilon(t)]. \end{aligned}$$

From the boundedness assumed for F_φ and $F(x, 0)$, $|F(x, \varphi^\epsilon)| \leq C(1 + |\varphi^\epsilon|)$ for some C . Using Gronwall's inequality we get from (4.5)

$$\|\varphi^\epsilon(t, \cdot)\| \leq (\|\Phi\| + Ct)e^{Ct},$$

where $\|\cdot\|$ is the sup norm. Therefore, φ^ϵ has a bound depending on t , but not ϵ .

We next use Lemma 2.2 with $\psi = \theta_1^\epsilon$ and $A = A^\epsilon$, $g = \text{tr } \alpha\varphi_{xx}^0$, G any open bounded set. We get

$$(4.6) \quad \begin{aligned} \theta_1^\epsilon(t, x) &= E_x \int_0^t D^\epsilon(s) \text{tr } \alpha\varphi_{xx}^0 ds \\ &+ E_x [D^\epsilon(\tau) \theta_1^\epsilon(t - \tau, \xi^\epsilon(\tau))], \end{aligned}$$

$$D^\epsilon(s) = \exp \int_0^s A^\epsilon(t - u, \xi^\epsilon(u)) du,$$

where $\text{tr } \alpha\varphi_{xx}^0$ is evaluated at $(t - s, \xi^\epsilon(s))$. Given a compact set $K \subset R^n$ and $t_1 > 0$, choose G large enough that $\xi^0(s) \in G$ for $0 \leq s \leq t_1$

if $\xi^0(0) = x$ and $x \in K$. Let a be the minimum distance to ∂G from any point $\xi^0(s)$, $0 \leq s \leq t$, with $x \in K$. By Lemma 2.1

$$(4.7) \quad P_x(\tau < t) \leq 2n \exp(-a\epsilon^{-1/2}\lambda)$$

for $0 \leq t \leq t_1$ and small ϵ . Since φ^ϵ is bounded, $|\theta_1^\epsilon| \leq C_1\epsilon^{-1}$ for some C_1 ; and since A^ϵ is bounded (by boundedness of F_φ), $|D^\epsilon \theta_1^\epsilon| \leq C\epsilon^{-1}$ for some C . Moreover, $\theta_1^\epsilon(t - \tau, \xi^\epsilon(\tau)) = 0$ if $\tau = t$. It follows that

$$\lim_{\epsilon \rightarrow 0} |E_x [D^\epsilon(\tau) \theta_1^\epsilon(t - \tau, \xi^\epsilon(\tau))] \leq C \lim_{\epsilon \rightarrow 0} \epsilon^{-1} P_x(\tau < t) = 0.$$

Thus, the second term on the right side of (4.6) tends to 0. The first term there is bounded. Hence θ_1^ϵ is bounded, which implies

$$\lim_{\epsilon \rightarrow 0} (\varphi^\epsilon - \varphi^0) = \lim_{\epsilon \rightarrow 0} \epsilon \theta_1^\epsilon = 0.$$

This holds uniformly on $[0, t_1] \times K$. By Lemma 2.1, $\|\xi^\epsilon - \xi^0\|_t$ tends to 0 in probability as $\epsilon \rightarrow 0$. Moreover, τ tends to t in probability. Therefore,

$$\lim_{\epsilon \rightarrow 0} E_x \int_0^\tau D^\epsilon(s) \operatorname{tr} \alpha \varphi_{xx}^0 ds = \int_0^t D^0(s) \operatorname{tr} \alpha \varphi_{xx}^0 ds,$$

$$D^0(s) = \exp \int_0^s A^0(s - u, \xi^0(u)) du,$$

where $\operatorname{tr} \alpha \varphi_{xx}^0$ on the right side is evaluated at $(t - s, \xi^0(s))$. From the method of characteristics the right side is then the desired solution of (4.3⁰) when $k = 1$. Thus, by (4.6), $\theta_1^\epsilon \rightarrow \theta_1$ as $\epsilon \rightarrow 0$. The convergence is uniform on $[0, t_1] \times K$. This proves Theorem 4.1 for $k = 1$.

The proof now continues by induction on k . Again by Lemma 2.2, we have

$$(4.8) \quad \begin{aligned} \theta_k^\epsilon(t, x) = & E_x \int_0^\tau D^\epsilon(s) [\Gamma_k^\epsilon + \operatorname{tr} \alpha (\theta_{k-1})_{xx}] ds \\ & + E_x [D^\epsilon(\tau) \theta_k^\epsilon(t - \tau, \xi^\epsilon(\tau))]. \end{aligned}$$

Since φ^ϵ is bounded, $|\theta_k^\epsilon| \leq C_k \epsilon^{-k}$ for some C_k . This, together with (4.7), suffices to show that the extreme right hand side of (4.8) tends to 0 as $\epsilon \rightarrow 0$. Using the induction hypothesis, it can be shown that Γ_k^ϵ tends to Γ_k^0 uniformly on compact sets. It then follows as before that $\theta_k^\epsilon \rightarrow \theta_k$ uniformly on $[0, t_1] \times K$ as $\epsilon \rightarrow 0$. This proves Theorem 4.1.

REMARK. In Theorem 4.1 we imposed conditions which insure that

φ^ϵ is uniformly bounded for bounded t . The same method would give the expansion (3.5) under any weaker set of assumptions which guarantee that the solutions φ^ϵ of (4.1 ϵ)–(4.2) are uniformly bounded on compact sets.

EXAMPLE. Consider the equation in 1 space variable

$$(4.9) \quad -\varphi_t^\epsilon + \epsilon\varphi_{xx}^\epsilon + F(\varphi^\epsilon) = 0.$$

Then $\alpha(x) = 1, f(x) = 0$. For $\epsilon = 0, \varphi^0$ is a solution of the ordinary differential equation $\varphi_t^0 = F(\varphi^0)$, depending on x as a parameter. The function θ_1 is a solution of the linear ordinary differential equation

$$(\theta_1)_t = F'(\varphi^0)\theta_1 + \varphi_{xx}^0.$$

One interpretation of equation (4.9) is as a simplified model of population growth with small diffusion. The population is distributed along an infinite linear habitat, and $\varphi^\epsilon(t, x)$ represents the population density at place x and time t . Typical choices for F are $F(\varphi) = m\varphi$ (linear growth rate) and $F(\varphi) = m_1\varphi - m_2\varphi^2$ (logistic growth rate). Another equation of type (4.9) arising from a population genetics model with diffusion was considered in [12]. That model neglects the effects of random sampling of genes in a population of finite size.

Expansion of $E_x\Phi[\xi^\epsilon(t)]$. We now apply Theorem 4.1 to obtain the expansion (3.5) already announced in §3. There is a slight technical difficulty because we do not know, under the rather mild growth assumptions (3.1), that the function φ^ϵ defined in §3 is actually a solution of the partial differential equation (4.1 ϵ).

THEOREM 4.2. *If Φ satisfies assumptions (3.1), then the expansion (3.5) holds for $\varphi^\epsilon(t, x) = E_x\Phi[\xi^\epsilon(t)]$ uniformly for (t, x) in any compact set.*

PROOF. *Step 1.* Suppose for the moment that f, σ, Φ are bounded together with their first and second order partial derivatives. Results about smooth dependence of solutions of stochastic differential equations on the initial state x then imply that φ^ϵ has continuous partial derivatives $\varphi_t^\epsilon, \varphi_{x_i}^\epsilon, \varphi_{x_i x_j}^\epsilon$ and satisfies

$$-\varphi_t^\epsilon + \mathcal{L}^\epsilon\varphi^\epsilon = 0.$$

See [9, Chapter VIII.5] or [10, Part I, §11, Theorem 1]. Moreover, $\|\varphi^\epsilon\| \leq \|\Phi\|$. By Theorem 4.1 with $F = 0$, expansion (3.5) holds uniformly on compact sets.

Step 2. Now let Φ be as in Step 1, while f, σ satisfy (2.3)–(2.4). Choose f_m, σ_m satisfying the conditions in Step 1 for $m = 1, 2, \dots$, such that $f_m = f, \sigma_m = \sigma$ for $x \in G_m$, where G_m is an expanding sequence of open sets with union R^n . Let

$$\varphi_m^\epsilon(t, x) = E_x \Phi[\xi_m^\epsilon(t)],$$

where ξ_m^ϵ satisfies (2.1 $^\epsilon$) with f_m, σ_m replacing f, σ . These solutions can be so defined that $\xi_m^\epsilon(t) = \xi^\epsilon(t)$ for $0 \leqq t \leqq \tau_m^\epsilon$, where the exit time τ_m^ϵ from G_m tends to infinity as $m \rightarrow \infty$. The following stronger estimate holds, in fact. Given a compact set K and t_1 , fix $G_\mu = G$ with μ large enough that the situation in the proof of Theorem 4.1 holds. Let τ be as in Lemma 2.2. Then

$$\varphi_m^\epsilon(t, x) = E_x \varphi_m^\epsilon(t - \tau, \xi^\epsilon(\tau)), \quad m \geqq \mu,$$

for $x \in G_\mu$. Moreover, $\|\varphi_m^\epsilon\| \leqq \|\Phi\|$. Let $m \rightarrow \infty$; by dominated convergence

$$\varphi^\epsilon(t, x) = E_x \varphi^\epsilon(t - \tau, \xi^\epsilon(\tau)).$$

When $\tau = t, \varphi_\mu^\epsilon = \varphi^\epsilon = \Phi$. Thus

$$|\varphi_\mu^\epsilon(t, x) - \varphi^\epsilon(t, x)| \leqq 2\|\Phi\|P_x(\tau < t)$$

for $x \in G$. By (4.7), if $x \in K, 0 \leqq t \leqq t_1$, then

$$|\varphi_\mu^\epsilon(t, x) - \varphi^\epsilon(t, x)| = o(\epsilon^k)$$

for any k . Since expansion (3.5) holds for φ_μ^ϵ , it also holds for φ^ϵ , uniformly on $[0, t_1] \times K$.

Step 3. It remains to remove the assumption that Φ is bounded. Choose Φ_m bounded for each $m = 1, 2, \dots$, such that $\Phi_m = \Phi$ in G_m and $|\Phi_m| \leqq |\Phi|$. Let

$$\tilde{\varphi}_m^\epsilon(t, x) = E_x \Phi_m[\xi^\epsilon(t)].$$

Then by (3.1a) and dominated convergence, $\tilde{\varphi}_m^\epsilon \rightarrow \varphi^\epsilon$ as $m \rightarrow \infty$; also

$$|\tilde{\varphi}_m^\epsilon(t, x)| \leqq M(1 + E_x V[\xi^\epsilon(t)]^k).$$

By formula (A.2), the right side is bounded for (t, x) in any compact set. A repetition of the reasoning in Step 2 shows that

$$|\tilde{\varphi}_\mu^\epsilon(t, x) - \varphi^\epsilon(t, x)| \leqq 2M_1 P_x(\tau < t),$$

for suitable M_1 . Consequently, by (4.7)

$$|\tilde{\varphi}_\mu^\epsilon(t, x) - \varphi^\epsilon(t, x)| = o(\epsilon^k)$$

for any k . Since expansion (3.5) holds for $\tilde{\varphi}_\mu^\epsilon$, it also holds for φ^ϵ , uniformly on $[0, t_1] \times K$. This proves Theorem 4.2.

5. Results for stopped processes. Let us consider now the autonomous form of (4.1 $^\epsilon$):

$$(5.1^\epsilon) \quad \mathcal{L}^\epsilon \varphi^\epsilon + F(x, \varphi^\epsilon) = 0, \quad x \in B,$$

with boundary data

$$(5.2) \quad \varphi^\epsilon(x) = \Phi(x), \quad x \in \partial B.$$

Here B is a given open set. For $\epsilon = 0$, we get the first order equation

$$(5.1^0) \quad \mathcal{L}^0 \varphi^0 + F(x, \varphi^0) = 0.$$

The method of characteristics gives a C^∞ solution to (5.1 0) with the boundary data only for x in certain portions of B for which the trajectories ξ^0 exit from B nontangentially. More precisely, let \mathcal{O} be an open set, $N = \mathcal{O} \cap \bar{B}$, such that:

- (5.3) (i) $N \cap \partial B$ is a C^∞ manifold;
 (ii) if $x \in N$, then for the solution ξ^0 of (2.1 0) with $\xi^0(0) = x$ we have $\xi^0(t) \in N$ for $0 \leq t \leq \tau^0$ and $\dot{\xi}^0(\tau^0)$ not tangent to ∂B at $\xi^0(\tau^0)$.

The method of characteristics defines a C^∞ solution φ^0 to (5.1 0) in N , with $\varphi^0 = \Phi$ on $N \cap \partial B$.

We suppose that φ^ϵ is a bounded solution of (5.1 $^\epsilon$)–(5.2), of class C^2 in B and continuous in the closure \bar{B} . We seek an expansion (3.5) valid in N , where the coefficients $\theta_1(x), \theta_2(x), \dots$ satisfy in N the first order partial differential equations obtained by putting $(\theta_k)_t = 0$ in (4.3 0). For boundary data, $\theta_k(x) = 0$ on $N \cap \partial B$.

Formula (3.5) is the “regular expansion” in the theory of singular perturbations [4].

The proof of the following lemma is elementary. To emphasize the dependence on the initial data x , we write in this lemma $\tau^0 = \tau^0(x)$.

LEMMA 5.1. *Let K be a compact subset of N , and $t_1 > \tau^0(x)$ for all $x \in K$. Let $\tau = \tau(x)$ denote the minimum of t_1 and the exit time of ξ^ϵ from N . Then given $\eta > 0$ there exists a $\delta > 0$ such that $\|\xi^\epsilon - \xi^0\|_{t_1} \leq \delta$ implies $\xi^\epsilon(\tau(x)) \in \partial B$ and $|\tau(x) - \tau^0(x)| < \eta$.*

LEMMA 5.2. *If there exists M such that $|\varphi^\epsilon(x)| \leq M$ for all $x \in N$, then the expansion (3.5) holds uniformly on any compact subset of N .*

To prove Lemma 5.2, only the following slight changes in the proof of Theorem 4.1 are needed. Define τ as in Lemma 5.1, and θ_k^ϵ as in §4. We have as in (4.8)

$$\theta_k^\epsilon(x) = E_x \int_0^\tau D^\epsilon(s) [\Gamma_k^\epsilon + \text{tr } \alpha(\theta_{k-1})_{xx}] dx + E_x [D^\epsilon(\tau) \theta_k^\epsilon(\xi^\epsilon(\tau))],$$

where the integrand is evaluated at $\xi^\epsilon(s)$. When $\xi^\epsilon(\tau) \in \partial B$, $\theta_k^\epsilon[\xi^\epsilon(\tau)] = 0$; and $|\theta_k^\epsilon[\xi^\epsilon(\tau)]| \leq C_k \epsilon^{-k}$ when $\xi^\epsilon(\tau) \notin \partial B$. Let $a \leq$ minimum distance to $B - N$ from any point $\xi^0(s)$, $0 \leq s \leq t_1$, on a trajectory ξ^0 starting from $x \in K$. Then

$$(5.4) \quad P_x(\xi^\epsilon(\tau) \notin \partial B) \leq 2n \exp(-a\epsilon^{-1/2}\lambda)$$

for small ϵ , by Lemma 2.1. Moreover, by Lemmas 2.1 and 5.1, $\|\xi^\epsilon - \xi^0\|_{t_1} \rightarrow 0$ and $\tau \rightarrow \tau^0$ in probability as $\epsilon \rightarrow 0$, uniformly for initial data $x \in K$. By the same reasoning as for Theorem 4.1, $\theta_k^\epsilon(x) \rightarrow \theta_k(x)$ as $\epsilon \rightarrow 0$ uniformly on K . This proves Lemma 5.2.

For the homogeneous equation

$$(5.5^\epsilon) \quad \mathcal{L}^\epsilon \varphi^\epsilon = 0 \text{ in } B$$

with $\varphi^\epsilon = \Phi$ on ∂B , the maximum principle for elliptic equations implies that $\|\varphi^\epsilon\| \leq \|\Phi\|$. Therefore, by Lemma 5.2:

THEOREM 5.1. *Let Φ be bounded, and let N satisfy (5.3). Then the expansion (3.5) holds for bounded solutions of (5.5 $^\epsilon$)–(5.2), uniformly on each compact subset of N .*

We note that, for some choices of f , Theorem 5.1 cannot be applied. At points $y \in N \cap \partial B$, we must have $f(y) \cdot \nu(y) > 0$, where ν is the exterior unit normal to ∂B at y . One possibility examined in [20], is when $f(y) \cdot \nu(y) < 0$ for all $y \in \partial B$. In this case the drift term in (2.1 $^\epsilon$) impedes the movement of $\xi^\epsilon(t)$ towards ∂B . It is found in [20] that under certain assumptions $\varphi^\epsilon(x)$ tends to $\Phi(y^*)$ as $\epsilon \rightarrow 0$, where y^* is a point of ∂B making a certain functional on the unperturbed system trajectories minimum.

We next turn to the inhomogeneous equation. To obtain the result in this case an additional assumption about the mean exit time starting from any point in B is needed.

THEOREM 5.2. *Assume that: (1) F, Φ are bounded; (2) N satisfies (5.3); and (3) $E_x \tau^\epsilon \leq C\epsilon^{-1}$ for all $x \in B$, where τ^ϵ is the exit time of $\xi^\epsilon(t)$ from B . Then the expansion (3.5) holds uniformly on each compact subset of N .*

PROOF. Let K be a compact subset of N . There exists N_1 with properties (5.3) such that $K \subset N_1$ and $\bar{N}_1 \subset N$. We show that φ^ϵ is uniformly bounded on N_1 . The theorem will then follow from Lemma 5.2, with N replaced by N_1 there.

First of all, by Lemma 2.3

$$\varphi^\epsilon(x) = E_x \int_0^{\tau^\epsilon} F[\xi^\epsilon(s), \varphi^\epsilon(\xi^\epsilon(s))] ds + E_x \Phi[\xi^\epsilon(\tau^\epsilon)].$$

By assumptions (1) and (3) the crude estimate

$$(*) \quad |\varphi^\epsilon(x)| \leq C_1 \epsilon^{-1}, \quad x \in B,$$

holds for some C_1 . For $x \in N_1$, we next let $\tau = \min(t_1, \text{exit time of } \xi^\epsilon(t) \text{ from } N)$, where $t_1 > \tau^0(x)$ for all $x \in \bar{N}_1$. Again by Lemma 2.3,

$$\varphi^\epsilon(x) = E_x \int_0^\tau F[\xi^\epsilon(s), \varphi^\epsilon(\xi^\epsilon(s))] ds + E_x[\varphi^\epsilon(\xi^\epsilon(\tau))].$$

For $\xi^\epsilon(\tau) \in \partial B, \varphi^\epsilon(\xi^\epsilon(\tau)) = \Phi(\xi^\epsilon(\tau))$. Then

$$|\varphi^\epsilon(x)| \leq t_1 \|F\| + \|\Phi\| + \|\varphi^\epsilon\| P_x(\xi^\epsilon(\tau) \notin \partial B).$$

Let $a \leq \text{distance}(N_1, B - N)$ be small enough that $\|\xi^\epsilon - \xi^0\|_{t_1} \leq a$ implies $\xi^\epsilon(\tau) \in \partial B$ (Lemma 5.1). By Lemma 2.1 and (*)

$$\|\varphi^\epsilon\| P_x(\xi^\epsilon(\tau) \notin \partial B) \leq C_1 \epsilon^{-1} 2n \exp(-a\epsilon^{-1/2}\lambda),$$

which tends to 0 as $\epsilon \rightarrow 0$. Hence, $|\varphi^\epsilon(x)| \leq M$ for all $x \in N_1$ and suitable constant M . This proves Theorem 5.2.

REMARK. The proof shows that, instead of (3) we could equally well assume that

$$E_x \tau^\epsilon \leq C \epsilon^{-p} \quad \text{for some } p > 1.$$

It remains to discuss condition (3). Roughly speaking this condition is satisfied if the drift term f in (2.1 ϵ) does not have too strong a tendency to keep $\xi^\epsilon(s)$ inside B . For instance (3) cannot hold if there is some $B_1 \subset B$ such that $f(y) \cdot \nu_1(y) < 0$ for all $y \in \partial B_1$, where ν_1 is the exterior unit normal to ∂B_1 at y . Suppose, however, that there is a function h such that:

- (5.6) (i) In B, h is C^2 and $h(x) \geq 0$;
- (ii) In \bar{B}, h is continuous and bounded;
- (iii) $\mathcal{L}^\epsilon h \leq -c\epsilon$ in B , where $c > 0$.

By Lemma 2.3,

$$h(x) = -E_x \int_0^{\tau^\epsilon} \mathcal{L}^\epsilon h[\xi^\epsilon(s)] ds + E_x h[\xi^\epsilon(\tau^\epsilon)],$$

$$h(x) \geq c\epsilon E_x \tau^\epsilon.$$

This gives the bound $E_x \tau^\epsilon \leq c^{-1} \epsilon^{-1} h(x)$. Therefore, the existence of h with properties (5.6) implies condition (3).

EXAMPLE. Let $n = 1$, $B = (a, b)$ a finite interval. Suppose $\alpha(x) > 0$ on $[a, b]$, $f(x) < 0$ for $a \leq x < \bar{x}$, $f(x) > 0$ for $\bar{x} < x \leq b$. Choose h with $h(a) = h(b) = 0$, $h'' < 0$, $h'(\bar{x}) = 0$. Since $fh' \leq 0$,

$$\mathcal{L}^\epsilon h = \epsilon \alpha(x) h'' + f(x) h' \leq \epsilon \alpha(x) h'' \leq -c\epsilon$$

for some c . There is a regular expansion on $[a, x_1)$ and also one on $(x_2, b]$ if $x_1 < \bar{x} < x_2$. By Theorem 5.2 with $F = 1$, $E_x \tau^\epsilon$ is bounded except for x near \bar{x} . However, if $x = \bar{x}$ (an unstable critical point for the unperturbed system (2.1⁰)), $E_x \tau^\epsilon \rightarrow \infty$ as $\epsilon \rightarrow 0$.

6. Population genetics models. In population genetics theory, several factors are considered as affecting the change in genetic make-up of a population with time. Among them are selective advantages held by individuals of one genotype over another, and rates of new mutations and of migration between habitats. Another factor is the random fluctuation in gene frequencies due to sampling at each generation. When the population size N is so large that it can be taken as infinite, this factor is negligible; a deterministic model is then appropriate. However, for finite population size N a stochastic model is more appropriate.

The following model, taking into account finite population size has been studied considerably [2, Chaps. 8, 9]. Consider one gene locus at which genes of either type A_1 or type A_2 may occur. In a diploid population (chromosomes in pairs) the three possible genotypes are A_1A_1 , A_1A_2 , A_2A_2 . Let x denote the frequency of gene A_1 initially, and $\xi(t)$ the frequency of gene A_1 in the t th generation ($0 \leq \xi(t) \leq 1$). The gene frequency process ξ is modeled as a Markov chain, which is then approximated by a 1-dimensional diffusion process. The generator of this diffusion turns out to be of the form

$$(6.1) \quad \mathcal{L}\phi = \frac{1}{4N} x(1-x)\phi'' + [x(1-x)\ell(x) + L(x)]\phi',$$

where $\ell(x) = c_1x + c_2$, $L(x) = c_3x + c_4$ are linear functions. (Actually in (6.1) instead of N one should put the effective population size N_e .) The constants c_1, c_2 depend on the relative fitnesses of the three genotypes A_1A_1, A_1A_2, A_2A_2 . When these are equally fit (no selective advantage), $\ell = 0$. The constants c_3, c_4 depend on rates of new mutations of A_1 to A_2 or vice versa, or of migrations between the population in question and some outside pool of genes. When the effects of new

mutations and migration are ignored, $L = 0$. In the notation of the previous sections, $\mathcal{L} = \mathcal{L}^\epsilon$, $\xi = \xi^\epsilon$, with

$$(6.2) \quad \epsilon = \frac{1}{4N}, \alpha(x) = x(1-x), f(x) = x(1-x)\ell(x) + L(x).$$

It is assumed that N is constant (not dependent on t).

The probability distribution of $\xi(t)$ has been calculated in several cases, as an infinite series involving classical special functions [2, Chap. 8.4-8.6]. Instead of the probability distribution itself, one often wants expectations $E_x\Phi[\xi(t)]$ for various functions Φ . In many instances these expectations can be found (exactly or approximately) more directly, with the aid of formula (3.2'). To illustrate this, we give two examples. In the second example we get at the same time, a known approximation to the smallest eigenvalue associated with the problem.

EXAMPLE 1. Suppose $\ell(x) = 0$, $f(x) = L(x)$. In population genetics literature, the selection pressures are then said to be linear. Let

$$\begin{aligned} \mu(t) &= E_x\xi(t) \\ H(t) &= E_x\xi(t)(1 - \xi(t)). \end{aligned}$$

Then $\mu(t)$ is the mean gene frequency at time t ; the quantity $2H(t)$ is called the heterozygosity at time t . From μ, H one also knows the variance $v(t) = \mu(t) - H(t) - \mu(t)^2$. By (3.2') with $\Phi(x) = x$,

$$\frac{d\mu}{dt} = E_x\mathcal{L}\Phi = E_x f.$$

Since $f = L$, μ satisfies the differential equation

$$\frac{d\mu}{dt} = c_3\mu + c_4, \mu(0) = x.$$

To get a differential equation for H , we take $\Phi(x) = x(1-x)$. Then

$$\frac{dH}{dt} = E_x\mathcal{L}\Phi = E_x(f\Phi') + \epsilon E_x(\alpha\Phi''),$$

$$f\Phi' = c_4 - (c_3 + 2c_4)x + 2c_3x(1-x), \alpha\Phi'' = -2\Phi,$$

$$\frac{dH}{dt} = c_4 - (c_3 + 2c_4)\mu + (2c_3 - 2\epsilon)H, H(0) = x(1-x).$$

In case $f = 0$ (no selection pressure), this gives in particular the well known rate of decrease of heterozygosity $H(t) = H(0)\exp(-t/2N)$ since $2\epsilon = (2N)^{-1}$. See [2, p. 329].

EXAMPLE 2. Suppose $L = 0$, $f(x) = x(1-x)\ell(x)$. In this case new mutations and migrations are neglected. The nonlinearity of f makes matters more difficult. Let us indicate a technique for calculating the mean heterozygosity approximately when $\ell \ll \epsilon$ (i.e., $N\ell \ll 1$). For simplicity we carry out the calculations only when $c_1 = 0$, $\ell = \text{constant}$. In genetical terms, this means that the fitness of the heterozygote A_1A_2 is the average of the fitness of the two homozygotes A_1A_1 and A_2A_2 , [2, p. 396]. Besides the function H in Example 1, we introduce two other functions

$$H_1(t) = E_x(f\Phi') = \ell E_x(\Phi\Phi'),$$

$$H_2(t) = E_x(f^2) = \ell^2 E_x\Phi^2,$$

$$\Phi(x) = \alpha(x) = x(1-x),$$

where each expression inside an expectation E_x is evaluated at $\xi(t)$. After calculating $\mathcal{L}(\Phi\Phi')$, $\mathcal{L}\Phi^2$, and using (3.2') we get

$$\frac{dH}{dt} = -2\epsilon H + H_1,$$

$$\frac{dH_1}{dt} = \ell^2 H - 6\epsilon H_1 - 6H_2,$$

$$\frac{dH_2}{dt} = 2\epsilon\ell^2 H - 12\epsilon H_2 + 2\ell^3 E_x(\Phi^2\Phi').$$

Since $\ell \ll \epsilon$, it can be seen that the term $2\ell^3 E(\Phi^2\Phi')$ can be neglected. One has then three linear differential equations for H, H_1, H_2 . The solutions are linear combinations of exponentials $e^{\lambda_i t}$, where $\lambda_1, \lambda_2, \lambda_3$ are the characteristic roots of the coefficient matrix

$$\begin{pmatrix} -2\epsilon & 1 & 0 \\ \ell^2 & -6\epsilon & -6 \\ 2\epsilon\ell^2 & 0 & -12\epsilon \end{pmatrix}.$$

They satisfy the equation

$$0 = (\lambda + 2\epsilon)(\lambda + 6\epsilon)(\lambda + 12\epsilon) - \ell^2\lambda.$$

If we denote by λ_1 the root of smallest absolute value, then with the aid of the implicit function theorem one finds that approximately for $\ell \ll \epsilon$

$$\lambda_1 = -\frac{1}{2N} \left(1 + \frac{2}{5} (N\ell)^2 \right).$$

For $|\lambda_1|t$ large, the mean heterozygosity $2H(t)$ decreases exponentially at rate nearly $|\lambda_1|$. Another interpretation for $|\lambda_1|$ is as the smallest eigenvalue of a boundary problem for the forward partial differential equation of the ξ processes, stopped when the gene frequency $\xi(t)$ reaches either endpoint of the interval $[0, 1]$. This follows from an asymptotic formula [2, p. 397] for the probability density of $\xi(t)$ when $|\lambda_1|t$ is large. Our formula for λ_1 agrees with the one in [2, p. 398].

If λ is not constant, the same technique leads to five linear ordinary differential equations.

If $\epsilon \ll \lambda$ (i.e., $N\lambda \gg 1$), the mean heterozygosity in Example 2 could be estimated using the expansion (3.5) in powers of ϵ . Times t of order λ^{-1} are of interest in this case; for larger times the gene frequency $\xi(t)$ has reached 0 or 1 with probability very nearly 1. (When $\epsilon \ll \lambda$, the accuracy of the diffusion approximation to the genetical model is in doubt.)

Other questions about the gene frequency process ξ can be answered by solving an ordinary differential equation

$$(6.3) \quad \mathcal{L}\phi + g(x) = 0, \quad 0 < x < 1,$$

with given data for $\phi(0)$ and $\phi(1)$. Here \mathcal{L} is the second order operator in (6.1). As in Example 2 we take $f = \lambda x(1 - x)$. If $g = 0$, $\phi(0) = 0$, $\phi(1) = 1$, then $\phi(x)$ is the probability that gene A_2 becomes extinct eventually when A_1 initially has frequency x . If $g(x) = 1$, $\phi(0) = \phi(1) = 0$, then $\phi(x)$ is the mean time until A_1 or A_2 becomes extinct. If $g(x) = x(1 - x)$, $\phi(0) = \phi(1) = 0$, then

$$\phi(x) = \int_0^\infty E_x \xi(t)(1 - \xi(t)) dt$$

and $2N\phi(x)$ is the total mean number of heterozygotes.

An effective solution of the seemingly innocuous equation (6.3) is not generally easy. However, $\phi(x)$ is known at least approximately for various cases of interest [11, Chap. 1] [15]. We discuss here some cases when $\phi(x)$ can be found approximately from the expansion in § 5, when $\epsilon = (4N)^{-1}$ is small. Let us take as boundary data $\phi(0) = \phi(1) = 0$. Then the probabilistic solution ϕ of (6.3) is

$$(6.4) \quad \phi(x) = E_x \int_0^\tau g[\xi(t)] dt,$$

where τ is the time when $\xi(t)$ first reaches 0 or 1. Let the relative fitness of the three genotypes A_1A_1 , A_1A_2 , A_2A_2 be respectively 1, $1 - h$,

$1 - s$, where $s > 0$. Then $\ell(x) = s[h + (1 - 2h)x]$; in our previous notation, $c_1 = sh, c_2 = s(1 - 2h)$.

Case 1. $0 \leq h \leq 1$. Then $\ell(x) > 0$ for $0 < x \leq 1$. If $g(1) \neq 0$, there is no regular expansion (3.5) of $\varphi(\approx \varphi^\epsilon)$ in powers of ϵ . In fact, the first term in such an expansion could not be finite, since $0 < \xi^0(t) < 1$ for every t and $\xi^0(t) \rightarrow 1$ as $t \rightarrow \infty$. Let us assume, therefore, that $g(1) = 0$, and show that there is a regular expansion (3.5) on $[b, 1]$ for any $b > 0$. Let us divide (6.2) by $1 - x$, obtaining

$$\begin{aligned} \mathcal{L}^*\phi + g^* &= 0, \quad 0 < x < 1, \\ \mathcal{L}^*\phi &= \epsilon x \phi'' + x \ell(x) \phi', \quad g(x) = (1 - x)g^*(x). \end{aligned}$$

Consider, instead of ξ , a new diffusion ξ^* with generator \mathcal{L}^* , $f^*(x) = x \ell(x)$, $\alpha^*(x) = x$. The assumptions (5.3) are satisfied by f^* on $(b, 1]$. By the Remark after Theorem 5.2 there is a regular expansion on $(b, 1]$ provided $E_x \tau^* \leq C\epsilon^{-2}$ for some C , and for $0 < x < 1$. Here τ^* is the exit time of $\xi^*(t)$ from $(0, 1)$. Let

$$\psi(x) = \epsilon^2 E_x \tau^*, \quad m(x) = \epsilon^{-1} \int_0^x \ell(y) dy.$$

Since $\ell > 0$, m is increasing. Moreover,

$$(\#) \quad \psi'' + m' \psi' + \frac{\epsilon}{x} = 0.$$

The following function $\tilde{\psi}$ is also a solution of (#):

$$\tilde{\psi}(x) = -\epsilon \int_0^x e^{-m(y)} \left[\int_0^y \frac{e^{m(z)} - 1}{z} dz + \log y \right] dy.$$

By the mean value theorem, monotonicity of m , and $m' = \epsilon^{-1}\ell$, we have

$$\frac{e^{m(z)} - 1}{z} \leq \epsilon^{-1} \|\ell\| e^{m(z)}.$$

It follows that $|\tilde{\psi}(x)| \leq (1/2)C$, for some C not depending on ϵ . Then $(1/2)C + \tilde{\psi}$ is a nonnegative solution of (#). Since ψ is also a solution of (#), with $\psi(0) = \psi(1) = 0$, we have $\psi \leq (1/2)C + \tilde{\psi} \leq C$. Thus $E_x \tau^* \leq C\epsilon^{-2}$ as required.

Case 2. $h > 1$. Assume that $g(0) = g(1) = 0$, and divide (6.2) by $x(1 - x)$. Then

$$\begin{aligned} \mathcal{L}^{**}\phi + g^{**} &= 0, \quad 0 < x < 1, \\ \mathcal{L}^{**}\phi &= \epsilon \phi'' + \ell(x) \phi', \quad g(x) = x(1 - x)g^{**}(x). \end{aligned}$$

This is the situation in the Example at the end of § 5, if we take $a = 0$, $b = 1$ there $\alpha^{**}(x) = 1$, $f^{**}(x) = \ell(x)$, $\bar{x} = h(2h - 1)^{-1}$. In this case, there is a regular expansion about each of the endpoints 0, 1.

Case 3. $h < 0$. The heterozygote A_1A_2 is more fit than either homozygote (overdominance). The term $f dt$ in (2.1 $^\epsilon$) has a strong tendency to drive $\xi(t)$ toward \bar{x} , which is a stable equilibrium point when $\epsilon = 0$ (in Case 2, \bar{x} is unstable). The mean exit time $E_x\tau^\epsilon$, for x outside a neighborhood of the endpoints 0, 1, is at least of order $\exp(c\epsilon^{-1})$ for some $c > 0$. There is no regular expansion. On the contrary, from (6.4) one can see that $\phi(x)$ is also extremely large if $g(x) > 0$ for $0 < x < 1$.

Multidimensional genetic models. The model considered above dealt with two kinds of gene A_1 or A_2 (called alleles) at a single locus. The resulting diffusion is 1-dimensional, if the population size N is assumed fixed. Models leading to multidimensional diffusions arise in several ways. For instance, the diffusion approximation will be multi-dimensional in case of several alleles at a single locus, or if more than one gene locus is considered, or if different gene frequencies at different habitats are taken into account.

If \mathcal{L} denotes the generator of an n -dimensional diffusion approximation in one of these models, frequently the second order coefficients $\alpha_{ij}(x)$ are quadratic in $x = (x_1, \dots, x_n)$. If, moreover, f is linear, then the means and covariances of $\xi_1(t), \dots, \xi_n(t)$ satisfy first order differential equations. These equations are found in a way similar to that for Example 1, § 3 and Example 1, § 6. There are also interesting results for 2 locus models. In particular, the mean and variance of the linkage disequilibrium have been computed, by methods in the same spirit as for Example 2, § 6. See [11, Chap. 7]. However, it seems that much remains to be done in the study of multidimensional genetic models.

APPENDIX

We outline here proofs of some statements made in § 2. It is always assumed that $0 \leq \epsilon \leq 1$. For notational simplicity, we write in the Appendix $\xi^\epsilon = \xi$, $\mathcal{L}^\epsilon = \mathcal{L}$, not showing their dependence on ϵ .

First of all, given any smooth coefficients f, σ , a solution of (2.1 $^\epsilon$) can be constructed in the following way, up to some "explosion time" τ^∞ . Let $G_1 \subset G_2 \subset \dots$ be open bounded sets with union R^n , and f_m, σ_m satisfying the Ito conditions for $m = 1, 2, \dots$ such that

$$f_m(x) = f(x), \sigma_m(x) = \sigma(x), \quad x \in G_m.$$

For these coefficients there is a solution process ξ_m . Moreover, if $x \in G_\mu$ with $\mu < m$, then with probability 1, $\xi_\mu(t) = \xi_m(t)$ for $0 \leq t \leq$

τ_μ where τ_μ is the exit time from G_μ . See [3, Chap. 11, § 3]. Define τ^∞ as the limit of the monotone sequence τ_m , and $\xi(t) = \xi_m(t)$ for large enough m and $0 \leq t < \tau^\infty$. It is known that assumptions (2.4a)(2.4c) guarantee that $\tau^\infty = \infty$, and in fact that $E_x V[\xi(t)] < \infty$ for any $t \geq 0$. See [13, Chap. II.2, Chap. III]. For completeness, we include the following proof. Take

$$f_m = f\beta_m^2, \sigma_m = \sigma\beta_m, G_m = \{x : V(x) < m\},$$

where $0 \leq \beta_m \leq 1$, $\beta_m(x) = 1$ for $x \in G_m$, $\beta_m(x) = 0$ for $x \notin G_{m+1}$. The generator \mathcal{L}_m of the process ξ_m satisfies $\mathcal{L}_m = \beta_m^2 \mathcal{L}$, and hence $\mathcal{L}_m V \leq M(1 + V)$ with the same constant M in (2.4a). Let

$$\rho_m(t) = E_x V[\xi_m(t)].$$

From the Ito stochastic differential rule we have (as in formula 3.2)

$$\rho_m(t) = \rho_m(0) + \int_0^t E_x \mathcal{L}_m V[\xi_m(s)] ds,$$

and therefore by (2.4a)

$$\rho_m(t) \leq \rho_m(0) + M \int_0^t (1 + \rho_m(s)) ds.$$

Since $\rho_m(0) = V(x)$, Gronwall's inequality gives the bound independent of m :

$$(*) \quad \rho_m(t) \leq (V(x) + Mt)e^{Mt}.$$

Let

$$W(t, x) = e^{Mt}V(x) + e^{Mt} - 1.$$

Then $-W_t + \mathcal{L}_m W \leq 0$, which implies that $W(t - s, \xi_m(s))$ is a non-negative supermartingale. Therefore,

$$P_x \left(\max_{0 \leq s \leq t} W(t - s, \xi_m(s)) \geq m \right) \leq \frac{1}{m} E_x W(0, \xi_m(t)).$$

The right side is $m^{-1}\rho_m(t)$. Since $V \leq W$ and τ_m is the exit time from $\{V < m\}$,

$$P_x(\tau_m > t) \leq m^{-1}\rho_m(t).$$

By (*), $P_x(\tau_m > t) \rightarrow 0$ as $m \rightarrow \infty$. Hence $\tau^\infty = \infty$. Since $V[\xi_m(t)] = V[\xi(t)]$ for large enough m (with probability 1), Fatou's lemma together with (*) imply

$$(A.1) \quad E_x V[\xi(t)] \leq (V(x) + Mt)e^{Mt}.$$

Let us next show, for any positive integer ℓ , a similar estimate

$$(A.2) \quad E_x V^\ell [\xi(t)] \leq (V^\ell(x) + M_\ell t) e^{M_\ell t}.$$

For this it suffices to show that V^ℓ also satisfies (2.4a) with some constant M_ℓ . By direct calculation,

$$\mathcal{L}V^\ell = \ell V^{\ell-1} \mathcal{L}V + \ell(\ell - 1) V^{\ell-2} V_x \alpha V_x.$$

By (2.3), $|\alpha(x)| \leq M^2(1 + |x|^2)$; while by (2.4b)

$$(1 + |x|)^2 |V_x|^2 \leq M^2(1 + V)^2,$$

$$V^{\ell-2} V_x \alpha V_x \leq V^{\ell-2} |V_x|^2 |\alpha|,$$

which gives with suitable M_ℓ

$$\mathcal{L}V^\ell \leq M_\ell (1 + V^\ell),$$

as required.

PROOF OF LEMMA 2.1. Given any compact set $K \subset R^n$ and $t_1 > 0$ let K_1 be the compact set consisting of all points $\xi^0(s)$, $0 \leq s \leq t_1$, on the solution ξ^0 of (2.1⁰) starting from some $x \in K$. Given $a > 0$ and $0 \leq t \leq t_1$, $P_x(\|\xi^\epsilon - \xi^0\|_t > a)$ is unaffected if we change f, σ outside the a -neighborhood of K_1 . Hence we may assume that f, σ are bounded and Lipschitz on R^n .

Let us write (2.1^{\epsilon}), (2.1⁰) in integrated form with the initial data (2.2) and subtract them. We get

$$\xi^\epsilon(t) - \xi^0(t) = \int_0^t [f(\xi^\epsilon(s)) - f(\xi^0(s))] ds + \epsilon^{1/2} \zeta(t),$$

$$\zeta(t) = 2^{1/2} \int_0^t \sigma(\xi^\epsilon(s)) dw.$$

By standard estimates,

$$\|\xi^\epsilon - \xi^0\|_t \leq \epsilon^{1/2} \|\zeta\|_t e^{kt},$$

where k is a Lipschitz constant for f . Thus,

$$P_x(\|\xi^\epsilon - \xi^0\|_t > a) \leq P_x(\epsilon^{1/2} \|\zeta\|_t e^{kt} > a).$$

To estimate the right side, we use an exponential martingale inequality. For $\gamma \in R^n$ with $|\gamma| = 1$, let

$$\chi(t) = \gamma \cdot \zeta(t) - \int_0^t |\gamma \sigma|^2 ds.$$

Then [16, p. 25]

$$P_x(\max_{0 \leq s \leq t} \chi(s) > c) \leq e^{-c}.$$

Since σ is bounded, $|\gamma\sigma|^2 \leq B$ for some B . Hence $\gamma\xi(s) > c + Bt$ implies $\chi(s) > c$. Take $\gamma = \pm e_i$, where e_1, \dots, e_n is an orthonormal basis for R^n . Then $\|\xi\|_t > d$ implies $\max_{0 \leq s \leq t} \gamma\xi(s) > n^{-1/2}d$ for at least one of these $2n$ choices for γ . By taking $d = n^{1/2}(c + Bt)$, we get

$$P_x(\|\xi\|_t > n^{1/2}(c + Bt)) \leq \sum_{i=1}^n [P_x(\max \zeta_i(s) > c + Bt) + P_x(\min \zeta_i(s) < -c - Bt)] \leq 2ne^{-c}.$$

Take

$$c = a\epsilon^{-1/2}\lambda, \lambda = \frac{1}{2}n^{-1/2}e^{-kt_1}.$$

For small enough ϵ ,

$$n^{1/2}(c + Bt) \leq a\epsilon^{-1/2}e^{-kt}, \quad 0 \leq t \leq t_1,$$

$$P_x(\|\xi^\epsilon - \xi^0\|_t > a) \leq P_x(\|\xi\|_t > n^{1/2}(c + Bt)) \leq 2n \exp(-a\epsilon^{-1/2}\lambda).$$

This proves Lemma 2.1.

PROOF OF LEMMA 2.2. *Step 1.* Suppose that $A, \psi_t, \psi_{x_t}, \psi_{x_t x_t}$ are continuous and bounded everywhere. Then

$$d(D\psi) = (dD)\psi + Dd\psi,$$

where ψ is evaluated at $(t - s, \xi(s))$ for fixed t . The ordinary product rule holds since the paths $D(\cdot)$ are C^1 functions of time. Now

$$dD = AD \, ds,$$

$$d\psi = (-\psi_t + \mathcal{L}\psi) \, ds + (2\epsilon)^{1/2}\psi_x \sigma \, dw,$$

$$d(D\psi) = D[-\psi_t + \mathcal{L}\psi + A\psi] \, ds + D(2\epsilon)^{1/2}\psi_x \sigma \, dw.$$

We take $E_x \int_0^\tau$ of both sides, and use $D(0) = 1, \xi(0) = x$, and (2.5). Since $D\psi_x \sigma$ is bounded and τ nonanticipative

$$E_x \int_0^\tau D\psi_x \sigma \, dw = 0.$$

This gives (2.6) under our present assumptions on A, ψ_t, \dots .

Step 2. To obtain (2.6) under the assumptions in Lemma (2.2), consider open sets G_1', G_2', \dots with union $G, \bar{G}_m' \subset G_{m+1}'$, and \bar{G}_m' compact. There exist A_m, ψ_m satisfying the assumptions in Step 1 and agreeing on G_m' with A, ψ . Then (2.6) holds with τ replaced by

$\tau_m' = \min(t, \text{exit time of } \xi(s) \text{ from } G_m')$. Moreover, τ_m' increases to τ as $m \rightarrow \infty$. We get (2.6) from the dominated convergence theorem. This proves Lemma 2.2.

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