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ANALYSIS OF TRUNCATION ERROR OF APPROXIMATIONS BASED ON THE PADÉ TABLE AND CONTINUED FRACTIONS

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1. **Introduction.** In the study and application of continued fractions

$$f = K(a_n/b_n) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots \tag{1}$$

it is important to have realistic estimates of the truncation error $|f - f_n|$ when (1) is approximated by its nth approximant f_n. Truncation error bounds are of two main types: (a) *A priori bounds* are expressed directly in terms of the elements a_n, b_n or parameters associated with these elements. (b) *A posteriori bounds* are generally of the form

$$|f - f_n| \leqq M_n |f_n - f_{n-1}| \tag{2}$$

and are obtained only after calculating the approximants $f_1, f_2, \cdots, f_n$. (There are also asymptotic estimates of the truncation error as given by [4, **15**]; however, such estimates will not be dealt with here.) Bounds of a priori type can be found in [**5**, **6**, **8**, **9**, **10**, **12**, **14**, **17** and **18**] and of a posteriori type in [**1**, **2**, **3**, **6**, **7**, **9**, **11**, **13**, **16**]. Most of the known truncation error bounds for continued fractions have been obtained either by studying inclusion regions for the approximants (Section 2) or by showing that the approximants form simple sequences (Section 3). In some cases both of these approaches have been used (Section 2). This paper provides a brief summary of the two approaches and reviews some of the main results. Proofs are omitted; however, proofs, application, and numerical examples may be found in references cited. These results have a strong connection with Padé tables. As an illustration of this connection we note that in a normal Padé table the approximants of the corresponding continued fraction of the form

$$\frac{a_0}{1} + \frac{a_1 z}{1} + \frac{a_z z}{1} + \cdots \text{(complex } a_n \neq 0)$$

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fill the stairlike sequence of squares $[0,0]$, $[1,0]$, $[1,1]$, $[2,1]$, $[2,2]$, $\cdots$ ([**19**], Theorem 96.1).

The following definitions and terminology are used. An *(infinite) continued fraction* is a triple of sequences $[\{a_n\}_1^\infty, \{b_n\}_1^\infty, \{f_n\}_1^\infty]$ where $a_1, a_2, \cdots$, and $b_1, b_2, \cdots$ are complex numbers, $a_n \neq 0$, and where f_n is an element of the extended complex plane defined as follows: If s_n denotes the linear fractional transformation (l.f.t.)

$$s_n(z) = a_n/(b_n + z), n = 1, 2, \cdots \tag{3a}$$

and

$$S_1(z) = s_1(z); S_n(z) = S_{n-1}(s_n(z)), n = 2, 3, \cdots \tag{3b}$$

then

$$f_n = S_n(0), n = 1, 2, \cdots. \tag{3c}$$

The a_n and b_n are called *elements* of the continued fraction and f_n is called the nth *approximant*. An infinite continued fraction is said to *converge* if its sequence of approximants converges; $f = \lim f_n$ is called its *value*. For convenience the infinite continued fraction $[\{a_n\}, \{b_n\}, \{f_n\}]$ may also be denoted by each of the three expressions in (1); in addition the expressions in (1) are used to denote the value of the continued fraction when it exists. The nth approximant may also be represented by $f_n = K_{k=1}^n (a_k/b_k)$. A *finite continued fraction* is a triple of finite sequences $[\{a_n\}_1^N, \{b_n\}_1^N, \{f_n\}_1^N]$, where the approximants f_n are defined as in (3). Its *value* is given by $f_N = K_{k=1}^N (a_n/b_n)$, which may also be used to denote the continued fraction itself.

2. **Inclusion Regions.** Consider the problem: Given the first n pairs of elements (a_1^*, b_1^*), $\cdots$, (a_n^*, b_n^*) of a continued fraction $K(a_n^*/b_n^*)$ that belongs to a certain class K of continued fractions, what can be said about the truncation error when $K(a_n^*/b_n^*)$ is approximated by its nth approximant? For this problem it is natural to study inclusion regions defined as follows:

Definition 1. For each $n = 1, 2, 3, \cdots$ let D_n be a subset of the complex, $2n$ dimensional, Cartesian space C^{2n}. Let K denote the class consisting of:

(A) All finite continued fractions $K_{n=1}^N (a_n/b_n)$, such that

$$(a_1, b_1, \cdots, a_n, b_n) \in D_n, \; n = 1, \cdots, N, \; N = 1, 2, \cdots, \tag{4}$$

and

(B) all infinite continued fractions $K(a_n/b_n)$, such that

$$\text{(5)} \qquad \text{(i) } (a_1, b_1, \cdots, a_n, b_n) \in D_n, \; n = 1, 2, \cdots.$$

Let $K(a_n^*/b_n^*)$ be a given continued fraction contained in the class K. For each $n = 1, 2, 3, \cdots$ let Φ_n be the set consisting of the values of all finite continued fractions in K of the form

$$\text{(6)} \qquad K_{k=1}^{n+m}(a_k/b_k),\, m \geqq 0, \text{ such that } (a_k, b_k) = (a_k^*, \mathrm{b}_k^*),\, k = 1, \cdots, n.$$

Then every closed set of complex numbers Ω_n containing Φ_n is called an nth *inclusion region for* $K(a_n^*/b_n^*)$ *relative to the class* K. The closure of Φ_n is called the *best* nth *inclusion region for* $K(a_n^*/b_n^*)$ *relative to the class* K.

REMARK. As can be seen the closure of Φ_n contains the nth approximant of $K(a_n^*/b_n^*)$ as well as all possible values of continued fractions in K having the first n pairs of elements $(a_1^*, b_1^*), \cdots, (a_n^*, b_n^*)$. Moreover, every closed set containing the above points must contain Φ_n. Thus we obtain,

THEOREM 1. *Let* $K(a_n^*/b_n^*)$ *be a convergent continued fraction belonging to a class* K *as described in Definition* 1. *Let* f_n *be the* nth *approximant of* $K(a_n^*/b_n^*)$ *and* $f = \lim f_n$. *If* $\{\Omega_n\}$ *is a sequence of inclusion regions for* $K(a_n^*/b_n^*)$ *relative to class* K, *then*

$$\text{(7)} \qquad |f - f_n| \leqq \text{diameter } \Omega_n.$$

In studies made so far $\{\Omega_n\}$ has been a nested sequence of compact subsets of C such that diameter $\Omega_n \to 0$. Clearly the error bounds in (7) can be best (on the basis of what is given) only if the Ω_n are best inclusion regions and f_n lies on the boundary of Ω_n. In no case can the error bound be less than half the diameter of Ω_n.

The first error bounds for continued fractions (with complex elements) based on inclusion regions were obtained by Thron [18] in 1958. For the class of continued fractions $K(a_n/1)$ with elements contained in the bounded subset of a parabolic region

$$|a_n| - \mathrm{Re}(a_n e^{-2i\alpha}) \leqq \tfrac{1}{2} \cos 2\alpha, \; -\pi/2 < \alpha < \pi/2.$$

Thron obtained a nested sequence of circular (disk) inclusion regions. His sharp estimates of the diameters of these disks provided a priori error bounds. More recently Thron's result has been extended by Snell and the author [12] to include variable parabolic regions and increased speed of convergence of the error bounds. An example is given by

THEOREM 2 [12]. *Let* $\{P_n\}$ *be a sequence of complex numbers* $P_n = p_n e^{i\psi_n}$ *such that*

$$(8) \qquad |P_n - 1/2| \leqq 1/2 - \epsilon, 0 < \epsilon < 1/2, n = 0, 1, \cdots.$$

Let $\{E_n\}$ *be the sequence of parabolic regions defined by*

$$(9)\ E_n = \{\zeta : |\zeta| - \mathrm{Re}[\zeta \exp(-i(\psi_n + \psi_{n-1}))] \leqq 2kp_{n-1}(\cos\psi_n - p_n)\},$$

where $0 \leqq k < 1$. *If* $K(a_n/1)$ *is a continued fraction with elements satisfying*

$$(10) \qquad a_n \in E_n,\ 0 < |a_n| \leqq M,\ n = 1, 2, \cdots,$$

for some constant $M > 0$, *then* $K(a_n/1)$ *converges to a value* f *and*

$$(11) \qquad |f - f_n| \leqq \frac{|a_1|(\cos\psi_1 - p_1)}{[1 + \epsilon^2(1 - k)/M]^{n-1}},\ n = 2, 3, \cdots.$$

REMARKS. (1) The right side of (11) estimates the diameter of the nth circular (disk) inclusion region. A decrease in the parameter k reduces the element regions E_n but increases the rate of convergence of the error bounds in (11).

(2) Thron's approach illustrated above has also been employed by Lange [14] for continued fractions $K(a_n/1)$ with twin-element-regions and by Hillam [10], Sweezy and Thron [17] and Field and Jones [5] for continued fractions $K(1/b_n)$ with the b_n contained in regions which are complements of open circular disks. In these cases the nth approximant of the continued fraction is in the interior of the nth inclusion region and hence resulting error bounds are best possible. Examples of best inclusion regions and best error bounds for continued fractions will be given in the next section on simple sequences.

3. Simple Sequences.

DEFINITION 2. A sequence of complex numbers $\{w_n\}$ is called a *simple sequence* if there exists a positive number C (called a *simple sequence constant*) such that

$$(12) \qquad |w_{n+m} - w_n| \leqq C|w_n - w_{n-1}|,\ m \geqq 0,\ n \geqq 2.$$

Simple sequences generalize real number sequences with the nesting property

$$(13) \qquad w_{2n-2} \leqq w_{2n} \leqq w_{2n+1} \leqq w_{2n-1},\ n \geqq 2.$$

A simple sequence may not converge (e.g. $w_n = (-1)^n$) and, moreover, a convergent sequence may not be simple (e.g., $w_n = 1/n$, which does not converge fast enough to be simple). However, knowledge of a simple sequence constant provides an immediate a posteriori truncation error bound as shown by

THEOREM 3. *If $\{w_n\}$ is a simple sequence converging to w, with simple sequence constant C, then*

$$|w - w_n| \leqq C|w_n - w_{n-1}|, n \geqq 2. \tag{14}$$

Most of the known a posteriori error bounds for continued fractions have been obtained in this way. Simple sequence constants for these cases are given in Table 1. The left column of Table 1 contains references (in brackets) and author's initials used hereafter to identify a particular result.

Perhaps the first known examples of continued fractions with simple sequences of approximants are those of the form $K(a_n/1)$ with $a_n > 0$ and $K(1/b_n)$ with $b_n > 0$; in such cases the approximants are positive real numbers satisfying (13) with simple sequence constant $C = 1$. These results are obtainable as special cases of HP in Table 1, with z real and positive. The first examples with complex elements a_n, b_n were obtained in Blanch [2] in 1964, using comparison relations for continued fractions (B_1 and B_2 in Table 1). An improvement of these results by Merkes [**16**] in 1966 was made from an analysis based on chain sequences (M in Table 1). In the same year Henrici and Pfluger [9] developed error bounds (HP in Table 1) for S-fractions (or series of Stieltjes) by considering inclusion regions Ω_n as follows: For an arbitrary but fixed S-fraction let Γ_n denote the circle passing through the approximants f_{n-2}, f_{n-1}, f_n and let $\Gamma_n{}^*$ denote the union of Γ_n and its interior. (If z is not real, the points f_{n-1}, f_n, f_{n+1} are not collinear and so Γ_n is a non-degenerate circle. In the limiting case that z is real and positive, the lens-shaped inclusion region Ω_n becomes a real line segment and the situation reduces to (13)). Then Ω_n is the lens-shaped region $\Gamma^*{}_{n-1} \cap \Gamma_n{}^*$ (see Figure 1). For each $n \geqq 3$, they proved the following results relative to the class of all S-fractions (finite or infinite) having the given set of first n elements $a_1{}^*, \cdots, a_n{}^*$: (1) Ω_n is the best nth inclusion region in the sense of Definition 1. (2) $\Omega_{n+1} \subset \Omega_n$. (3) Ω_n is convex. (4) Γ_n and Γ_{n-1} intersect in an angle (interior to Ω_n) of magnitude $|\arg z|$, independent of n. From these facts it was shown that $\{f_n\}$ is a simple sequence with constant $C(z)$ given in Table 1 (HP) and diameter $\Omega_{n+1} = C(z)|f_n - f_{n-1}|$. Thus both the ideas of simple sequences and inclusion regions were employed.

Using a modification of the method of HP, Jefferson [11] obtained best inclusion regions for the class of T-fractions in Table 1 (J) and showed that the approximants form simple sequences. A generalization of HP and J was obtained by Thron and the author [13] (see Table 1, JT); it applies also to continued fractions of Gauss, to large classes of J-fractions, to continued fractions $K(1/b_n)$ where $|\arg b_n| \leqq (\pi/2) - \epsilon$, $\epsilon > 0$ or $b_n = 0$, and to other examples. They developed inclusion regions Ω_n as follows: Let θ be a given real number such that $0 < |\theta| < \pi$ and let $\{\gamma_n\}$ be a given sequence of real numbers. Let K denote the class of all (finite or infinite) continued fractions $K(a_n/b_n)$ satisfying the conditions in Table 1 (JT). Let $K(a_n^*/b_n^*)$ be an arbitrary but fixed infinite continued fraction in K, with nth approximant f_n. For each $n \geqq 1$ let Γ_n denote the circle defined by

$$\Gamma_n : w = S_n(te^{i\gamma_n}), \; -\infty \leqq t \leqq \infty,$$

where $\{S_n\}$ is the sequence of l.f.t.'s (3b) associated with $K(a_n^*/b_n^*)$. It is shown that there exists a sequence $\{\zeta_n\}$ such that each Γ_n passes through $f_{n-1} = S_n(\infty)$, ζ_n, $f_n = S_n(0)$ and ζ_{n-1} in the given order. Then Ω_n is defined as the closed, lens-shaped region bounded on one side by the arc of Γ_n with end points f_{n-1}, ζ_{n-1} passing through ζ_n and w_n and on the other side by the arc of Γ_{n-1} with the same end points f_{n-1}, ζ_{n-1} and not passing through f_{n-2} or ζ_{n-2} unless $\zeta_{n-1} = f_{n-2}$ or $f_{n-1} = \zeta_{n-2}$ (see Figure 2). For each $n \geqq 2$ they proved the following: (1) Ω_n is an nth inclusion region for $K(a_n^*/b_n^*)$ relative to the class K. (2) $\Omega_{n+1} \subset \Omega_n$. (3) Ω_n is convex. (4) Γ_n and Γ_{n-1} intersect in an angle (interior to Ω_n) of magnitude $|\theta|$, independent of n. From this it was shown that $\{f_n\}$ is a simple sequence with constant $C(\theta)$ given in Table 1 (JT) and diameter $\Omega_n = C(\theta)|\zeta_{n-1} - f_{n-1}| \leqq C(\theta)|f_n - f_{n-1}|$. It was further shown that the conditions imposed on the a_n, b_n in Table 1 (JT) are invariant in form under equivalence transformations of continued fractions. The inclusion regions Ω_n could not be shown to be best in the sense of Definition 1, without assuming more about the structure of the class K.

We conclude with some remarks on subclasses of S-fractions (or series of Stieltjes) which represent functions $f(z)$ holomorphic at the origin. For such cases smaller inclusion regions and sharper error estimates than those of HP have been obtained. In 1968 Common [3] obtained such bounds for real values of z and Baker [1] extended this to complex z by considering best inclusion regions. However, Baker's analysis did not provide a simple method to calculate error bounds, since it did not show that the lens-shaped inclusion regions

Ω_n were convex (a property that greatly simplifies the calculation of diameter Ω_n). This was done in an independent study at about the same time by Gragg [6] for the case with $f(z)$ holomorphic for $|z| < 1$. In a second paper Gragg [7] extended his analysis to include functions holomorphic in the complex plane cut along an arbitrary finite interval of the real axis.

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TABLE 1. Simple Sequences $\{f_n\}$ from Continued Fractions

	f_n = nth Approximant of:	C = Simple Sequence Constant
B_1 [2]	$\frac{a_1}{1}+\frac{a_2}{1}+\cdots, \quad \lvert a_n\rvert \leqq \frac{1}{4}-\epsilon, 0<\epsilon<1/4$	$C=(1/2\epsilon)-1$
B_2 [2]	$\frac{1}{b_1}+\frac{1}{b_2}+\cdots, \quad \lvert b_n\rvert \geqq 2+\epsilon, \epsilon>0$	$C=d/(1-d),\ d=1+\frac{\epsilon}{2}-\left[\left(1+\frac{\epsilon}{2}\right)^2-1\right]^{1/2}$
M [16]	$\frac{a_1}{b_1}+\frac{a_2}{b_2}+\cdots, \quad \left\lvert\frac{a_n}{b_nb_{n-1}}\right\rvert \leqq r(1-r), 0<r<1/2$	$C=r/(1-2r)$
HP [9]	$\frac{a_1}{z}+\frac{a_2}{1}+\frac{a_3}{z}+\frac{a_4}{1}+\cdots, \quad a_n>0,\ \lvert\arg z\rvert<\pi$ (S-fraction)	$C(z)=\begin{cases}1, & \lvert\arg z\rvert\leqq\pi/2\\ \tan[(1/2)\arg z] & \pi/2<\lvert\arg z\rvert<\pi\end{cases}$
J [11]	$1+d_0z+\frac{z}{1+d_1z}+\frac{z}{1+d_2z}+\cdots, \quad d_n>0,\ \lvert\arg z\rvert<\pi$ (T-fraction)	$C(z)=\begin{cases}1, & \lvert\arg z\rvert\leqq\pi/2\\ \sec[(1/2)\arg z], & \pi/2<\lvert\arg z\rvert<\pi\end{cases}$
JT [13]	$\frac{a_1}{b_1}+\frac{a_2}{b_2}+\cdots, \quad \gamma_0,\ \theta$ real, $0<\lvert\theta\rvert<\pi$ $\gamma_n=\arg a_n-\gamma_{n-1}-\theta \pmod{2\pi}$ $b_ne^{-i\gamma_n}\in \begin{matrix} D[0,\theta], & 0<\theta<\pi\\ D[\theta,0], & -\pi<\theta<0\end{matrix}$ where $D[\alpha,\beta]\equiv\{z: z=0 \text{ or } \alpha\leqq\arg z\leqq\beta\}$.	$C(\theta)=\begin{cases}1, & 0<\lvert\theta\rvert<\pi/2\\ \sec(\lvert\theta\rvert-\pi/2), & \pi/2<\lvert\theta\rvert<\pi.\end{cases}$

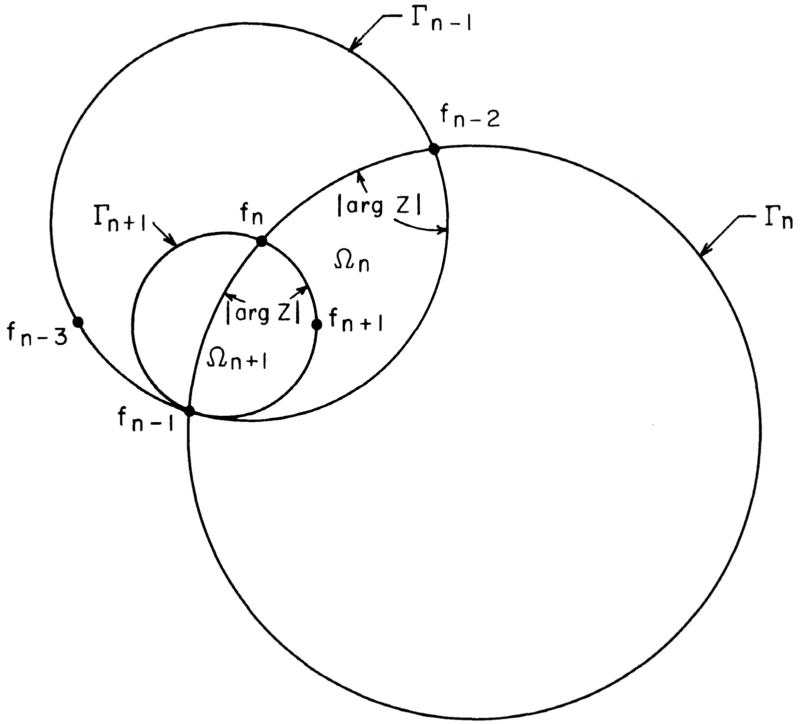

FIGURE 1. Schematic diagram of best inclusion regions Ω_n for S-fractions.

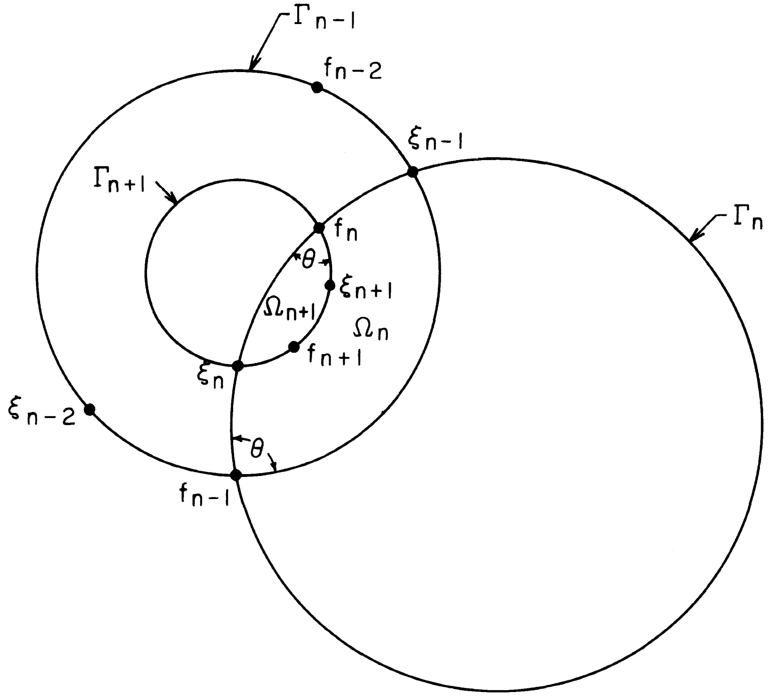

FIGURE 2. Schematic diagram for lens-shaped inclusion regions for class of continued fractions in Table 1 (JT).