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STIELTJES TYPE CONTINUED FRACTIONS IN QUANTUM ELECTRODYNAMICS

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ABSTRACT. The exactly solvable vacuum polarization by an external constant electromagnetic field is examined. It is proven that the Stieltjes type continued fraction corresponding to the divergent perturbation expansion in the general case does not converge to the exact solution, the convergence taking place only for the vacuum polarization for a pure electric or a pure magnetic field. The summability of the perturbation expansion to the exact solution under the Borel method is also proven.

In this communication some results will be reported concerning the divergent perturbation expansion of the vacuum polarization by an external constant electromagnetic field and its summability to the exact solution through the Stieltjes type continued fraction and the (generalized) Borel method.

The proof of the statements presented can be found in [1] and [2].

The complete Lagrangian due to the vacuum polarization by an external constant electromagnetic field, as it has been computed by Schwinger [3] reads:

$$L = -F - L_1 \tag{1}$$

$$L = -F - \frac{1}{8\pi^2}\int_0^\infty \frac{e^{-s}}{s^3}[(es)^2G\,\frac{\mathrm{Re}\cosh(esX)}{\mathrm{Im}\cosh(esX)} - 1 - \frac{2}{3}(es)^2F]\,ds \tag{2}$$

where : e is the electron charge; $F = 1/4 F_{\mu\nu}^2 = 1/2(\bar{H}^2 - \bar{E}^2)$ is the free electromagnetic field Lagrangian; $G = \bar{F} \circ \bar{H}$ is the pseudoscalar electromagnetic field invariant; $X = (2(F + iG))^{1/2}$, and the electron mass has been put equal to 1.

The first step in proving our statements is the following:

LEMMA 1. *The interaction Lagrangian* (2) *may be written under the form*:

$$L_I(\alpha) = 2\alpha^2\int_{-\infty}^{+\infty} \frac{\psi(t)\,dt}{1+\alpha t} \tag{3}$$

$\psi(t)$ being positive in $(-\infty, +\infty)$ with finite moments given by the formula:

$$A_m = \int_{-\infty}^{+\infty} t^m \psi(t)\, \mathrm{dt} = 2(2m+1)! \sum_{n=1}^{\infty} \frac{G^2}{n^2\pi^2} \tag{4}$$

$$\left[f\left(\left(\frac{x'}{y'}\right)^{1/2} n\pi\right) \left(-\frac{y'}{n^2\pi^2}\right)^m + f\left(\left(\frac{y'}{x'}\right)^{1/2} n\pi\right)\left(\frac{x'}{n^2\pi^2}\right)^m \right]$$

where $\alpha = e^2/4\pi$, $f(z) = \coth(z)/z$ and:

$$\begin{aligned} \psi(t) = {} & \frac{G^2}{y'} \sum_{n=1}^{\infty} f\left(\left(\frac{x'}{y'}\right)^{1/2} n\pi\right) \exp\left[\frac{-n\pi(-t)^{1/2}}{(y')^{1/2}}\right] \theta(-t) \\ & + \frac{G^2}{x'} \sum_{n=1}^{\infty} f\left(\left(\frac{y'}{x'}\right)^{1/2} n\pi\right) \exp\left[\frac{-n\pi(t)^{1/2}}{(x')^{1/2}}\right] \theta(t) \end{aligned} \tag{5}$$

with $x' = 4\pi(F + (F^2 + G^2)^{1/2})$; $y' = 4\pi(-F + (F^2 + G^2)^{1/2})$.

PROOF. See [2, Lemma 2.1].

As an immediate consequence of Lemma 1, we conclude that the expansion of $L_1(\alpha)$ in power series of α has zero radius of convergence. From (3), the formal expansion of $L_1(\alpha)$ is indeed given by:

$$L_l(\alpha) = 2\alpha^2 \sum_{m=1}^{\infty} A_m(-\alpha)^m \tag{6}$$

and from (4) it follows that: $\lim_{m\to\infty} A_m{}^{1/m} = +\infty$.

LEMMA 2. *The Hamburger moment problem:*

$$A_n = \int_{-\infty}^{+\infty} t^n \phi(t)\, dt \tag{7}$$

where A_n are given by (4), *and $\phi(t)$ is non-negative in $(-\infty, +\infty)$, is indeterminate.*

PROOF. See [2, Lemma 2.2].

As a direct consequence of the former Lemmas, we have now the following:

THEOREM 1. *The divergent perturbation expansion* (6) *does not sum to $L_1(\alpha)$ under the Stieltjes method, i.e., the Stieltjes type continued fraction corresponding to the power series* (6) *does not converge to $L_1(\alpha)$. This implies that no diagonal sequences of Padé approximants to* (6) *converge to the exact solution.*

PROOF. See [2, Theorem 2.3].

For the particular case of the vacuum polarization by a pure electric or a pure magnetic field we have, however, a positive result.

A pure magnetic field is obtained when $G = 0$, $F = \frac{1}{2}\bar{H}^2 > 0$. Alternatively one has a pure electric field when $G = 0$, $F = -\frac{1}{2}\bar{E}^2$. Consider now only the pure magnetic field, since for the pure electric one our considerations are the same. Taking the constant H equal to 1, the Lagrangian (2) becomes:

$$(8) \qquad L_I{}^H = -\frac{1}{8\pi^2}\int_0^\infty \frac{e^{-s}}{s^3}\left[(es)\coth(es) - 1 - \tfrac{1}{3}(es)^2\right] ds$$

and its divergent Taylor expansion in powers of α is given by:

$$(9) \qquad L_I{}^H(\alpha) = -\frac{1}{8\pi^2}\sum_{n=2}^{\infty}(8\pi)^{2n}B_{2n}\frac{(2n-3)!}{(2n)!}\alpha^n$$

where B_{2n} are the Bernoulli numbers. We have now:

THEOREM 2. *The divergent expansion of* $(L_1{}^H(\alpha))/2\alpha^2$ *is summed to the exact solution by the Stieltjes method, i.e., the corresponding Stieltjes type continued fraction is convergent. This implies the convergence to* $L_1{}^H(\alpha)$ *of any* $[N, N+j]$ $j \geqq 1$, *sequence of Padé approximants to the divergent expansion* (9).

PROOF. See [1].

We have seen so far that the Stieltjes method fails, in the general case, to sum the divergent perturbation expansion to the exact solution. This is not the case for the Borel one, as we will now state. We define, as usual, the Borel transform of order 2 of the formal perturbation expansion (6) of $(L_1(\alpha))/2\alpha^2$ through the convergent power series:

$$(10) \qquad F(\alpha) = \sum_{m=0}^{\infty}\frac{A_m}{(2m+1)!}(-\alpha)^m.$$

The Borel sum of order 2 of the series (6) being defined by the integral $\int_0^\infty \exp(-s^{1/2})F(\alpha s)\,ds$, we have:

THEOREM 3. *The divergent perturbation expansion* (6) *is Borel summable of order* 2 *to the exact solution* $L_1(\alpha)$ *in the whole* α*-plane cut along the real axis.*

PROOF. See [2, Theorem 3.1].

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