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MONOTONE SURJECTIONS HAVING MORE THAN ONE FIXED POINT

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1. **Introduction.** Suppose X is a continuum and f is a self-mapping of X which has a fixed point e. Under what circumstances is there another point of X, distinct from e, which is also fixed under f? Very simple considerations suffice to indicate that in order to establish any kind of satisfactory theorem, e must be an endpoint in some appropriate sense, f must be surjective, and f must be more than merely continuous.

Questions of this type were studied as long ago as 1930 by W. L. Ayres [**1**]. In 1944 the first direct antecedent of the present paper appeared in a theorem of G. E. Schweigert [**16**]. Schweigert proved that if X is a dendrite, if e is an endpoint of X (that is, e is a point of order one), and if $T(X) = X$ is a homeomorphism such that $T(e) = e$, then $T(x) = x$ for some $x \in X - \{e\}$. Soon thereafter, A. D. Wallace [17] proved this theorem in case X is *any* locally connected continuum. When one attempts to prove this result for a larger class of mappings than the homeomorphisms then it becomes clear that the continua X must be drastically restricted. The author has shown [19] that for monotone surjections on locally connected continua, the existence of a fixed endpoint e implies the existence of a "small" invariant subcontinuum not containing e. The Schweigert theorem for monotone surjections is an immediate corollary. More recently W. J. Gray [9] has studied the same class of questions for finitely generated commutative semigroups of monotone surjections. He further generalized the Schweigert theorem by showing that if such a semigroup of mappings on a dendrite has a common fixed endpoint e, then it must have a common fixed point distinct from e.

It is the primary purpose of this paper to study these questions for monotone surjections on dendroids. It is proved that if two monotone surjections have a common fixed endpoint, and if they commute, then

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they have another common fixed point. It is necessary to devise arguments which are somewhat more prolix than is required for the case of dendrites, but the proofs are still elaborations of a technique exploited by the author in [20] and [22]. This technique lends itself most readily to spaces which contain no simple closed curves, but it is possible to adapt them to other continua, especially in the locally connected case. The final result states that if a monotone surjection on a hereditarily locally connected continuum has a fixed endpoint, then it must have another fixed point.

2. **Preliminaries.** If Γ is a partial order on a set S, we write $x \leqq y$, $(x, y) \in \Gamma$, $x \in \Gamma y$ and $y \in x\Gamma$ as synonyms. As usual, we write $x < y$ if $x \leqq y$ and $x \neq y$. If $A \subset S$ then

$$A\Gamma = \bigcup \{a\Gamma : a \in A\},$$

$$\Gamma A = \bigcup \{\Gamma a : a \in A\}.$$

A *chain* (relative to Γ) is a subset of S which is simply ordered with respect to Γ. If S is a topological space then Γ is *lower* (*upper*) *semi-continuous* provided $\Gamma x (x\Gamma)$ is a closed set for each $x \in S$. If $A \subset S$ and if there exists $z \in A$ such that $z \leqq a$ for each $a \in A$, then z is a *zero* of A. Zeroes of sets, when they exist, are necessarily unique.

A *dendroid* is an arcwise connected compactum with the property that any two of its closed connected subsets have a connected intersection. The following properties of dendroids are well-known and easy to prove. (See, for example, [3], [7], [13] and [14].)

(2.1) *If x and y are distinct elements of a dendroid D, then there is a unique arc in D whose endpoints are x and y.*

Hereafter, the unique arc jointing the points x and y of a dendroid will be denoted $[x, y]$.

(2.2) *Every subcontinuum of a dendroid is a dendroid.*

(2.3) *If D is a dendroid, K is a subcontinuum of D and x and y are members of K, then $[x, y] \subset K$.*

(2.4) *If D is a dendroid and $\mathscr{N}$ is a nested family of arcs contained in D then there exists an arc A such that $\bigcup \mathscr{N} \subset A \subset D$.*

The uniqueness of the arc $[x, y]$ in (2.1) permits us to define a partial order on a dendroid which facilitates its study (see [11], [20] and [22]). If D is a dendroid and $p \in D$, define

$$\Gamma_p = \{(x, y) \in D \times D : x \in [p, y]\}.$$

It is a simple exercise to verify that Γ_p is a partial order. The following properties of Γ_p were established in [22].

(2.5) *If D is a dendroid and $p \in D$ then*

(2.5.1) $D = p\Gamma_p$.

(2.5.2) *If $x < y$ in D then $x\Gamma_p \cap \Gamma_p y$ is the arc $[x, y]$. In particular, Γ_p is lower semicontinuous.*

(2.5.3) *If A is a totally unordered subset of D and if P is a continuum contained in $A\Gamma_p$, then $P \subset a\Gamma_p$, for some $a \in A$.*

(2.5.4) *Each subcontinuum of D has a zero relative to Γ_p, and each chain has a supremum.*

(2.5.5) *If $x \in D$, if Y is a continuum contained in $D - \{x\}$ and if Y meets $x\Gamma_p$, then $Y \subset x\Gamma_p$.*

It is worth noting that the existence of a partial order Γ on a compactum D which satisfies conditions (2.5.1), (2.5.2), (2.5.3) and (2.5.5) implies that D is a dendroid [22]. Some related characterizations are found in [10].

The next preliminary result was established first by Borsuk [3]. We note that there have been several recent extensions of this theorem which are relevant to this paper, especially those of Charatonik [5] and [6], and Mohler [15].

(2.6) *A dendroid has the fixed point property.*

Recall [23] that a mapping f is *monotone* if $f^{-1}(y)$ is a connected set for each element y of the range of f. If f is a monotone mapping on a compact space, and if B is a closed, connected subset of the range of f, then $f^{-1}(B)$ is connected.

Recently Charatonik and Eberhart [7] have studied dendroids, especially a class of well-behaved ones called smooth, and they have noted some results concerning monotone mappings on dendroids. We state a few of these, not always with the greatest possible generality, and we include proofs, in part for completeness but also to impart some of the flavor of the arguments to follow.

(2.7) *The Hausdorff monotone image of a dendroid is a dendroid.*

Proof. Let D be a dendroid, Y a Hausdorff space and suppose $f : D \to Y$ is a monotone surjection. Since the Hausdorff continuous image of an arc is arcwise connected, it is clear that Y is arcwise connected. If A and B are subcontinua of Y then $f^{-1}(A)$ and $f^{-1}(B)$ are subcontinua of D, and therefore $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ is connected. It follows that $A \cap B$ is connected, and therefore that Y is a dendroid.

(2.8) *If D is a dendroid, Y is a Hausdorff space, $f : D \to Y$ is a monotone surjection and a and b are members of D, then $f([a, b]) = [f(a), f(b)]$.*

PROOF. By (2.7) Y is a dendroid, so there is a unique (possibly degenerate) arc $[f(a), f(b)]$. Since $f([a, b])$ is arcwise connected, it contains $[f(a), f(b)]$. Since f is monotone the set $f^{-1}([f(a), f(b)])$ is a continuum containing a and b, so that by (2.3), it contains $[a, b]$. Therefore, $[f(a), f(b)] \supset f([a, b])$, completing the proof.

(2.9) *If D is a dendroid, $p \in D$ and $f : D \to D$ is a monotone mapping such that $p = f(p)$, then f is order-preserving with respect to Γ_p.*

PROOF. If $x \leqq y$ then $x \in [p, y]$ and therefore by (2.8), $f(x) \in [f(p), f(y)] = [p, f(y)]$. That is, $f(x) \leqq f(y)$.

The next result is a technical lemma which will be essential in what is to follow. A *simple triod* is the union of three arcs which are mutually disjoint except for a common endpoint called the *emanation point.*

(2.10) *If T is a simple triod with emanation point v, if T is contained in a dendroid D, and if $f : D \to D$ is a monotone mapping which maps the set of endpoints of T onto itself, then $f(T) = T$ and $f(v) = v$.*

PROOF. Let the endpoints of T be denoted x_1, x_2 and x_3, and let σ be the permutation of $\{1, 2, 3\}$ such that $f(x_i) = x_{\sigma(i)}$. By (2.8), $f([x_i, x_j]) = [x_{\sigma(i)}, x_{\sigma(j)}]$ and therefore $f(T) = T$. Further, $f(v) \in [x_1, x_2] \cap [x_1, x_3] \cap [x_2, x_3] = \{v\}$.

Suppose X and Y are spaces and F is a set-valued mapping on X such that $F(x)$ is a closed subset of Y, for each $x \in X$. Such a mapping is called *upper semicontinuous* if, for each $x \in X$ and each open set V of Y such that $F(x) \subset V$, there exists an open set U such that $x \in U \subset X$ and $F(t) \subset V$ for each $t \in U$. The following proposition was established in [22].

(2.11) *If X and Y are compact Hausdorff spaces, $F : X \to Y$ is upper semicontinuous and continuum-valued, and if K is a continuum contained in X, then $F(K)$ is a continuum.*

From the definition of a monotone mapping and [12, p. 174] the last result of this section is immediate.

(2.12) *If X and Y are compact Hausdorff spaces and $f : X \to Y$ is a monotone surjection, then f^{-1} is upper semicontinuous and continuum-valued.*

3. **A fixed point theorem of dead-end type.** We will establish a

preliminary result whose proof is a paradigm for a large class of fixed point arguments. It is typical of the kind of proof which employs what Bing [2] has called the "dead end" method. Crudely put, the idea is to locate a point x whose image $f(x)$ is "ahead" of x relative to some inherent order structure. As x pursues $f(x)$ the heterogeneous character of the space is invoked to trap $f(x)$ in a "dead end," producing a fixed point. The theorem proved here is closely related to an earlier one of the author's [19].

(3.1) THEOREM. *Let X be a compact Hausdorff space endowed with a lower semicontinuous partial order Γ, and suppose that each maximal chain is compact. Suppose in addition that if C is a maximal chain and $a \in C$ then $a\Gamma \cap C$ is a closed set. If $x_0 \in X$ and if $f: X \to X$ is a continuous, order-preserving function, then a necessary and sufficient condition for $x_0\Gamma$ to contain a fixed point of f is that there exists $x \in x_0\Gamma$ such that $x \leqq f(x)$.*

PROOF. The necessity of the condition is obvious. On the other hand, if $x_0 \leqq x \leqq f(x)$ then, because f is order-preserving, it follows that

$$x \leqq f(x) \leqq f^2(x) \leqq \cdots \leqq f^n(x) \leqq \cdots,$$

and so all of these elements lie in a compact maximal chain C. Therefore, the sequence $f^n(x)$ has a cluster point $y \in C$. Because the sets $f^n(x)\Gamma \cap C$ are closed it follows that $f^n(x) \leqq y$ for each n. If there exists an element z such that $f^n(x) \leqq z < y$ for each n, then the sequence $f^n(x)$ is never in $X - \Gamma z$, a neighborhood of y. Therefore y is the (unique) least upper bound of the elements $f^n(x)$ and hence $f^n(x)$ converges to y. Since f is continuous the sequence $f^{n+1}(x)$ (and hence also the sequence $f^n(x)$) converges to $f(y)$, and therefore $y = f(y)$.

(3.2) COROLLARY. *If D is a dendroid and $p \in D$, suppose $f: D \to D$ is a monotone mapping such that $p = f(p)$. If there exists $x \in D - \{p\}$ such that $(x, f(x)) \in \Gamma_p$ then f has another fixed point in the set $x\Gamma_p$.*

(3.2) is an immediate consequence of Theorem (3.1), together with (2.4), (2.5.2) and (2.9).

4. **The Schweigert-Wallace fixed point theorem for monotone surjections on a dendroid.** If X is an arcwise connected space then an element e of X is an *endpoint* of X provided e is an endpoint of any arc in X which contains e. The set of endpoints of X is denoted $E(X)$. It follows from (2.4) and a simple maximality argument that if D is a dendroid then $E(D)$ contains at least two elements.

(4.1) THEOREM. *If D is a dendroid, $e \in E(D)$, $F : D \to D$ is an upper semicontinuous, continuum-valued mapping, $e \in F(e)$ and if there exists $x \in D - \{e\}$ such that $F(x)$ meets $x\Gamma_e$, then F has a fixed point distinct from e.*

PROOF. We give D the partial order $\Gamma = \Gamma_e$ and let C be a maximal chain of the set $\{x \in D : F(x) \cap x\Gamma \neq 0\}$. By (2.5.4), $x_1 = \sup C$ exists and $x_1 \neq e$.

Supposed there exists $x_0 \in C$ such that for all $x \in C - \Gamma x_0$ the sets $F([x, x_1])$ and $[x, x_1] \cup x_1\Gamma$ are disjoint. (In particular, then, we are assuming that $x_1 \notin C$.) By (2.11) $F([x, x_1])$ is a continuum, and since $F([x, x_1])$ meets $x\Gamma - \{x\}$, it follows from (2.5.5) that $F([x, x_1]) \subset x\Gamma - \{x\}$. Therefore, if $z(x)$ denotes the zero of $F([x, x_1])$, then $x < z(x)$ for all $x \in C - \Gamma x_0$. Since $z(x) \notin x_1\Gamma$ we can choose $y \in C$ with $x < y \notin \Gamma z(x)$. Since $y < z(y)$ it follows that $y \in [z(x), z(y)] \subset F([x, x_1])$, and hence $y \in F([x, x_1]) \cap ([x, x_1] \cup x_1\Gamma)$, contrary to our hypothesis.

Therefore, there exists an increasing sequence x_n which is cofinal in C and which has the property that $F([y_n, x_1])$ meets $[y_n, x_1] \cup x_1\Gamma$ for each n. If each $F([y_n, x_1])$ meets $[y_n, x_1]$, then it is easy to see from the upper semicontinuity of F that $x_1 \in F(x_1)$ and the theorem is proved. Otherwise, there exists n such that $F([y_n, x_1])$ meets $x_1\Gamma - \{x_1\}$, and hence $F(x_1) \subset F([y_n, x_1]) \subset x_1\Gamma - \{x_1\}$. In this case $x_1 \in C$, and if z_1 denotes the zero of $F(x_1)$ then $x_1 < z_1$. We choose a sequence w_n in $[x_1, z_1] - \{x_1\}$ such that w_n converges to x_1. By the maximality of C there exists $v_n \in F(w_n) - w_n\Gamma$ for each n, and since $F([x_1, w_n])$ is arcwise connected we have $w_n \in [v_n, z_1] \subset F([x_1, w_n])$. By upper semicontinuity, $x_1 = \lim w_n \in F(x_1)$.

Theorem (4.1) will be used to prove the Schweigert-Wallace theorem for monotone surjections on a dendroid:

(4.2) THEOREM. *If D is a dendroid, $e \in E(D)$ and $f : D \to D$ is a monotone surjection such that $e = f(e)$, then there exists $x \in D - \{e\}$ such that $x = f(x)$.*

PROOF. By (2.12) f^{-1} is upper semicontinuous and continuum-valued, so by (3.2) and (4.1) it is sufficient to show that one of the following holds: (1) there exists $x \in D - \{e\}$ such that $x \leqq f(x)$, or (2) there exists $x \in D - \{e\}$ such that $f^{-1}(x) \cap x\Gamma$ is nonempty, where $\Gamma = \Gamma_e$.

Select $m \in D$ such that $[e, m]$ is a maximal arc of D and $f(m) \neq e$. Since f is surjective and order-preserving, it is clear that this choice is possible. If $f(m) \leqq m$ then (2) is satisfied. Otherwise, m and $f(m)$ are not comparable, and since $e \in E(D)$ there exists $t \in D$

such that $t = \sup(\Gamma m \cap \Gamma f(m)) > e$. Since $t < m$ it follows that $f(t) \leqq f(m)$, and therefore t and $f(t)$ are comparable. If $t \leqq f(t)$ then (1) is satisfied. If $f(t) \leqq t$ and $f(t) \neq e$, then (2) is satisfied by letting $x = f(t)$. In case $f(t) = e$, we note that $f([t, m]) = [f(t), f(m)] = [e, f(m)]$, and hence there exists $t' \in [t, m]$ such that $f(t') = t$. Then (2) is satisfied by letting $x = t$, and the theorem is proved.

5. **Commuting monotone surjections on a dendroid.** In this section we consider the fixed point properties of a pair of monotone surjections on a dendroid D which commute with respect to composition, i.e. $f(g(x)) = g(f(x))$ for each $x \in D$.

(5.1) *If D is a dendroid, $e \in E(D)$ and if f and g are commuting monotone surjections such that $e = f(e) = g(e)$, then f has a fixed point in $D - g^{-1}(e)$.*

Proof. By Theorem (4.2) there exists $x_1 \in D - \{e\}$ such that $x_1 = f(x_1)$. By (2.2) and the monotonicity of g it follows that $g^{-1}(x_1)$ is a dendroid. By commutativity we have $fg^{-1}(x_1) \subset g^{-1}(x_1)$, and since dendroids have the fixed point property (2.6), there exists $x_2 \in g^{-1}(x_1)$ such that $f(x_2) = x_2$. Since $x_1 \neq e$ and $x_2 \in g^{-1}(x_1)$, we conclude that $x_2 \in D - g^{-1}(e)$.

For the remainder of this section, whenever D is a dendroid and $e \in E(D)$, the partial order employed is $\Gamma = \Gamma_e$.

(5.2) *If D is a dendroid, $e \in E(D)$ and if f and g are commuting monotone surjections such that $e = f(e) = g(e)$, then there exists $x \in D - g^{-1}(e)$ such that $x = f(x)$ and x and $g(x)$ are comparable.*

Proof. By (5.1) there exists $x_1 = f(x_1) \in D - g^{-1}(e)$, so we may assume that x_1 and $g(x_1)$ are not comparable. Since $e \in E(D)$ there exists $t \in D$ such that

$$t = \sup(\Gamma x_1 \cap \Gamma g(x_1)) > e.$$

Since $g(t) \in g([e, x_1]) = [e, g(x_1)] = \Gamma g(x_1)$, it follows that t and $g(t)$ are comparable. Moreover f keeps each of the points x_1, $g(x_1)$ and e fixed. Therefore, by (2.10), $f(t) = t$. Thus the proof is complete unless $g(t) = e$. But then there exists $a \in [t, x_1]$ such that $g(a) = t$. That is, $g^{-1}(t)$ meets $t\Gamma - \{t\}$ and hence $g^{-1}(t) \subset t\Gamma - \{t\}$ by (2.5.5). Since $fg^{-1}(t) \subset g^{-1}(t)$ and the dendroid $g^{-1}(t)$ has the fixed point property, f has a fixed point $t_1 \in g^{-1}(t)$ and $e \neq g(t_1) = t < t_1$.

(5.3) Theorem. *If D is a dendroid, $e \in E(D)$ and if f and g are* *commuting monotone surjections on D such that $e = f(e) = g(e)$, then there exist comparable elements x_1 and x_2 such that $e \neq x_1 = f(x_1)$ and $e \neq x_2 = g(x_2)$.*

PROOF. By (5.2) there exists $x_1 \in D$ such that $x_1 = f(x_1)$, x_1 and $g(x_1)$ are comparable and $g(x_1) \neq e$. A routine maximality argument shows that we may assume x_1 is maximal relative to this property, and therefore by this maximality it is not the case that $x_1 < g(x_1)$. Therefore we may assume that $g(x_1) < x_1$.

Since $g^{-1}(x_1)$ is a dendroid and since, by commutativity, $fg^{-1}(x_1) \subset g^{-1}(x_1)$, there exists $y_1 \in g^{-1}(x_1)$ such that $f(y_1) = y_1$. By the maximality of x_1 it follows that $y_1 \notin x_1\Gamma$, and therefore $g^{-1}(x_1)$ and $x_1\Gamma$ are disjoint. If z_1 denotes the zero of $g^{-1}(x_1)$ and $z_1 < x_1$, then $g(z_1) = x_1 > z_1$, and since g is order-preserving, $x_1 = g(z_1) \leqq g(x_1)$ which is contrary to the assumption that $g(x_1) < x_1$. Consequently z_1 and x_1 are not comparable and there exists $t_1 \in D$ such that $t_1 = \sup(\Gamma x_1 \cap \Gamma z_1) \neq e$. Since $t_1 \in [x_1, z_1]$ it follows from (2.8) that $g(t_1) \in [g(x_1), g(z_1)] = [g(x_1), x_1]$. Since $g(x_1) < x_1$ we infer that $g(x_1) \leqq g(t_1)$, but since $t_1 \leqq x_1$ it must be that $g(t_1) \leqq x_1$, and therefore $g(t_1) = g(x_1)$. If $t_1 \leqq g(x_1)$, then we set $x_2 = g(x_1)$, and we note (because $t_1 \leqq x_2 \leqq x_1$ and $g(t_1) = g(x_1)$) that $g(x_2) = g(x_1) = x_2$ and that x_1 and x_2 are comparable. The theorem is proved in this case.

On the other hand, if $t_1 \nleqq g(x_1)$ then $g(x_1) < t_1$ and hence $g(t_1) \leqq g(x_1) < t_1$. Since $t_1 \in [g(t_1), x_1] = g([t_1, z_1])$, there exists $t_2 \in [t_1, z_1] - \{t_1\}$ such that $g(t_2) = t_1$. The mapping g^{-1} now satisfies the hypotheses of Theorem (4.2), and so g^{-1} (and hence g) has a fixed point $x_2 > t_1$. Thus $f(g(x_1)) = g(x_1)$, $g(x_2) = x_2$ and $g(x_1) < x_2$, and the proof is complete.

(5.4) COROLLARY. *If D, e, f and g satisfy the hypotheses of Theorem (5.3), then one of f and g, say f, has a fixed point $x_1 \neq e$ which is maximal with respect to preceding some fixed point of g. Moreover, g has a fixed point x_2 which is minimal with respect to $x_2 \geqq x_1$.*

We omit the proof of Corollary (5.4) which follows from Theorem (5.3) in a straightforward way.

Now suppose x_1 and x_2 satisfy the conclusion of (5.4) and that $x_2 > x_1$. Then $g(x_1) \leqq g(x_2) = x_2$ and $f(g(x_1)) = g(x_1)$, so by the maximality of x_1 we know that $g(x_1) \leqq x_1$. By the minimality of x_2 we know that $g(x_1) < x_1$, and since f is order-preserving we know that $x_1 < f(x_2)$.

If x_2 and $f(x_2)$ are not comparable, we let $t_1 = \sup(\Gamma x_2 \cap \Gamma f(x_2))$, and we note that $t_1 \geqq x_1 > e$, so that t_1 is the emanation point of a simple triod whose endpoints are e, x_2 and $f(x_2)$. By (2.10), $g(t_1) = t_1$, and this contradicts the minimality of x_2.

If $f(x_2) < x_2$ then by commutativity, $g(f(x_2)) = f(x_2)$, and again the minimality of x_2 is contradicted.

Therefore, it follows that $x_2 < f(x_2)$. But in this case the sequence $f^n(x_2)$ is increasing and converges to a common fixed point of f and g, and this contradicts the maximality of x_1. Apparently $x_1 = x_2$, and we may state the main result of this section.

(5.5) THEOREM. *If D is a dendroid, $e \in E(D)$ and if f and g are commuting monotone surjections on D such that $e = f(e) = g(e)$, then there exists $x \in D - \{e\}$ such that $x = f(x) = g(x)$.*

6. **Monotone surjections on hereditarily locally connected continua.** If X is the plane continuum consisting of the 2-cell $\{(x, y) : x^2 + y^2 \leqq 1\}$ and the line segment joining (1,0) to (2,0), then it is easy to construct a monotone surjection on X which has only $(2, 0)$ as a fixed point. However, if Y is the continuum consisting of the circle $\{(x, y) : x^2 + y^2 = 1\}$ and the line segment joining $(1, 0)$ and $(2, 0)$, then a monotone surjection on Y which keeps $(2, 0)$ fixed must also keep $(1, 0)$ fixed. These examples motivate the next result.

(6.1) THEOREM. *If X is a hereditarily locally connected continuum, $e \in E(X)$ and f is a monotone surjection on X such that $f(e) = e$, then there exists $x_0 \in X - \{e\}$ such that $f(x_0) = x_0$.*

In order to simplify the proof, it is helpful to recall some results from [19] concerning partial order in locally connected continua. If X is a connected space and $p \in X$, we let $(x, y) \in \Gamma_p$ if and only if $x = p$ or $x = y$ or x separates p and y in X. Then we have

(6.2) *If X is a locally connected continuum and $p \in X$ then*

(6.2.1) *Γ_p is a partial order with closed graph, and hence Γ_p is both upper and lower semicontinuous,*

(6.2.2) *p is a zero with respect to Γ_p,*

(6.2.3) *if $x \in X$ then $\Gamma_p x$ is a chain and $x\Gamma_p$ is connected,*

(6.2.4) *if f is a monotone surjection on X and $f(p) = p$, then f is order-preserving with respect to Γ_p.*

PROOF OF (6.1). Give X the partial order $\Gamma = \Gamma_e$. If there exists $x \in X - \{e\}$ such that $x \leqq f(x)$, then the desired fixed point exists in $x\Gamma$ by Theorem (3.1), so we assume that no such x exists. We wish to prove that there exists $q \in X$ such that $e \neq f(q) < q$. If $x \in X - f^{-1}(e)$ and x does not have the desired property, then x

and $f(x)$ are not comparable. Therefore, $\Gamma x \cap \Gamma f(x)$ is a closed chain and hence has a maximal element, p. Since f is order-preserving, it follows that $f(p) \in \Gamma f(x)$. By assumption, $p \not\leq f(p)$, and thus $f(p) < p$. It follows that p separates $f(p)$ and $f(x)$ in X, and consequently, if A is an arc connecting p and x in the set $p\Gamma$, we infer that $p \in f(A)$. Thus there exists $q \in X$ such that $e \neq p = f(q) < q$.

It follows that $f(q)$ separates e and the continuum $K_1 = f^{-1}f(q)$. Inductively, we obtain a sequence of continua K_n such that K_n separates e and K_{n+1}, where $f(K_{n+1}) = K_n$. Indeed, we may assert that K_n separates e and $\bigcup \{K_{n+m} : m = 1, 2, \cdots\}$. To see this, let $X - \{f(q)\} = A \cup B$, $e \in A$, $K_1 = f^{-1}f(q) \subset B$, with A and B disjoint open sets. Then $X - K_1 = f^{-1}(A) \cup f^{-1}(B)$, $e \in f^{-1}(A)$, $K_2 = f^{-1}(K_1) \subset f^{-1}(B)$. We claim that $A \subset f^{-1}(A)$. For if $f(q) \in f^{-1}(B)$, then $f^2(q) \in B$ and hence $f(q) < f^2(q)$, which contradicts (6.2.4). Therefore $f(q) \in f^{-1}(A)$ and, since $f^{-1}(B)$ meets B, it follows that $f^{-1}(B) \cap A$ is empty. Therefore, $A \subset f^{-1}(A) \subset f^{-2}(A) \subset \cdots$, and thus $K_n = f^{-1}(K_{n-1})$ separates e and each K_{n+m} as claimed.

We now invoke the hereditary local connectedness of X. By a result of Whyburn [23; V (2.6)], K_n is a null sequence. In particular, $\lim K_n$ exists and is a single point, x_0. Since $f(K_{n+1}) = K_n$, it is clear that $f(x_0) = x_0$.

7. **Concluding remarks.** It is natural to inquire whether (5.5) is true for a family of more than two commuting, monotone surjections. W. J. Gray [9] has shown that if S is a compact commutative semigroup of monotone surjections on a dendrite and if S has a fixed endpoint, then it does not necessarily follow that S has another fixed point, but if S is finitely generated then a second fixed point exists. I conjecture that the same is true for dendroids.

Following Wallace [18] we may use the term *arc* to describe a compact connected Hausdorff space with exactly two non-cutpoints. (In this terminology, an arc homeomorphic to $[0, 1]$ is called a *real arc.*) Let us call a space an *arboroid* if it is an arcwise connected, compact Hausdorff space with the property that any two of its closed connected subsets have a connected intersection. Thus a dendroid may be characterized as a metrizable arboroid. It is worth noting that all of the results stated in this paper for dendroids are also true for arboroids, and in most cases the proofs are quite similar. (An exception is (2.7), which requires that the Hausdorff continuous image of an arc be arcwise connected. The validity of this statement for general (not necessarily metrizable) arcs has been established by Harris [10].)

ADDED IN PROOF. Since submission of this paper, the author has become aware of three related papers which warrant mention. In [25] Helga Schirmer has made a study of the fixed point sets of monotone surjections and homeomorphisms on dendrites which sharpens Theorem (4.2) for the case where D is a dendrite. In [26] she proved a coincidence theorem for set-valued mappings on a tree which has a special case of Theorem (4.1) as a corollary. Finally in [24] Muenzenberger and Smithson obtained the same corollary by a different proof.

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