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ON SOME OLD PROBLEMS OF FIXED POINT THEORY

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It is not uncommon to begin an expository paper with the modest admission that the paper contains no new mathematics. I must make the even more modest admission that much of what I will write overlaps other expositions; especially the excellent paper of Fadell [17]. The purpose of this paper, however, is quite different from Fadell's. He described some of the new discoveries in fixed point theory; I wish to call attention to some of the problems which are, nevertheless, still with us.

1. **Continua in the Plane.** Let X be a space and $f: X \to X$ a map (continuous function), then $x \in X$ is a *fixed point* of f if $f(x) = x$. A space X has the *fixed point property* [*for homeomorphisms*] if every map [homeomorphism] $f : X \to X$ has a fixed point. A *continuum* is a compact connected Hausdorff space. Denote the plane by R^2.

PROBLEM 1. *If $X \subset R^2$ is a continuum such that $R^2 - X$ is connected, does X have the fixed point property?*

I do not know who first asked the question, but its age can be estimated from the reference to it in 1929* as a "well-known problem" [1]. In order to discuss the history of this problem, we must introduce its little brother:

PROBLEM 1½. *Under the hypotheses of Problem* 1, *does X have the fixed point property for homeomorphisms?*

The first solution to Problem 1½ was by Ayers [1] (1929), but under the additional hypothesis that X be locally connected. It was soon

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*The date of publication of a journal article can be very misleading information when one is trying to understand the history of a mathematical problem because there is so much variation in the length of time it takes for a paper to be published after it is submitted to a journal. Therefore, the dates used in this paper are always the earliest that can be associated with the referenced paper. Whenever possible, we have used the date the manuscript was received by the editors, or the date of presentation to a scholarly meeting, in preference to the publication date.

recognized that results of Borsuk [4] (1929) implied the same partial solution to Problem 1 itself. Hamilton [21] (1937) settled Problem 1½ in the affirmative assuming that X is *hereditarily decomposable,* that is, that every subcontinuum of X can be expressed as the union of two of its proper subcontinua. Seiklucki [38], [39] (1967) solved Problem 1 under the hypothesis that the *boundary* of X is hereditarily decomposable; thus improving Hamilton's result by getting a stronger conclusion out of a weaker hypothesis. In the most recent advance, Hagopian [20] (1971) has announced a solution to Problem 1 assuming that X is arcwise connected. However, that is not the end of the story of Problem 1½ because in the meantime, Choquet [13] (1941) proved that *some* homeomorphisms of X have fixed points, namely those which extend to periodic homeomorphisms of the plane, of period other than two. Then, in 1949, Cartwright and Littlewood [12] proved that all homeomorphisms X which extend to orientation preserving homeomorphisms of R^2 have fixed points. Therefore, we know how to solve Problem 1 if we are permitted to put some further restriction on X and we know that some homeomorphisms of X have fixed points anyway, but there the matter rests at present.

2. **Products of Manifolds.** Denote euclidean n-space by R^n and let $R^n_+ = \{(x_1, \cdots, x_n) \in R^n \mid x_n \geqq 0\}$. By a *manifold* we mean a compact metric space, every point of which has a neighborhood homeomorphic either to R^n or, in the case of boundary points, to R^n_+.

PROBLEM 2. *If manifolds M and N have the fixed point property does $M \times N$ have the fixed point property?*

This problem has a long history, in the sense that the very first important fixed point theorem was actually a result of this kind. Let I be the unit interval and let $I^n = I \times I \times \cdots \times I$. It is an easy exercise for a calculus student to prove that I has the fixed point property. It requires more sophisticated mathematics to prove the "Brouwer Fixed Point Theorem" [7] (1910) (but, according to Bing [2], first published in 1904 by Bohl [3]) which states that I^n has the fixed point property. Of course the Brouwer Theorem would follow from an affirmative answer to Problem 2 using induction.

Problem 2 has a famous ancestor, which was formally posed by Kuratowski in 1930 [27]. It is:

PROBLEM II. *If X and Y are locally connected metric continua with the fixed point property, does $X \times Y$ have the fixed point property?*

Dyer [16] (1955) proved that the answer to Problem II is "yes" even if the continua are not metric, provided that they are *chainable* that is, that every open cover has a refinement $\{U_1, \cdots, U_n\}$ such that $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leqq 1$. On the other hand, Connell [14] (1959) proved that the answer to Problem II is "no" for metric spaces in general, even if they are subsets of the plane, and Klee [25] (1960) did the same with bounded subsets of R^2. Knill [26], in 1965, gave a "near counterexample" for Problem II in which $X = I$ and Y is even a contractible metric continuum, but Y was not locally connected. Finally, in 1967, Lopez [34] used observations of Fadell to give a resounding "no" answer to Problem II because in his example, $X = I$ and Y is a finite polyhedron with the fixed point property, and yet $X \times Y$ does not have the fixed point property.

A simplified version of Lopez's original example, due to Bredon, is easy to describe. Let Γ be either the complex numbers $\underset{\sim}{C}$ or the quaternions $\underset{\sim}{H}$, topologized by identification with R^2 or R^4. Let $\Gamma^{n+1} = \Gamma \times \cdots \times \Gamma$ and call $x, y \in \Gamma^{n+1} - 0$ *equivalent* if there exists $\gamma \in \Gamma$ such that $x = \gamma y$, then the resulting quotient space is denoted by ΓP^n. For a space A, let $\sum A$ be $A \times I$ with $A \times \{0\}$ and $A \times \{1\}$ each identified to a point. In Bredon's example, $X = I$ and Y consists of $\sum \underset{\sim}{C}P^4$ and $\underset{\sim}{H}P^3$ joined at a single point. In a neighborhood of the point where $\sum \underset{\sim}{C}P^4$ and $\underset{\sim}{H}P^3$ meet, the example looks just about as *unlike* a manifold as it is possible for a polyhedron to look. Fadell, perhaps encouraged by this fact, posed Problem 2 in [17].

Incidentally, the answer to Problem 2 is "yes" if one of the manifolds is the unit interval (see [11, VIII.F]) so the Brouwer Fixed Point Theorem can indeed be proved by induction using the right form of Problem 2. Actually, it's a lot easier to prove the Brouwer Theorem directly than it is to verify that $M \times I$ has the fixed point property whenever M does.

We have seen that Problem 2 represents an effort to state the "right" version of Kuratowski's question; after Fadell and Lopez wiped out the original version. Such an effort has a very honorable and famous precedent. Recall that Poincaré originally conjectured that a 3-dimensional manifold without boundary whose first integer homology vanishes is a sphere. A counterexample, the so-called "Poincaré sphere" was soon discovered. A corrected version of the Poincaré conjecture, with "simply-connected" replacing the homology condition, is still unsettled. Of course, this is not to suggest that Problem 2 is in the same league as the Poincaré conjecture! In fact, it is really too early to say whether Problem 2 is difficult or not.

For recent advances concerning the fixed point property on "nice" spaces such as polyhedra or manifolds, as opposed to the pathological spaces one meets in connection with Problem 1, see Fadell [18] and Bredon [6].

3. **The Lefschetz Fixed Point Theorem.** Part of the difficulty of this next problem comes from the fact that there is no general agreement about how it should be stated. Therefore, it will be necessary to outline the history of the subject before we can discuss what the problem is.

Let X be a space and denote its rational homology by $H_*(X) = \{H_p(X) \mid p \geqq 0\}$. Each $H_p(X)$ is a vector space. Call $H_*(X)$ *finitary* if each $H_p(X)$ is finite dimensional and all but a finite number of the $H_p(X)$ are trivial. A map $f: X \to X$ induces linear transformations $f_{*p}: H_p(X) \to H_p(X)$. If $H_*(X)$ is finitary, we can define $L(f)$, the *Lefschetz number* of f by

$$L(f) = \sum_{p=0}^{\infty} (-1)^p \operatorname{trace}(f_{*p})$$

where $\operatorname{trace}(f_{*p}) = 0$ if $H_p(X)$ is trivial. A "Lefschetz Fixed Point Theorem" is a statement of the form: if $f: X \to X$ is a map such that $L(f) \neq 0$, then f has a fixed point.

Of course, the Lefschetz Theorem is not going to be true without some hypotheses. First of all, we would like to know that $H_*(X)$ is finitary so that $L(f)$ is defined. Even very well-behaved spaces, such as the plane with all points represented by two integer coordinates removed, lack this property. That requirement alone is not enough, since the fixed point free map $f: R^1 \to R^1$ defined by $f(x) = x + 1$ turns out to have a nonzero Lefschetz number. Compactness of X is not enough either, even when $H_*(X)$ is finitary. In 1934, Borsuk [5] constructed an example of a map $f: X \to X$ without fixed points, where X is a locally connected metric continuum, and yet $L(f) = 1$. Lefschetz [28] first announced his fixed point Theorem in 1923, with the hypotheses that X is both a finite polyhedron and a manifold without boundary. He was soon able to include manifolds with boundary [29] (1926). In 1928, Hopf [23] proved the Lefschetz Theorem for all finite polyhedra. Lefschetz extended his fixed point theorem to compact ANR's (*absolute neighborhood retracts,* a generalization of polyhedra) in a disguised form in 1930 [30] and explicitly in 1937 [31]. It was proved by Hanner [22] in 1950 that if X is an ANR then X is ϵ*-dominated* by polyhedra, that is, given $\epsilon > 0$ there exists a polyhedron P and maps $\phi: X \to P$ and $\psi: P \to X$ such that $\psi\phi$ is homotopic to the identity map on X by a homotopy that moves no point more than ϵ. Using this result, it is easy to reduce the Lefschetz Theorem for compact ANR's to the corresponding theorem for finite polyhedra (see [11, III.C]). Thus although Lefschetz's 1937 version of his theorem does include some infinite-dimensional spaces (for example, the Hilbert cube), it works because these spaces are so much like finite-dimensional ones that the tools of algebraic topology can't tell the difference.

For a long time after 1937, research related to the Lefschetz Fixed Point Theorem was concerned with generalizations of the theorem to various kinds of compact spaces other than ANR's and to other matters which are not of concern to us here. However, one important development was Dugundji's discovery [15] in 1951 that a convex subset of a normed linear space is an ANR. Consequently, many of the fixed point problems that are of interest in analysis are questions about maps on (in general noncompact) ANR's. This suggests, if your mind works that way, that it would be desirable to prove the Lefschetz Theorem for all ANR's.

The humble example $f(x) = x + 1$ on the real line shows that you can't do it. In fact, the "punctured plane" example above proves that a noncompact ANR need not have finitary homology, so the Lefschetz number need not even be defined. The way around these difficulties is suggested by the approach to Problem 1 that Cartwright and Littlewood used [12]. They observed that when, in a problem in differential equations, we want to know that a map on a continuum satisfying the hypotheses of Problem 1 has a fixed point, we are not really confronted with just any old map, but with a very special kind which arises from the differential equations setting. In Cartwright and Littlewood's paper, we recall that the maps were homeomorphisms which extend to orientation-preserving homeomorphisms of the plane. Before discussing what kind of maps are appropriate to the present situation, let us state our problem in the form suggested by this discussion.

PROBLEM $3 - \epsilon$. *Prove the Lefschetz Fixed Point Theorem for all analytically important maps on ANR's.*

Of course the vague phrase "analytically important maps" makes this form of the problem most unsatisfying (and thus not worthy of being called "Problem 3"), but at least it suggests a direction in which to proceed: try first to prove the Lefschetz Theorem for *some* "analytically important maps" and then attempt to improve your result until you can include all of them.

What are some "analytically important maps"? The fixed point results which seem to have been used most frequently in analysis are the fixed point theorems of Banach and of Schauder. It is the latter result [36] (1930) that interests us; even though its proof is, in fact, a rather easy application of the Lefschetz Fixed Point Theorem for compact ANR's. A set A in a space X is a *relatively compact* subset if the closure of A in X is compact. Let X be a space, then $f : X \to X$ is a *compact* map if $f(X)$ is relatively compact. The Schauder Fixed Point Theorem states that if $f : X \to X$ is a compact map on a Banach space, then f has a fixed point.

With the Schauder Theorem in mind, we ask if the Lefschetz Fixed Point Thorem is true for compact maps on ANR's. An affirmative answer was supplied by Granas [19] in 1967. The Lefschetz number of a compact map $f : X \to X$ on an ANR can still be defined because f_{*p} has finite-dimensional image for all p and is nonzero for only a finite number of integers p. The proof is a variant of the compact ANR case: use ϵ-domination to reduce the situation to the finite polyhedral case.

The next part of the story of Problem 3 also goes back to the Schauder Fixed Point Theorem. An equivalent form of the theorem states that if $f : X \to X$ is a map on a Banach space such that f maps bounded sets to relatively compact sets and $f(X)$ is bounded, then f has a fixed point. In 1958, F. Browder [8] generalized this result in the following manner. Given $f : X \to X$, define $f^1(x) = f(x)$, $f^2(x) = f(f(x))$, and, in general, $f^m(x) = f(f^{m-1}(x))$. Browder proved that if $f : X \to X$ is a map on a Banach space that maps bounded sets to relatively compact sets and if $f^m(X)$ is bounded for some m, then f has a fixed point. Since, as we noted above, the Schauder Theorem is a direct consequence of the Lefschetz Fixed Point Thorem, Browder was led to look for a generalization of the Lefschetz Theorem along the lines of his generalization of Schauder's Theorem.

Browder's solution rests on some important concepts due to Leray. Leray wrote a paper [32] in the early 1940's concerning fixed point theory on compact spaces in which, for a map $f : X \to X$, the set $\bigcap_{m>0} f^m(X)$, now called the *core* of f, played a major role. In part of this work, Leray had to consider the restriction $(f \mid \text{core}) :$ core $\to$ core of f to its core. In order to have a Lefschetz theory for this map, he had to add the hypothesis to several of his theorems that the homology of the core of f was finitary. In order to eliminate this unnatural and quite restrictive hypothesis from his results, Leray much later [33] (1959) developed a theory of a "generalized trace" and the corresponding "generalized Lefschetz number". In the setting of Leray's earlier paper, the map $(f \mid \text{core})$ does have a generalized

Lefschetz number even when $H_*(\text{core})$ is not finitary, so he was able to remove that annoying hypothesis from his previous results.

Browder's form of the Lefschetz Fixed Point Theorem [10] (1967) states that if $f: X \to X$ is a map on an ANR such that f is "locally compact" in the sense that every point x has a neighborhood whose image under f is relatively compact and such that $f^m(X)$ is relatively compact for some m, then the generalized Lefschetz number $L(f)$ exists and if it is not zero, then f has a fixed point. The proof consists of showing that the hypotheses imply the existence of K, a compact ANR in X, such that $f(K) \subseteq K$ and $L(f)$ equals the ordinary Lefschetz number of $f \mid K$. Thus Browder's theorem reduces, ultimately, to the classical Lefschetz Fixed Point Theorem.

The most recent chapter in the story of the efforts to solve Problem 3 is very helpful in showing us where we stand with regard to the problem. This version of the Lefschetz Fixed Point Theorem was presented by Eells at a conference in 1969 (unpublished). It is a generalization of Browder's Theorem. Like Browder, Eells uses Leray's generalized Lefschetz number and a line of argument that reduces the theorem to the compact ANR case. Eells' hypotheses for a map $f: X \to X$ on an ANR are:

(1) The core C of f is compact and not empty.

(2) There is a family $\mathcal{U} = \{U\}$ of neighborhoods of C in X such that

(a) if an open set V contains C then $U \subseteq V$ for some $U \in \mathcal{U}$,

(b) $f(U) \subset U$ for all $U \in \mathcal{U}$,

(c) if $U \in \mathcal{U}$ and $x \in X$ then $f^m(x) \in U$ for some m.

(3) There exists a neighborhood U of C in X such that $f(U)$ is relatively compact.

The reason this result is not considered a satisfactory solution to Problem $3 - \epsilon$ is that one of the hypotheses is so restrictive that Eells' Theorem does not apply to all, or even most, "analytically important maps". The offending hypothesis is not, as one might first imagine, the complicated number (2), but rather the last hypothesis. Note that hypothesis (3) is really Granas' hypothesis, that f be a compact map, except that the entire space has been replaced by a smaller open set. The trouble with such a hypothesis is that in many interesting ANR's, the compact sets may be very "thin" in the sense that they must have empty interiors. Therefore, a hypothesis that requires maps to take a "fat" (i.e., open) set into a compact one eliminates many "analytically important maps".

It is tempting to restate Problem $3 - \epsilon$ just by saying "prove Eells' result without using hypothesis (3)". There is no doubt that most people would consider that a satisfactory solution to Problem $3 - \epsilon$, but to require Eells' other hypotheses without modification doesn't leave us much room to maneuver. Browder's requirement, that the map be locally compact, implies hypothesis (3), so we can not seek help from that direction. A better form of Problem 3 seems to be:

PROBLEM $3 - \epsilon/2$. *Prove the Lefschetz Fixed Point Theorem for an impressively large class of maps on ANR's; without requiring that the maps take any open sets to relatively compact sets.*

Of course the statement is still rather vague, but I don't think it will be hard to recognize the correct solution once someone finds it.

As I look at the various generalizations of the Lefschetz Theorem since Hopf's polyhedral version of 1928, I wonder if perhaps the real challenge is: prove any version of the Lefschetz Fixed Point Theorem by a method other than reducing the question to the Lefschetz-Hopf Theorem. In other words, the problem is to find a truly infinite-dimensional version of the Lefschetz Fixed Point Theorem.

4. **The Nielsen Number.** Let $f: X \to X$ be a map on a finite polyhedron, then f can be approximated, according to a theorem of Hopf [23] (1928), by a map $\bar{f}$ with a finite number of fixed points, each in a neighborhood of X homeomorphic to a euclidean space. Identify a neighborhood of a fixed point of $\bar{f}$ with euclidean space, with that point at the origin, and call the fixed point "positive" or "negative" depending on which the Jacobian of the map of euclidean space "$\bar{f}$ minus identity" is at that point (it can't be zero). Fixed points x and x' of $\bar{f}$ are said to be *equivalent* if there is a path P in X from x to x' such that P and $f(P)$ are homotopic by a homotopy keeping x and x' fixed. The *index* of an equivalence class of fixed points of $\bar{f}$ is the number of positive points in the class minus the number of negative ones. The *Nielsen number* $N(f)$ of f is the number of equivalence classes of fixed points of $\bar{f}$ which have non-zero index. It is true, but by no means obvious, that homotopic maps have the same Nielsen number. Therefore, the definition of the Nielsen number of a map f is independent of the choice of approximating map $\bar{f}$. Furthermore, every map homotopic to f has at least $N(f)$ fixed points. This last statement, known as the Nielsen Fixed Point Theorem, is the reason we are interested in the Nielsen number. Nielsen introduced his theory in 1927 for maps on closed surfaces [35]. The generalization to all finite polyhedra was due to Wecken [41] (1940). It might seem that the Nielsen Theorem, which offers information on the number of fixed points of a map, is so much more powerful than the Lefschetz Fixed Point Theorem, which only promises the existence of a single fixed point, that it is a waste of time to discuss the

Lefschetz Theorem. However, in order to compute the Lefschetz number of a map $f: X \to X$, we need to know only enough about the homomorphisms of the homology of X induced by f to compute traces. On the other hand, it should not surprise the reader that the computation of the Nielsen number of f directly from the definition we gave above is almost always impossible. Thus, although the Nielsen Fixed Point Theorem is, in theory, more powerful than the Lefschetz Fixed Point Theorem, in practice the Nielsen Theorem is much more difficult to apply.

Until 1962, virtually the only cases for which the Nielsen number of $f: X \to X$ could be computed were those, such as where X is simply-connected or f is the identity map, for which $N(f) \leqq 1$. In that year, more sophisticated methods for computing Nielsen numbers were developed by Jiang [24]. Jiang proved that if $f: X \to X$ is a map and X is a polyhedron whose fundamental group satisfies a certain technical condition, then $N(f)$ can be computed algebraically just from a knowledge of the homomorphism of the fundamental group of X induced by f. In that case, the computation of $N(f)$ is of the same order of difficulty as that of $L(f)$. The Jiang condition is satisfied by some familiar spaces, Lie groups for example, but unfortunately it isn't a widespread phenomenon. There have been refinements of Jiang's techniques (see [17] for references) but, in general, it is still very difficult to compute a Nielsen number.

The problem we are concerned with is not just the computation of the Nielsen number, although that is our central concern. In 1960, Browder [9] suggested that it should be possible to characterize the Nielsen number of a map $f: X \to X$ algebraically from a knowledge of the homomorphisms induced by f, just as the Lefschetz number is. If this could be done, the computation of the Nielsen number would be similar to the computation of the Lefschetz number. Furthermore, we would have an easy proof that the Nielsen number is invariant under homotopy. Most likely, other, as yet unsuspected, properties of the Nielsen number would emerge from this new level of understanding of the concept. But just what do we mean by "induced homomorphisms"? Do we mean homomorphisms of homology groups, of homotopy groups, or of something else? Browder, in a letter in 1964, suggested that the problem was to express the Nielsen number of $f: X \to X$ in terms of the homomorphisms induced by f on the homology groups of X, considered as modules over the group ring of the fundamental group of X. Although this formulation, in which the fundamental group plays a major role, fits in well with Jiang's results, it seems rather restrictive. Anyhow, no one has succeeded in solving the problem in this form.

The formulation of the problem which seems to me to offer the widest scope, while preserving the essential elements of Browder's idea, is the following. A *functor* from the category of homotopy classes of maps of finite polyhedra to the category of morphisms of graded groups is a function which assigns to a map a sequence of homomorphisms in such a way that

(1) homotopic maps have the same image under the functor,
(2) the image of the identity function is a sequence of identity functions,
(3) composition is always preserved or always reversed.

Homology, whether or not over the group ring of the fundamental group, is such a functor. So also is homotopy theory and the other tools used in algebraic topology.

PROBLEM 4. *Characterize the Nielsen number of a map on a finite polyhedron by means of algebraic operations on the images of the map under functors from the category of homotopy classes of maps to the category of morphisms of graded groups.*

It is possible, as Scholz has done [37] (1970), to define the Nielsen number in a very general setting. However, I believe that the essential difficulty of Problem 4 has little to do with the generality of the spaces and maps considered. If Problem 4 can be solved in the form given, the generalization to cover every case in which the Nielsen number has been defined should present no great problem.

In spite of the lack of an answer to Problem 4, the Nielsen number has played an important part in fixed point theory; for example in connection with Problem 2 [17]. A solution to Problem 4 would probably give us a tool more powerful than any we now possess in fixed point theory.

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