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FIXED POINT THEOREMS FOR CONTRACTION MAPPINGS WITH APPLICATIONS TO NONEXPANSIVE AND PSEUDO-CONTRACTIVE MAPPINGS

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ABSTRACT. The principal result states that if H is a closed and convex subset of a Banach space X, G a subset of H open relative to H with the origin 0 in G, and if $U : \bar{G} \to H$ is a contraction mapping satisfying (i) $U(x) = \alpha x$ implies $\alpha \leqq 1$ for $x \neq 0$ in the boundary of G relative to the closed subspace $\mathcal{H}$ of X spanned by H, then U has a fixed point in $\bar{G}$. This result is then used to obtain some fixed point theorems for nonexpansive and pseudo-contractive mappings. It may be compared with a recent result of W. V. Petryshyn for "condensing mappings," a class of mappings more general than the contraction mappings. Petryshyn has shown that if G is bounded with 0 in the interior of G and if $U : \bar{G} \to X$ is a condensing mapping satisfying $U(x) = \alpha x$ implies $\alpha \leqq 1$ for x on the boundary of G, then U has a fixed point in $\bar{G}$. This paper shows that for contraction mappings G need not be bounded, and under certain circumstances, 0 may be on the boundary of G.

1. **Introduction.** Although we restrict our attention in this paper to contraction, nonexpansive, and pseudo-contractive mappings, we will compare our principal results with a recent theorem of W. V. Petryshyn for "condensing mappings." We begin with a description of Petryshyn's result.

For a bounded subset A of a real Banach space X, let $\gamma(A)$ denote the measure of noncompactness of A [17], that is,

$$\gamma(A) = \inf\{d : A \text{ can be covered by a finite number of sets each of diameter} \leqq d\}.$$

Following [22] we say that a continuous mapping $T : G \to X$, $G \subset X$, is *condensing* if for every bounded set $A \subset G$ such that $\gamma(A) \neq 0$ it is the case that $\gamma(T(A)) < \gamma(A)$. This class of mappings includes mappings of the form $T = S + C$ where S is a contraction mapping (i.e., there exists $k < 1$ such that $\|S(x) - S(y)\| \leqq k\|x - y\|$ for all x, $y \in G$) and C a compact mapping of G into X.

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It has been shown independently by Furi and Vignoli [8] and by R. D. Nussbaum [20] that if T is a condensing mapping of a bounded closed convex subset of X into itself then T has a fixed point. More recently, using the theory of topological degree developed for condensing mappings by Nussbaum in his University of Chicago thesis [19], Petryshyn has obtained the following result (here $\bar{G}$ and ∂G denote, respectively, the closure and boundary of G).

PROPOSITION 1.1 (PETRYSHYN [22]). *If G is a bounded open subset of a Banach space X with 0 in G and if T is a condensing mapping of $\bar{G}$ into X which satisfies*

(i) $$T(x) = \alpha x \text{ for } x \in \partial G \Rightarrow \alpha \leqq 1,$$

then the fixed point set of T in $\bar{G}$ is nonempty and compact.

The very mild boundary condition (i) is equivalent to the Leray-Schauder condition imposed by F. E. Browder in his study [6] of semicontractive mappings. For convex G it is much weaker than the frequently used assumption $T : \partial G \to \bar{G}$.

The two principal observations of this paper are contained in Section 2. One consequence of Theorem 2.1, the main result of this section, is that Proposition 1.1 remains true *for contraction mappings* without the assumption of boundedness of G (although boundedness of G is essential when T is merely assumed condensing). A second consequence of this theorem is even more significant. We see that it is permissible under certain circumstances for the origin 0 to be on the boundary of G. Our conditions permit the mapping, for example, to have for its domain the intersection of a ball centered at the origin with a cone, with suitable conditions imposed on the boundary implying existence of a fixed point.

The results of Section 2 are used in Section 3 to obtain some new theorems for nonexpansive mappings. In Section 4 we obtain a much more general version of the principal result of Gatica-Kirk [9] for pseudo-contractive mappings, and then we formulate a theorem which may be viewed as an analogue of Borsuk's Antipodal Theorem for these mappings.

2. **Contraction Mappings.** In this section we establish our principal result (Theorem 2.1). The proof of Proposition 1.1 given in [22] amounts to a simple application of Nussbaum's degree theory, an approach which will yield only a special case of our theorem. We give a direct proof patterned after an argument of Browder (cf. [6, Theorem 5]).

To establish notation used below, let H be a closed subset of the Banach space X and let A be a subset of the closed subspace $\mathcal{H}$ of X spanned by H. We will use the symbols $\partial_{\mathcal{H}} A$ and $A_{\mathcal{H}}{}^0$, respectively, to denote the boundary of A relative to $\mathcal{H}$ and the interior of A relative to $\mathcal{H}$.

THEOREM 2.1. *Let H be a closed and convex subset of a Banach space X, G a subset of H open relative to H, and suppose 0 is in G (although possibly $0 \in \partial_{\mathcal{H}} G$). Let $U : \bar{G} \to H$ be a contraction mapping satisfying*

(i) $$U(x) = \alpha x \text{ for } x \in \partial_{\mathcal{H}} G,\ x \neq 0, \Rightarrow \alpha \leqq 1.$$

Then U has a fixed point in G.

We obtain Theorem 2.1 from the following theorem.

THEOREM 2.2. *Let G be a subset of a Banach space with $0 \in \bar{G}$, $U : \bar{G} \to X$ a contraction mapping satisfying*

(i) $$U(x) = \alpha x \text{ for } x \in \partial G,\ x \neq 0, \Rightarrow \alpha \leqq 1.$$

Suppose further that for some $\beta \in (0, 1]$, βU has a fixed point in $\bar{G}$. Then U has a fixed point in $\bar{G}$.

PROOF. Let $k < 1$ be a Lipschitz constant for U. If $U(x) = \lambda x$ for $\lambda > 1$ then

$$\begin{aligned} \|x\| &\leqq \|\lambda x\| \\ &\leqq \|\lambda x - U(0)\| + \|U(0)\| \\ &= \|U(x) - U(0)\| + \|U(0)\| \\ &\leqq k\|x\| + \|U(0)\|. \end{aligned}$$

Hence $(1 - k)\|x\| \leqq \|U(0)\|$ and thus the set

$$E = \{x \in \bar{G} : U(x) = \lambda x \text{ for some } \lambda > 1\}$$

is bounded. Choose M_0 so that $\|x\| < M_0$ if $x \in E$ and let

$$B = \{x \in X : \|x\| < M_0\}.$$

Letting $D = B \cap G$, clearly D is bounded and $U : \bar{D} \to H$. Suppose $x \in \partial D$ with $U(x) = \alpha x$. Then either $x \in \partial B$, in which case $\|x\| = M_0$, or $x \in \partial G$. If $\|x\| = M_0$ then $x \notin E$ so $\alpha \leqq 1$. On the other hand, if $x \in \partial G$ and $x \neq 0$ then $\alpha \leqq 1$ by (i). It follows that D satisfies all the assumptions of G in the theorem and in addition D is bounded. *Thus there is no loss in generality in assuming G is bounded.*

Since the interior G^o of G is open, $(I - U)(G^o)$ is open and $\partial[(I - U)(G^o)] \subset (I - U)(\partial G)$, (cf. [6]). Also, for each $t \in [0, 1]$, $U_t = tU$ is a contraction mapping of $\bar{G} \to X$ so $G_t = (I - U_t)(G^o)$ is open and $\partial G_t \subset (I - U_t)(\partial G)$. Thus if $0 \in \partial G_t$ for $t \in (0, 1)$, then (i) implies $U(0) = 0$.

Now assume that U has no fixed point in $\bar{G}$. Let $M > 0$ be chosen so that $\|U(x)\| \leqq M$ for $x \in \bar{G}$, and let

$$T = \{t \in [0, 1] : 0 \in G_t\}.$$

Observe that if $t \in (0, 1]$ then by (i) either $t \in T$ or $0 \notin \bar{G}_t$. Also, for $\beta \in (0, 1]$, βU has a fixed point $x_0 \in \bar{G}$. Since $U(x_0) = \beta^{-1}x_0$, (i) implies $x_0 \in G^o$ and thus $\beta \in T$. Therefore $T \neq \emptyset$ and $\sup T > 0$.

The argument is completed by successively establishing $\sup T = 1$ and $1 \in T$, thus contradicting our assumption.

Suppose $\alpha = \sup T < 1$. Let $\{\alpha_n\}$ be a sequence of points of T chosen so that $\alpha_n \to \alpha$, and for each n let $x_n \in G$ be such that $x_n - \alpha_n U(x_n) = 0$. Then

$$\begin{aligned}\|x_n - x_m\| &= \|\alpha_n U(x_n) - \alpha_m U(x_n) + \alpha_m U(x_n) - \alpha_m U(x_m)\| \\ &\leqq |\alpha_n - \alpha_m| M + \alpha_m \|x_n - x_m\|.\end{aligned}$$

Hence $(1 - \alpha)\|x_n - x_m\| \to 0$ as $n, m \to \infty$ and since $\alpha < 1$ there exists a point $x \in H$ such that $x_n \to x$. Further, since $\bar{G}$ is bounded, the sequence of functions $\{\alpha_n U\}$ converges uniformly to αU on $\bar{G}$, and hence

$$\alpha U(x) = \lim_{n \to \infty} \alpha_n U(x_n).$$

This implies $x - \alpha U(x) = 0$ and thus $0 \in \bar{G}_\alpha$ which, because of (i) and the fact that 0 is not a fixed point of U, in turn implies $0 \in G_\alpha$. Thus $\alpha \in T$.

Therefore there exists $x_0 \in G^o$ satisfying $x_0 - \alpha U(x_0) = 0$. Then for $t \in [0, 1]$,

$$\begin{aligned}\|x_0 - tU(x_0)\| &= \|x_0 - tU(x_0) - x_0 + \alpha U(x_0)\| \\ &= |t - \alpha|\, \|U(x_0)\| \\ &\leqq |t - \alpha| M.\end{aligned}$$

Now, $0 \notin G_t$ for all t such that $\alpha < t \leqq 1$. Hence $0 \notin \bar{G}_t$ for these t. We may choose $t_0 \in (\alpha, 1)$ and a sequence $\{t_n\}$ such that $t_n \downarrow \alpha$ and $t_n < t_0$ for all n. Further, since $0 \notin \bar{G}_{t_n}$ and $x_0 - t_n U(x_0) \in G_{t_n}$ the segment joining 0 and $x_0 - t_n U(x_0)$ must intersect ∂G_{t_n} yielding a point $y_n \in \partial G_{t_n}$ such that $\|y_n\| \leqq |t_n - \alpha|M$, $n = 1, 2, \cdots$. Now $\partial G_{t_n} \subset (I - t_nU)(\partial G)$ so there exists $x_n \in \partial G$ such that $x_n - t_nU(x_n) = y_n$; thus

$$\|x_n - t_nU(x_n)\| \leqq |t_n - \alpha|M.$$

Therefore

$$\begin{aligned}
&\|x_n - x_n\| \\
&\quad \leqq \|x_n - t_nU(x_n)\| + \|t_nU(x_n) - x_m\| \\
&\quad \leqq |t_n - \alpha|M + \|t_nU(x_n) - t_mU(x_n)\| + \|t_mU(x_n) - x_m\| \\
&\quad \leqq |t_n - \alpha|M + |t_n - t_m|M + \|t_mU(x_n) - t_mU(x_m)\| + \|x_m - t_mU(x_m)\| \\
&\quad \leqq (|t_n - \alpha| + |t_n - t_m|)M + t_m\|x_n - x_m\| + |t_m - \alpha|M \\
&\quad \leqq (|t_n - \alpha| + |t_n - t_m| + |t_m - \alpha|)M + t_0\|x_n - x_m\|.
\end{aligned}$$

Thus:

$$(1 - t_0)\|x_n - x_m\| \to 0 \text{ as } n, m \to \infty$$

and we conclude that $\{x_n\}$ is a Cauchy sequence of points of ∂G. Letting $\lim_{n\to\infty} x_n = x \in \partial G$ one has, as before, $x - \alpha U(x) = \lim_{n\to\infty}(x_n - t_nU(x_n)) = 0$ and this contradicts (i). Therefore the assumption $\alpha < 1$ leads to a contradiction.

To complete the proof we need only show $1 \in T$. Suppose the contrary. Then there exists $r > 0$ such that if $\|z\| \leqq r$ then $z \notin G_1$ (because $0 \notin \bar{G}_1$ under the assumption U has no fixed point in $\bar{G}$). Select $\alpha_n \in T$ such that $\alpha_n \uparrow 1$. Then for each n there exists $x_n \in G$ such that $x_n - \alpha_nU(x_n) = 0$. Further, an integer N can be chosen so that $1 - \alpha_N < r/M$. Thus

$$\begin{aligned}
\|x_N - U(x_N)\| &= \|\alpha_NU(x_N) - U(x_N)\| \\
&= (1 - \alpha_N)\|U(x_N)\| \\
&< r.
\end{aligned}$$

This contradicts $x_N - U(x_N) \in G_1$. Therefore $1 \in T$, completing the proof.

Proof of Theorem 2.1. As before $\bar{G}$ may be assumed to be bounded. Since 0 is an interior point of $\bar{G}$ relative to H, for $\beta \in (0, 1)$ sufficiently small, $\beta U : \bar{G} \to \bar{G}$. Hence by the Contraction Principle βU has a fixed point in $\bar{G}$. The proof is completed by applying Theorem 2.2 with the subspace $\mathscr{H}$ taken as the setting.

By taking $H = X$ in Theorem 2.1 we obtain the following:

COROLLARY 2.3. *Let G be an open subset of X with $0 \in G$ and let $U : \bar{G} \to X$ be a contraction mapping satisfying* (i) *of Proposition* 1.1 *on ∂G. Then U has a fixed point in G.*

As we remarked in the introduction, boundedness of G is essential in Proposition 1.1 for condensing mappings. In fact, this is easily verified by considering a translation of the plane.

Corollary 2 of Assad-Kirk [1] states that if H is a closed convex subset of X, K a closed subset of H, and $F : K \to H$ a contraction mapping which maps the relative boundary of K in H back into K, then F has a fixed point in K. This, with Theorem 2.2, yields:

COROLLARY 2.4. *Let G be a subset of X with $0 \in \bar{G}$, and $U : \bar{G} \to X$ a contraction mapping of $\bar{G} \to X$ satisfying* (i) *of Theorem* 2.2. *Suppose that for some $\beta \in (0, 1]$, $\beta U : \partial G \to \bar{G}$. Then U has a fixed point in $\bar{G}$.*

We might remark that the results of this section guarantee only *existence* of a (unique) fixed point, and the question of how one might approximate this fixed point is left open. In contrast to this, the result of Assad-Kirk referred to above describes an iterative procedure for approximating the point in question.

3. **Nonexpansive mappings.** A mapping $T : K \to X$ is called *nonexpansive* if $\|Tx - Ty\| \leqq \|x - y\|$ for all $x, y \in K$. Extensive study of fixed point theory for nonlinear nonexpansive mappings was initiated in 1965 with the proof that if X is uniformly convex and $T : K \to K$ is nonexpansive, where K is a nonempty, bounded, closed and convex subset of X, then T has a fixed point in K (cf. F. E. Browder [3], D. Göhde [**10**], W. A. Kirk [13]). Since that time a number of papers have been published by several authors treating analogues, generalizations, and applications of the above result, as well as certain iterative procedures for approximating the fixed point when its existence is known.

In this section we obtain a new fixed point theorem for nonexpansive mappings by applying Theorem 2.1 of the preceding section. Notice that assumption (ii) below, suggested by the proof of Theorem 2.2, always holds if the range of the mapping is bounded. Also, as before, $\mathcal{H}$ denotes the closed subspace of X spanned by H.

THEOREM 3.1. *Let H be a closed and convex subset of a Banach space X, G a subset of H open relative to H, and suppose* 0 *is in G. Let $T : \bar{G} \to H$ be a nonexpansive mapping satisfying:*

(i) $Tx = \alpha x$ *for* $x \in \partial_{\mathcal{H}} G$, $x \neq 0, \Rightarrow \alpha \leqq 1$.

(ii) *The set* $E = \{x \in G : Tx = \lambda x \text{ for some } \lambda > 1\}$ *is bounded.*

(iii) $(I - T)(\bar{G})$ *is closed.*
Then T *has a fixed point in* $\bar{G}$.

Proof. Since E is bounded and T nonexpansive, $T(E)$ is bounded. Let $M > 0$ be such that $\|T(x)\| \leqq M$ for all $x \in E$ and let $\{\alpha_n\}$ be a sequence of numbers in the interval $(0, 1)$ such that $\alpha_n \to 1$. Then for each n there exists, by Theorem 2.1, an element $x_n \in \bar{G}$ such that $x_n = \alpha_n T(x_n)$. Since $\alpha_n \in (0, 1)$, $1/\alpha_n > 1$ and thus $x_n \in E$ for all n. Hence

$$\begin{aligned} \|x_n - T(x_n)\| &= \|\alpha_n(Tx_n) - T(x_n)\| \\ &\leqq |\alpha_n - 1|M. \end{aligned}$$

Therefore $x_n - T(x_n) \to 0$ yielding $0 \in (I - T)(\bar{G})$ by (iii). It follows that T has a fixed point in $\bar{G}$.

A Banach space is said to satisfy *Opial's condition* [18] if whenever a sequence $\{x_n\}$ converges weakly to $x_0 \in X$ then it is the case that

$$\liminf_{n\to\infty} \|x_n - x\| > \liminf_{n\to\infty} \|x_n - x_0\|$$

for all $x \in X$, $x \neq x_0$. (Opial has shown [21] that if X satisfies this condition and if C is a weakly compact subset of X with $T : C \to X$ nonexpansive, then $I - T$ is demiclosed.)

Corollary 3.2. *Suppose* X *is a reflexive space that satisfies Opial's condition, with subsets* G *and* H *as in the preceding theorem. If* $T : \bar{G} \to H$ *is a nonexpansive mapping with bounded range which satisfies* (i) *on* $\partial_H G$, *then* T *has a fixed point in* $\bar{G}$.

Proof. Select $\alpha_n \in (0, 1)$ so that $\alpha_n \to 1$. For each n, $\alpha_n T$ satisfies all the assumptions of Theorem 2.1 so there exists $x_n \in \bar{G}$ such that $\alpha_n T(x_n) = x_n$. Furthermore, because X is reflexive and the range of T bounded it may be assumed that the sequence $\{T(x_n)\}$ converges weakly, say to y. Since $\alpha_n \to 1$ it follows that $\alpha_n T(x_n) \to y$ weakly and thus $x_n \to y$ weakly. As in Theorem 3.1 one sees that $x_n - T(x_n) \to 0$ strongly. Thus

$$\begin{aligned} \liminf_{n\to\infty} \|x_n - y\| &\leqq \liminf_{n\to\infty} \|T(x_n) - T(y)\| \\ &= \liminf_{n\to\infty} \|x_n - T(y)\|, \end{aligned}$$

and by Opial's condition $T(y) = y$.

Spaces known to satisfy Opial's condition include all reflexive spaces which possess weakly continuous duality maps ([11]), in particular the Hilbert spaces and the ℓ^p spaces for $1 < p < \infty$ ([4]).

Finally, we remark that the hypotheses of Theorem 3.1 and Corollary 3.2 may be modified in the obvious ways if Theorem 2.2, Corollary 2.3, or Corollary 2.4 is used instead of Theorem 2.1. For example, Theorem 2.2 yields:

THEOREM 3.3. *Let G be an open subset of a Banach space X with $0 \in \bar{G}$, and let $T : \bar{G} \to X$ be a nonexpansive mapping satisfying* (i), (ii), (iii) *of Theorem* 3.1. *If βT has a fixed point in $\bar{G}$ for some $\beta \in (0, 1]$, then T has a fixed point in $\bar{G}$.*

PROOF. The same as the proof of Theorem 3.1, except that for $\alpha_n \to 1$, $\alpha_n \in (0, 1)$, one is only assured that $\alpha_n T$ satisfies the hypothesis of Theorem 2.2 for n sufficiently large.

4. **Pseudo-contractive mappings.** A mapping $U : D \to X$ is said to be pseudo-contractive [**5**] if for each $r > 0$ and $u, v \in D$,

$$\|u - v\| \leqq \|(1 + r)(u - v) - r(U(u) - U(v))\|.$$

These mappings, easily seen to be more general than the nonexpansive mappings, are of interest because of their close connection with "accretive" operators, a class of operators important in the study of certain nonlinear differential equations. Specifically, a mapping $U : D \to X$ is pseudo-contractive if and only if $(I - U)$ is accretive (see Browder [**5**], Kato [**12**]).

Fixed point theorems for pseudo-contractive mappings may be found in [**1, 5, 9, 15, 23**].

By following the argument given for Theorem 1 of Gatica-Kirk [**9**] (but using Theorems 2.1 and 3.1 of this paper instead of Petryshyn's Theorem 7 of [**22**]) one can obtain the following result for mappings which include the lipschitzian pseudo-contractive mappings. This theorem generalizes Theorem 1 of [**9**] in several ways.

THEOREM 4.1. *Let H be a closed and convex subset of a Banach space X, G a subset of H open relative to H, and suppose 0 is in G. Let $U : \bar{G} \to H$ be a lipschitzian mapping with Lipschitz constant k, and suppose U satisfies*:

(i) *$U(x) = \alpha x$ for $x \in \partial_H G$, $x \neq 0 \Rightarrow \alpha \leqq 1$.*

(ii) *The set $E = \{x \in \bar{G} : U(x) = \lambda x$ for some $\lambda > 1\}$ is bounded.*

(iii) *$(I - U)(\bar{G})$ is closed.*

(iv) *For some $r \in (0, 1)$ such that $rk < 1$,*

$$\|x - y\| \leqq \|(1 + r)(x - y) - r(U(x) - U(y))\|, \quad x, y \in \bar{G}.$$

Then U has a fixed point in $\bar{G}$.

OUTLINE OF PROOF. With r determined by (iv), rU is a contraction mapping. Define mappings S, T of $\bar{G}$ into $\mathcal{H}$ by $S = (1-r)I$ and $T = I - rU$. Then $T(G_{\mathcal{H}}{}^{o})$ is open in $\mathcal{H}$, $\partial_{\mathcal{H}} T(G) = T(\partial_{\mathcal{H}} G)$, and thus $T(\bar{G}) = \overline{T(G)}$. Also, since rU satisfies (i) on $\partial_{\mathcal{H}} G$, by Theorem 2.1 there exists $x \in \bar{G}$ such that $x = rU(x)$. Furthermore, (i) implies $x \in G_{\mathcal{H}}{}^{o}$ and thus $0 \in T(G_{\mathcal{H}}{}^{0})$, i.e., 0 is an interior point, relative to $\mathcal{H}$, of the set $B = T(\bar{G}) = \overline{T(G_{\mathcal{H}}{}^{o})}$. To complete the proof one shows that the mapping $H = ST^{-1}$ is a nonexpansive mapping of $B \to \mathcal{H}$ satisfying

(i)′ $H(x) = \alpha x$ for $x \in \partial_{\mathcal{H}} B \Rightarrow \alpha \leqq 1$.

(ii)′ The set $\{x \in B : H(x) = \lambda x \text{ for some } \lambda > 1\}$ is bounded.

(iii)′ $(I - H)(\bar{B})$ is closed.

The details of showing that H is nonexpansive and satisfies (i)′ and (iii)′ are found in [9], and (ii)′ is established in the same way as (i)′. We include the proof of (ii)′ here because it is not explicitly indicated in [9].

Observe that if $H(x) = \lambda x, \lambda > 1$, then $ST^{-1}(x) = \lambda x$ so $(1-r)T^{-1}(x) = \lambda x$ yielding $T(\lambda x/(1-r)) = x$. This implies

$$\frac{\lambda}{1-r}\, x - rU\left(\frac{\lambda}{1-r}\, x\right) = x$$

and thus

$$U\left(\frac{\lambda}{1-r}\, x\right) = \frac{(\lambda - 1 + r)}{r(1-r)}\, x$$

$$= \mu\left(\frac{\lambda}{1-r}\, x\right)$$

where $\mu = (\lambda - 1 + r)/\lambda r > 1$. By (ii) there exists $M > 0$ such that $\|\bar{x}\| \leqq M$ if $U(\bar{x}) = \mu\bar{x}, \mu > 1$. Thus

$$\|x\| \leqq \|\lambda x\| \leqq (1-r)M$$

and we see that (ii)′ follows from (ii).

Having established (i)′, (ii)′, (iii)′ for H, it follows from Theorem 3.1 that H has a fixed point $y \in B$. As in [9] we have, upon letting $x = T^{-1}(y)$, $S(x) = ST^{-1}(y) = H(y) = y = T(x)$ and thus $(1-r)x = x - rU(x)$ yielding $U(x) = x$.

Because Opial's condition is satisfied in Hilbert space, the following theorem extends a theorem of Browder [2].

THEOREM 4.2. *Let G be an open subset of X containing* 0 *such that if $x \in \partial G$ then $-x \in \bar{G}$. Suppose $U : \bar{G} \to X$ is a lipschitzian map-*

ping with Lipschitz constant k satisfying $U(x) = -U(-x)$ for $x \in \partial G$. In addition, suppose assumptions (ii), (iii), (iv) *of Theorem* 4.1 *are satisfied. Then U has a fixed point in $\bar{G}$.*

PROOF. In order to apply Theorem 4.1 we need only show that $U(x) \neq \lambda x$ if $x \in \partial G$ and $\lambda > 1$. If $U(x) = \lambda x, \lambda > 1$, then

$$\begin{aligned} \|U(x) - U(-x)\| &= 2\lambda\|x\| \\ &\leqq k\|x - (-x)\| \\ &= 2k\|x\| \end{aligned}$$

yielding $\lambda \leqq k$. On the other hand, by (iv),

$$\begin{aligned} 2\|x\| &= \|x - (-x)\| \\ &\leqq \|(1 + r)(x - (-x)) - r(U(x) - U(-x))\| \\ &= \|2(1 + r)x - 2r\lambda x\| \\ &= 2|1 + r - \lambda r|\ \|x\|. \end{aligned}$$

This implies $|1 + r - \lambda r| \geqq 1$. Since $1 + r - \lambda r \geqq 1$ is impossible (because $\lambda > 1$) we must have $1 + r - \lambda r \leqq -1$, and this implies $1 - \lambda \leqq -2/r$ which in turn implies $\lambda > 2k + 1$. This contradicts the fact that $\lambda \leqq k$, completing the proof.

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