## RECESSION OF SOME RELATIVISTIC MARKOV PROCESSES ${ }^{1}$

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1. Introduction. This paper is intended to be expository. Most of the precise formulations and all the proofs are to be published elsewhere [5].

Before taking up relativistic processes let us consider random motions in ordinary Euclidean spaces $R^{k}$. For simplicity let $x_{t}$ be the standard Brownian motion with values in $R^{k}, 0 \leqq t<\infty$, i.e. the coordinates of $x_{t}$ are independent 1-dimensional Wiener processes. Then for $k=1,2, x_{t}$ is recurrent, i.e. it returns to any non-empty open set for arbitrarily large times with probability 1 . For $k \geqq 3, x_{t}$ is no longer recurrent, and $\left|x_{t}\right| \rightarrow \infty$ as $t \rightarrow \infty$ (Ito and McKean [7], p. 236).

However, the angular or directional part $x_{t}| | x_{t} \mid$ is recurrent in the unit sphere $S^{k-1}$ for all $k$. I will give a proof of this known fact, to contrast it with the relativistic situation in which it fails, as will be seen.

For a fixed, even if large, time $T$, the asymptotic behavior of $x_{t}| | x_{t} \mid$ as $t \rightarrow \infty$ is independent of $x_{s}$ for $s \leqq T$, since $\left|x_{t}\right| \rightarrow \infty$. Thus the probability that $x_{t} /\left|x_{t}\right|$ visits a non-empty open set $U \subset S^{k-1}$ for arbitrarily large $t$ is at least $\sigma(U) / 2$ where $\sigma$ is the normalized orthogonally invariant Borel measure on $S^{k-1}$. This is true even if we condition the probability on any values of $x_{s}$ for $s \leqq T$. Thus there exist times $T_{n} \rightarrow \infty$ fast enough so that

$$
\begin{gathered}
\operatorname{Pr}\left\{x_{i} /\left|x_{t}\right| \notin U \text { for } T_{m} \leqq t<T_{m+1}, m=r, \cdots, n\right\} \\
\leqq(1-\sigma(U) / 4)^{n-r} \rightarrow 0 \text { as } n \rightarrow \infty
\end{gathered}
$$

The proof just concluded makes no use of the Gaussian distribution or other special properties of Brownian motion. We used an isotropic property to bring in $\boldsymbol{\sigma}(U)$, but the same conclusion holds for a broad class of motions in $R^{k}$, for which $\sigma(U)$ can be replaced by some fixed positive number depending on $U$, a condition we might call "quasiisotropic." We also used the facts that $\left|x_{t}\right| \rightarrow \infty$ and that $x_{t}$ has independent increments.

Thus, recurrence of $x_{t} /\left|x_{t}\right|$ in $S^{k-1}$ can be considered typical of reasonable random processes $x_{t}$ in $R^{k}$. The main point of this paper is

[^0]that if we replace Newtonian space $R^{3}$ by the relativistic velocity space or Lobachevsky space $\mathcal{U}$, then recurrence of the direction of motion becomes far from typical. In fact, for reasonable random processes the direction converges as $t \rightarrow \infty$. The reasons have to do with the metric structure of $\mathcal{U}$, which is homeomorphic to $R^{3}$ but metrically very different, especially in the neighborhood of $\infty$.
2. Relativity and the Markov property. Now let us consider what random motions are allowed by relativity theory together with the Markov property. The two basic principles of special relativity are:
(1) Physical motions faster than the speed $c$ of light are impossible (we assume henceforth units chosen so that $c=1$ ), and $c$ is an absolute constant.
(2) The laws of physics are the same in any two inertial coordinate systems (both in a state of uniform, unaccelerated motion).
In our inertial frames we have ordinary orthonormal space coordinates $(x, y, z)$ and a time $t$. Because of (1) and (2), it turns out that the transformation of co-ordinates between inertial systems affects not only the space variables but also the time. Thus, events can be called simultaneous only in a particular coordinate system.

Given a trajectory (in space-time) with speeds less than 1 everywhere, we can carry a clock along it and parametrize it by the time $\tau$ shown by the clock as it passes each point. Thus, on such a trajectory we have a time measurement which does not depend on the coordinate system. Given an inertial system $(x, y, z, t)$, it turns out that

$$
\begin{equation*}
d \tau=\left(d t^{2}-d x^{2}-d y^{2}-d z^{2}\right)^{1 / 2} \tag{3}
\end{equation*}
$$

([2], p. 44; [10], pp. 99-102). This $\tau$ is called the proper time on the trajectory.

As a side remark, we note that Arens [1] has classified certain sets of relativistic "particles." From his viewpoint, the kinds of particles we consider are just the "familiar point particles." He gives eight other types of particles, of which only one has speeds less than 1.

Now we consider random processes $\xi(t, \omega)$, where $\omega \in \Omega$ for some probability space $(\Omega, \delta, P)$. The Markov property asserts that conditional on $\xi(t, \omega)$, for each fixed $t$, the families $\{\xi(s, \omega): s<t\}$ and $\{\xi(u, \omega): u>t\}$ are independent.

If we let $\xi=(x, y, z)$ and consider processes in $R^{3}$, we find that the existence of finite velocities $d \xi / d t$ imposed by (1) gives a dependence between past and future which conflicts with the Markov property, except under special conditions such as flows along fixed trajectories,
where $\xi(t, \omega)$ determines $d \xi(t, \omega) / d t$. These seem uninteresting as random motions since there is not much room for randommess.

One way around this difficulty, which we will use, is to enlarge the state space of the processes to include a velocity component. Thus we might consider processes of the form $\langle\xi(t), d \xi(t) / d t\rangle$ in $R^{6}$. There is a large class of such processes satisfying (1) and the Markov property. Actually it is more convenient to reparametrize processes by proper time, giving us space-time functions

$$
p(\tau)=\langle x(\tau), y(\tau), z(\tau), t(\tau)\rangle
$$

and trajectories $\left\langle p(\tau), p^{\prime}(\tau)\right\rangle$ in phase space. By (3), $p^{\prime}(\tau)$ always belongs to the 3 -dimensional hyperboloid

$$
\mathcal{U}=\left\{(x, y, z, t): t=\left(x^{2}+y^{2}+z^{2}+1\right)^{1 / 2}\right\} .
$$

Since we can recover $p(\tau)$ from $p^{\prime}(\cdot)$ and $p(0)$, we may as well restrict attention to the process $p^{\prime}(\tau)(\omega)$ in $\mathcal{U}$.

Let $\mathcal{L}$ be the group of proper Lorentz transformations of $R^{4}$, each taking one inertial frame into another ([10], p. 41). This group acts on $\mathcal{U}$. We can represent $\mathcal{U}$ as the homogeneous, symmetric space $\mathcal{L} / K$ where $K$ is the maximal compact subgroup of $\mathcal{L}$ leaving the point $p=$ $(0,0,0,1)$ fixed. Thus $K$ is an orthogonal group $\mathrm{O}(3)$.

Markov processes in $U$ with stationary, $\mathcal{L}$-invariant transition probabilities were treated in [3] and [4] and classified by a generalized Lévy-Khinchin formula due to Tutubalin [11] and Gangolli [6]. Except in the trivial case $p^{\prime}(\tau)=$ constant, the $\mathcal{L}$-invariant processes are all transient, i.e. $p^{\prime}(\tau)$ goes to $\infty$ in $\mathcal{U}$ as $\tau \rightarrow \infty$ almost surely ([3], Theorem 9.2). Actually, for any compact $K \subset u, \operatorname{Pr}\left(p^{\prime}(\tau) \in K\right) \rightarrow 0$ exponentially fast as $\tau \rightarrow \infty$ [3, Lemma 9.3], unlike the case of homogeneous stationary processes $X$ in $R^{k}$ where $\operatorname{Pr}(X(t) \in K)$ is often of the order of $t^{-k / 2}$ as $t \rightarrow \infty$.

A further difference in the relativistic case is that as shown in [5, Theorem 3] and further discussed below, the direction of $p^{\prime}(\tau)$ converges as $\tau \rightarrow \infty$, for any of the Lorentz-invariant processes mentioned above. When this happens, we shall say that the process recedes.

To understand why processes in $u$ tend to recede, we consider the metric structure of $\mathcal{U}$. For each $u \in \mathcal{U}$ we can choose an inertial frame in which $u$ represents rest. Then the usual Euclidean metric induces a Riemannian metric on the tangent space to $\mathcal{U}$ at $u$. This Riemann structure defines a metric $\rho$ on $\mathcal{U}$.

For any $u \in \mathcal{U}$ and $r>0$, the sphere

$$
S_{r}(u)=\{q \in \mathcal{U}: \rho(u, q)=r\}
$$

is isometric to an ordinary Euclidean sphere, but of radius $\sinh r$ [3, p. 257]. Thus the circumference of such a sphere grows exponentially as $r \rightarrow \infty$. Hence a randomly moving point in $\mathcal{U}$, if it has no particular preference for any direction but moves through on the average about the same Riemann distance in different directions, will tend to move outward to larger spheres much more easily than it will circumnavigate spheres.

Recession holds not only for the invariant processes of [3] and [4] ([5], Theorem 3) but also for any process such that the transition probabilities starting from each point $u$ are invariant under the orthogonal group leaving $u$ fixed [5, Corollary of Theorem 2]. The hypotheses can be weakened substantially further. Instead of invariance we only need a partial balancing of transition probabilities in different directions, as follows.

Let $P_{\tau}(\xi, u, A)=\operatorname{Pr}\left\{p^{\prime}(\tau) \in A \mid p(0)=\xi, p^{\prime}(0)=u\right\}$, where the conditional probability is more precisely a transition probability associated with the Markov process $\left\langle p(\tau), p^{\prime}(\tau)\right\rangle$. We always have a decomposition

$$
P_{\tau}(\xi, u, \cdot)=\int_{0}^{\infty} Q_{\tau}(\xi, u, r, \cdot) P_{\tau}^{\prime}(\xi, u, d r)
$$

where $P_{\tau}{ }^{\prime}(\xi, u,[a, b))=P_{\tau}(\xi, u,\{w \in \mathscr{U}: a \leqq \rho(u, w)<b\})$, and $Q_{\tau}$ is a probability measure on the sphere $S_{r}(u)$.

We shall use the following three assumptions:
(I) Each $Q_{\tau}$ has a density $q_{\tau}(\xi, u, r, \cdot)$ with respect to the rotationally invariant Borel probability measure $d \sigma$ on $S_{r}(u)$, and $q_{\tau}$ is strictly positive and bounded away from 0 for $u$ in bounded sets.
(II) Let $\bar{q}(\rho)=\sup \left\{q_{\tau}(\xi, u, r, \sigma): \tau \geqq 0, \quad \xi \in R^{4}, \quad \rho(u, p) \leqq \rho\right.$, $\left.r \geqq 0, \sigma \in S_{r}(u)\right\}$. Then $\lim _{\rho \rightarrow \infty} e^{-\rho} \bar{q}(\rho)=0$.
(III) There is some $\beta>0$ such that for any $\tau \geqq 0, \xi \in R^{4}$, and $u \in \mathcal{U}$, letting $L$ be any 3 -dimensional linear subspace of $R^{4}$ containing $0, p$ and $u$, and either closed half-space $H$ of $R^{4}$ bounded by $L$, we have $P_{\tau}(\xi, u, H \cap u) \geqq \beta$.

A proof that (I), (II) and (III), together with standard regularity conditions for Markov processes, imply convergence of velocities $p^{\prime}(\tau)$ as $\tau \rightarrow \infty$, is given in [5, Theorem 2]. Here is a sketch of the proof.

Either $p^{\prime}(\boldsymbol{\tau})$ converges to a finite limit as $\tau \rightarrow \infty$, in which case of course its direction also converges, or it moves through some positive $\rho$-distance infinitely often. In the latter case, (I) implies that $p^{\prime}$ does not remain bounded as $\tau \rightarrow \infty$, since it always has a positive chance to change in any direction, including directions which carry it out
toward $\infty$. Then (II) is used to show that $p^{\prime}(\tau) \rightarrow \infty$, without returning infinitely often into compact sets. One might say that the space $\mathcal{U}$ has an exponential bias toward recession, and (II) says that the transition probabilities are less than exponentially biased, so they cannot overcome the inherent tendency toward recession. Finally, (III) is used to show that $p^{\prime}$ does not spiral around as it goes to $\infty$.

Here is a further explanation of what (III) says. If we use a coordinate system in which we start from rest, and later we have reached some velocity, then we may have some bias toward turning left rather than right, or up rather than down, but this bias must remain uniformly bounded. We may prefer deceleration to acceleration as long as the preference is less than exponential, according to (II).
3. Motion of galaxies. It is known that galaxies are receding from each other, with relatively few exceptions. It seems natural to suppose that galaxies, perhaps especially in their formative periods, are subject to influential random influxes of new matter. The process of accretion of matter into galaxies is apparently continuing up to the present [9]. It is also known that on occasion galaxies collide. These effects make it at least conceivable that the observed recession of galaxies need not be extrapolated all the way back to the extreme of a closely packed fireball. Since recession in a limiting direction is the natural mode of behavior for relativistic random processes under reasonable conditions, we may well ask whether such conditions may hold for the galaxies. We would need to consider general relativity, i.e. global gravitational effects, which are not yet included in this treatment.

If an infinite space were on the average uniformly filled with a gas of positive, finite density, in equilibrium, then the velocities of the particles in the gas should have a relativistic Maxwell-Boltzmann distribution [8]. At times far enough apart to allow for many collisions on the average for one particle, its velocities would be nearly independent. Thus, for positive density we do not have recession. To allow for recession, then, it seems reasonable to suppose that the density of mass approaches zero as we consider larger and larger regions, as is true to the best of our knowledge [12].

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