ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 4, Number 1, Winter 1974

FIXED POINT THEORY VIA SEMICOMPLEXES

RICHARD B. THOMPSON[1]

ABSTRACT. In this paper we first establish a scheme which can be used to classify much of the work that has been done in fixed point theory and then survey the current status of knowledge about semicomplexes in this context. Some recent discoveries are introduced and used to give strengthened forms of some earlier results. Finally, several open questions are posed, whose solutions would, in the opinion of the author, make valuable contributions to the theory of semicomplexes.

1. **Classification of fixed point theories.** As any check of the recent literature shows, work in fixed point theory has proliferated to such an extent that the field now encompasses a very broad range of mathematical activity. This diversity suggests the need for some type of framework to use in classifying both new and old results in fixed point theory. As an initial step in this direction, we will introduce two axes which seem to form natural dividing lines for the subject.

First note that one may approach fixed point questions by either *direct* or *indirect* methods. Work will be classified as using the direct approach if it studies fixed point problems in the natural settings where they arise outside of fixed point theory itself. Thus, for example, a question about fixed points for maps between absolute neighborhood retracts (ANR's) would be studied and solved by working entirely within the category of ANR's and their maps.

This is contrasted with the indirect approach, which we define as the study of fixed point questions in categories that were originally devised for the purpose of doing some type of fixed point theory. In this case, a fixed point question about maps of ANR's might be regarded as a special case of a question about maps of, for example, semicomplexes and then studied in the category of semicomplexes and their maps — a category specifically invented for fixed point theory.

Received by the editors February 2, 1972.
AMS 1970 *subject classification.* Primary 54-02, 54H25, 55C20; Secondary 47H10.
Key words and phrases. Semicomplex, quasi-complex, Q-simplicial spaces, local fixed point index, Lefschetz fixed point theorem, *lc** spaces, absolute neighborhood retract.
[1]This research was partially supported by the National Science Foundation under grants GP-8709 and GP-13623 and is an expansion of the author's address given at the Chicago meeting of the American Mathematical Society on March 27, 1971.

We will elaborate briefly on the distinction between these two approaches by giving examples of each and noting their primary advantages and disadvantages. The direct approach is the older and more obvious of the two methods. This covers the work done in the categories of finite polyhedra and compact metric ANR's as well as much of the fixed point work arising from operator theory and differential equations. In these latter situations fixed point questions are most commonly studied in categories consisting of certain subsets of topological vector spaces (often Banach spaces) and particular types of maps of these spaces. A final example is provided by the work of point set topologists on the fixed point property for various specific spaces and classes of spaces. This is clearly a direct approach, since their work is done entirely within the space or spaces under investigation.

The most notable advantage of direct methods is that solutions are given explicitly in terms of the original problem — if you ask a question about ANR's, you get an answer about ANR's. A further benefit is that one usually is working in categories that have been thoroughly investigated and are relatively well understood. Thus the answer to a particular fixed point question may not be known, but a large amount of information and machinery is available and may be brought to bear on the problem.

These advantages are balanced by the drawback that each fixed point question, however closely related to similar questions in other settings, must usually be answered by ad hoc methods peculiar to the category in which one has asked the question. For example, a result obtained in Banach spaces may depend heavily on the presence of a metric and be of no help in answering the corresponding question in locally convex linear spaces. A second aspect of this phenomenon is that useful methods are seldom employed to their greatest extent, since this would take one out of the original category in which the problem arose. These difficulties notwithstanding, it may be observed that a majority of the work in fixed point theory has employed direct methods.

Indirect methods comprise a somewhat more recent approach to fixed point questions — dating from the work of Lefschetz on quasi-complexes published in 1942 [**12**, p. 323]. This work was followed by the convexoids of Leray [**14, 15** and **16**], the semicomplexes of Browder [**8**], the weak semicomplexes of the author [**19** and **20**] and the Q-simplicial spaces of Knill [**11**].

All of these indirect fixed point theories share several common advantages as well as some common difficulties. On the plus side, there is the obvious point that a quite satisfactory fixed point theory (either local or global) is developed in each of these settings. This, of course, should come as no great surprise since these categories were designed for that purpose. A less obvious virtue is that each of these categories is so constructed that the fundamental techniques used in their developments are employed in what would appear to be almost their fullest generality. Thus by working in these abstract, indirect settings the full power of one's methods is utilized and one obtains a uniform treatment for large classes of related questions, each of which would have required separate investigations by direct methods. The last advantage of the indirect approach is one which does not seem to have borne much fruit as yet. This is the simple fact that new mathematical structures are created which may well prove to be of some intrinsic interest for reasons outside the realm of fixed point theory.

The main drawback to these indirect theories is that difficulty which is common in all cases of generalization and abstraction — problems must be translated into the indirect setting and solutions must be translated back into the original context of the problem. Thus, while one can build an elaborate machine, it is not always clear whether or not this machine can be applied to a given problem. An interesting example of this difficulty is that it is still unknown after almost thirty years whether or not all ANR's admit quasi-complex structures.

A second aspect of indirect methods must also be classed as a drawback. Namely, no really useful indirect fixed point theory will ever be found that covers all of the situations where fixed point questions arise. Hence some special direct methods will always be needed. This statement follows from the work of the author in [21], where it is proved that no general fixed point theory can be closed under products (i.e., apply to the cartesian product of any two spaces to which it is applicable) and still apply to all compact Hausdorff spaces that have the fixed point property.

In addition to the distinction between direct and indirect theories, we can also draw a second dividing line to help classify fixed point results. All theories deal with some class of maps (continuous functions) between members of certain specified classes of spaces. If the needed restrictions are placed only on the type of space considered so that all maps are covered by the theory, then we will refer to this as an *unrestricted mapping theory*. If, on the other hand, conditions must be placed on the type of maps to be studied, then we will speak of a *restricted mapping theory*. In practice, this difference is usually reflected by whether one places strong compactness conditions on the spaces considered and can then look at all maps or whether one uses weaker compactness conditions on the spaces and hence must place some limitations on the maps that may be considered.

Obviously, statements about the fixed point property for spaces can only be made in unrestricted theories. Thus, if we agree to date fixed point theory from Brouwer's theorem stating the fixed point property for the n-cell, then our subject began with an unrestricted theory. We can press this example somewhat further to illustrate the fact that as succeeding generalizations of a theorem unfold, they often alternate between restricted and unrestricted mapping results. In a purely topological direction, Ayres ([1], 1930) expanded the class of subsets of the plane to be covered from the 1- and 2-cells of Brouwer to all Peano continua that do not separate the plane. He then proved the restricted mapping result that all such spaces have the fixed point property for homeomorphisms. Subsequently, Borsuk ([3], 1932) strengthened this to an unrestricted result by showing that 'homeomorphisms' could be replaced by 'continuous functions'.

The same phenomenon occurs with the generalizations of Brouwer's theorem that have been used in analysis. For example, Schauder ([17], 1930) extended Brouwer's theorem by showing that every compact convex subset of a Banach space has the fixed point property. Interestingly enough, this unrestricted mapping result is equivalent to the following restricted theorem. If C is a closed convex subset of a Banach space B and $f : B \to B$ is a map such that the closure of $f(C)$ is compact, then C contains a fixed point of f. Finally, this latter form of the result has been extended by Browder ([7], 1959) to a restricted mapping result which requires conditions on the iterates of the mapping rather than on the mapping itself. This type of theorem is somewhat typical of the current fixed point theorems used in operator theory where most results are of the restricted mappings type.

To complete our classification picture, we note that recent direct theory results of primarily topological interest appear to be divided between those of a restricted and those of an unrestricted nature. Classification is much easier in the case of indirect theories, where all but the Q-simplicial theory are of an unrestricted nature.

Before illustrating these ideas with semicomplexes in the next section, we will make three further remarks. First, the 1963 monograph by Van der Walt [25] gives an interesting survey of much of the older fixed point theory work and provides a good place to apply our classification plan. Second, the various fixed point results for multi-valued functions all fall under the heading of direct theories and appear, in general, to require fairly strong restrictions on the type of maps considered. Finally, the following diagram may help to present a summary of the program outlined in this section.

	Direct Theories	Indirect Theories
Restricted Mapping Theories	Current work used in operator theory including both local and global fixed point indices. Multi-valued functions.	Q-simplicial spaces.
Unrestricted Mapping Theories	Work on the fixed point property by geometric and algebraic topologists. Questions concerning bounds for the number fixed points of maps.	Quasi-complexes, semicomplexes, weak semicomplexes, and convexoids.

Figure (1.1)

2. **Semicomplexes.**

The material in this section has been selected to serve a two-fold purpose. In the first place, we will mention some new results on semicomplexes and indicate how these can be used to strengthen several older theorems. On the other hand, we can use this summary of the present status of knowledge about semicomplexes to give concrete illustrations of the advantages and disadvantages mentioned for indirect fixed point theories in § 1.

We will begin by setting down the basic definitions and conventions. All topological spaces considered are assumed to be compact and Hausdorff and all chain complexes and homology groups are taken with rational (Q) coefficients. Given a space X, we denote by $\sum(X)$ the collection of all finite covers of X by open sets and recall that $\sum(X)$ is quasi-ordered by the relation of refinement. If $\alpha, \beta \in \sum(X)$ and α refines $\beta(\alpha > \beta)$ and we let N_α and N_β stand for the nerves of α and β, then $\pi_\beta{}^\alpha : C(N_\alpha) \to C(N_\beta)$ will denote any of the usual chain maps induced by a vertex transformation based on set inclusion. The support of a chain $c \in C(N_\alpha)$ is written as $\sup(c)$ and is the union of all sets in α which appear in simplexes of N_α that occur with non-zero coefficients in c. The following definition is given in [**20**, p. 9] along with a more detailed explanation of the general notation and terminology that we will use in the rest of this paper.

DEFINITION (2.1). A weak semicomplex (WSC), $S(X) = \{X, \Omega, C\}$, is a triple where X is a compact Hausdorff space; Ω is a function assigning to each $\lambda \in \sum(X)$ a cofinal subset Ω_λ of $\sum(X)$ which has a designated coarsest element $\alpha_0(\lambda)$ such that $\alpha_0(\lambda) > \lambda$; and C is a function assigning to each $\lambda \in \sum(X)$ a family, C_λ, of chain maps consisting of one or more chain maps $c_\alpha{}^\beta : C(N_\beta) \to C(N_\alpha)$ for every pair $\alpha, \beta \in \Omega_\lambda$ such that $\alpha > \beta > \alpha_0(\lambda)$. Each $c_\alpha{}^\beta \in C_\lambda$ has the property that if $\sigma \in N_\beta$, then there is a set $U \in \lambda$ with $\sup(\sigma) \cup \sup(c_\alpha{}^\beta(\sigma)) \subseteq U$. These chain maps are called antiprojections and are assumed to satisfy the following axioms.

(i) If $\alpha > \beta > \gamma > \alpha_0(\lambda)$, $\alpha, \beta, \gamma \in \Omega_\lambda$ and $c_\alpha{}^\beta$, $c_\alpha{}^\gamma \in C_\lambda$, then $c_\alpha{}^\beta$ is chain homotopic ($\sim$) to $c_\alpha{}^\lambda \pi_\gamma{}^\beta$.

(ii) If $\alpha > \beta > \gamma > \alpha_0(\lambda)$, $\alpha, \beta, \gamma \in \Omega_\lambda$ and $c_\beta{}^\gamma$, $c_\alpha{}^\gamma \in C_\lambda$, then $c_\beta{}^\gamma \sim \pi_\beta{}^\alpha c_\alpha{}^\gamma$.

(iii) If $\alpha > \alpha_0(\lambda)$, $\alpha \in \Omega_\lambda$ and $c_\alpha{}^\alpha \in C_\lambda$, then $c_{\alpha *}^\alpha : H(N_\alpha) \to H(N_\alpha)$ is an idempotent endomorphism whose image is exactly the image of the projection homomorphism $p_\alpha : H(X) \to H(N_\alpha)$ where $H(X)$ denotes Čech homology with rational coefficients.

A weak semicomplex, $\{X, \Omega, C\}$, is called simple (SWSC) if for each $\lambda \in \sum(X)$, $\alpha \in \Omega_\lambda$ and $c_\alpha{}^\alpha \in C_\lambda$, $c_\alpha{}^\alpha \sim 1_{C(N\alpha)} : C(N_\alpha) \to C(N_\alpha)$.

We will often refer to X as the underlying space of a WSC, $S(X)$, and will call $S(X)$ an SC-structure on X.

As was indicated in § 1 of this paper, WSC's constitute an indirect fixed point theory. In this particular case, the concept was formulated with the goal of having the Lefschetz number of a map serve as a global fixed point index, i.e., the underlying spaces of WSC's satisfy the Lefschetz fixed point theorem. This fact, which was proved in [**20**, p. 15], is stated in the following theorem.

THEOREM (2.2). *If $S(X)$ is a WSC and $f : X \to X$ is a fixed point free map, then its Lefschetz number $L(f)$ is zero.*

The motivation for introducing the concept of simple WSC's is that they provide a complete characterization of the class of quasi-complexes as a subclass of the WSC's. The theorem given below is proved in [**20**, p. 19].

THEOREM (2.3). *A space X is a quasi-complex if and only if it admits an SWSC-structure.*

In addition to the references given above, WSC's and SWSC's (quasi-complexes) are used or discussed in [**6**], [**9**], [**10**], [**18**], [**19**], [**22**] and [**23**].

Finding the "right" set of axioms of this same nature that would allow one to have a local fixed point index seems to have been an elusive goal. In 1960, Browder [**8**] defined the notions of semicomplex

(SC) and semicomplex morphism (SC-morphism) and proved that a local fixed point index existed for this category. Less restrictive axioms for both SC's and SC-morphisms which allow one to get a clearer picture of exactly which spaces admit SC-structures were given by the author in [**20**, sec. III]. However, these axioms for SC's still forced one to limit his attention to metric spaces for many key constructions. Finally, in 1970 [**24**] a replacement for one of the SC axioms was found which eliminates this difficulty while at the same time leaving valid all of the known proofs of facts about SC's. These new axioms are given in the following definition where, as before, Čech homology is used.

DEFINITION (2.4). A semicomplex, (SC), $S(X) = \{X, I, \Omega, \alpha_0, C\}$, is a quintuple where X is a compact Hausdorff space; I is a collection of finite covers of X by connected open sets which is cofinal in $\sum (X)$; Ω is a cofinal subset of $\sum (X)$; α_0 is a function from I into Ω such that for each $\lambda \in I$, $\alpha_0(\lambda) > \lambda$; and C is a function assigning to $\lambda \in I$ a family, C_λ, of chain maps consisting of one or more chain maps $c_\alpha{}^\beta : C(N_\beta) \to C(N_\alpha)$ for every pair $\alpha, \beta \in \Omega$ such that $\alpha > \beta > \alpha_0(\lambda)$. These chain maps $c_\alpha{}^\beta$ are called antiprojections and are assumed to preserve the Kronecker index as well as satisfy the following axioms.

(i) If $\alpha > \beta > \gamma > \alpha_0(\lambda)$, $\alpha, \beta, \gamma \in \Omega$ and $c_\alpha{}^\beta, c_\alpha{}^\gamma \in C_\lambda$ then there exists a chain homotopy $\Delta_\alpha{}^\beta$ connecting $c_\alpha{}^\beta$ and $c_\alpha{}^\gamma \pi_\gamma{}^\beta$ such that for each $\sigma \in N_\beta$ there is a set $U \in \lambda$ with $\sup(\sigma) \cup \sup(c_\alpha{}^\beta(\sigma)) \cup \sup(\Delta_\alpha{}^\beta(\sigma) \subseteq U$.

(ii) If $\alpha > \beta > \gamma > \alpha_0(\lambda)$, $\alpha, \beta, \gamma \in \Omega$ and $c_\beta{}^\gamma, c_\alpha{}^\gamma \in C_\lambda$ then there exists a chain homotopy $\Gamma_\beta{}^\gamma$ connecting $c_\beta{}^\gamma$ and $\pi_\beta{}^\alpha c_\alpha{}^\gamma$ such that for each $\sigma \in N_\gamma$ there is a set $V \in \lambda$ with $\sup(\sigma) \cup \sup(c_\beta{}^\gamma(\sigma)) \cup \sup(\Gamma_\beta{}^\gamma(\sigma)) \subseteq V$.

(iii) If $\alpha > \alpha_0(\lambda)$, $\alpha \in \Omega$ and $c_\alpha{}^\alpha \in C_\lambda$, then $c^\alpha_{\alpha *} : H(N_\alpha; Q) \to H(N_\alpha; Q)$ is an idempotent endomorphism whose image is exactly the image of the projection homomorphism $p_\alpha : H(X; Q) \to H(N_\alpha; Q)$.

(iv) If $\lambda, \mu \in I$ with $\mu > \lambda$; $\alpha > \beta$; $\alpha, \beta \in \Omega$ with β refining both $\alpha_0(\lambda)$ and $\alpha_0(\mu)$; $c_\alpha{}^\beta(\lambda) \in C_\lambda$; and $c_\alpha{}^\beta(\mu) \in C_\mu$; then there exists a chain homotopy $\theta_\alpha{}^\beta$ connecting these two antiprojections such that for each $\sigma \in N_\beta$ there is a set $W \in \lambda$ with $\sup(\sigma) \cup \sup(c_\alpha{}^\beta(\lambda)(\sigma)) \cup \sup(\theta_\alpha{}^\beta(\sigma)) \subseteq W$.

The concept of a simple SC has been defined [**20**] in a fashion analogous to that used for simple WSC's. However, we will not consider this aspect of the theory here.

It is clear that an SC-structure on a space can be used to derive a WSC structure. In this sense, the SC's are a subclass of the WSC's.

As we mentioned above, the concept of a map between two spaces which respects existing SC-structures on these spaces has also evolved.

DEFINITION (2.5). Suppose that $S(X) = \{X, I, \Omega, \alpha_0, C\}$ and $S(Y) = \{Y, J, \Psi, \beta_0, D\}$ are SC's and that $h : X \to Y$ is a continuous map of spaces. h is called an SC-morphism from $S(X)$ into $S(Y)$ and written as $h : S(X) \to S(Y)$ if for each $\lambda \in I$ and $\mu \in J$ there are covers $\lambda_1 = \lambda_1(\lambda, \mu) \in I$ and $\mu_1 = \mu_1(\lambda, \mu) \in J$ with $\lambda_1 > \lambda$ and $\mu_1 > \mu$ which have the following property. For any four covers $\phi, \psi \in \sum(X)$ and $\chi, \omega \in \sum(Y)$, four covers $\alpha, \beta \in \Omega$ and $\gamma, \delta \in \Psi$ can be picked successively so that $\delta = \delta(\mu, \omega)$ is a common refinement of $\beta_0(\mu_1)$ and ω; $\beta = \beta(\delta, \psi)$ is a common refinement of $\alpha_0(\lambda_1)$, $h^{-1}(\delta)$ and ψ; $\gamma = \gamma(\delta, \chi)$ is a common refinement of δ and χ; and $\alpha = \alpha(\beta, \gamma, \phi)$ is a common refinement of $\beta, h^{-1}(\gamma)$ and ϕ. Further, these are assumed to have the property that if

$$h_\gamma{}^\alpha : C(N_\alpha) \to C(N_\gamma) \text{ and } h_\delta{}^\beta : C(N_\beta) \to C(N_\delta)$$

are chain maps induced by $h, c_\alpha{}^\beta \in C_{\lambda_1}$ and $d_\gamma{}^\delta \in D_{\mu_1}$ then there exists a chain homotopy Δ connecting $h_\gamma{}^\alpha c_\alpha{}^\beta$ and $d_\gamma{}^\delta h_\delta{}^\beta$ such that for each $\sigma \in N_\beta$ there is a set $U \in \mu$ with

$$h(\sup(\sigma)) \cup \sup(\Delta(\sigma)) \subseteq U.$$

It is shown in [20] that the collection of all SC's and SC-morphisms is a category and several of the properties of this category are studied, including its behavior under an equivalence relation defined by saying that $S(X)$ is equivalent to an SC, $T(X)$, if the identity map on X is an SC-morphism from $S(X)$ to $T(X)$.

As with WSC's, SC's are the basis for an indirect fixed point theory, as shown by Browder in [8].

THEOREM (2.6). *A local fixed point index exists for the category of the underlying spaces and maps of SC's and SC-morphisms.*

In the remaining work we will sometimes use the term semicomplex as a generic name for any one of the types of structures, SC, WSC or SWSC.

We will now illustrate the remarks made in § 1 by giving as concrete examples some of the known results on SC's. Several of these represent strengthened forms of the previously published versions of the theorems. This is made possible by the improved set of axioms which may now be used. In some of these cases the reference given is for the proof of the theorem in its earlier form and the adaptation of that proof to the present axioms is left to the reader. In those instances where proof of the strengthened form requires any great amount of ingenuity, the reference given is to a proof of the new result.

In regard to the advantages claimed for indirect fixed point theories in § 1, we have already noted that either a global or a local fixed point theory does exist for each type of SC. It should be noted, however, that the existence of this fixed point theory does not follow as any immediate consequence of the definitions of SC's. This is particularly so in the case of the local index, whose derivation requires a lengthy and sophisticated argument. It is reasonably clear, however, that the techniques used in the proofs of the fixed point result for the various SC's are utilized in their full generality.

As was observed in § 1 of this paper, the most substantial problem with indirect fixed point theories is that of relating them with the standard categories in which one normally works. In view of the somewhat complex nature of the definitions of the various types of SC's and their morphisms, it is not surprising that much of the work on these categories has been devoted to finding sufficient conditions for a space to support an SC-structure. Some of these results will be outlined in the following material.

In the case of some elementary spaces, one can construct SC-structures without any great difficulty (as shown in [20, sec V]).

Proposition (2.7). *All finite polyhedra and the Hilbert cube admit SC-structures (and hence WSC-structures).*

The various categories of SC's are also closed under some of the standard topological constructions. In particular, the following results are established in [20, sec VI] and in [22, p. 259] respectively (with the second theorem strengthened via use of the new axioms).

Theorem (2.8). *If X and Y are two compact Hausdorff spaces that admit a given type of SC-structure, then $X \times Y$ supports the same type of SC-structure. Moreover, in the case of actual SC's (not WSC's) these structures may be taken so that the projection maps are SC-morphisms and equivalent structures on the factors induce equivalent structures on the product.*

Theorem (2.9). *If A is a retract of a compact Hausdorff space X and X has either a WSC-structure or an SC-structure, then the retraction induces a structure of the same type on A.*

The advantage of this type of result is that it can often be used to avoid direct constructions of SC-structures on certain classes of spaces. Thus, since any compact, metric ANR is homeomorphic to a retract of the product of a finite polyhedron and the Hilbert cube [5, p. 105], we have the following consequence of the last two theorems.

THEOREM (2.10). *Every compact, metric ANR admits an SC-structure (and hence a WSC-structure).*

It can be deduced further (see [22, p. 268]) that any map between two such ANR's is an SC-morphism between any SC-structures on its range and domain. In particular, any two SC-structures on such spaces are equivalent.

The above result has been generalized, by explicit construction of SC's, to the HLC^* spaces introduced by Lefschetz in [13, p. 122]. (We will refer to these as lc^* spaces for purposes of notational consistency.)

This notion arises in the study of generalized manifolds (see [27] and [2]) where it plays a role analogous to that of ANR's for ordinary manifolds. The following result is a strengthened form of a similar theorem in [20, p. 45] whose proof is given in [24, sec. 4].

THEOREM (2.11). *Every lc^* space admits an SC-structure and all maps between such spaces induce SC-morphisms between any SC-structures on their domains and ranges.*

We will digress briefly at this point to note that the last result gives rise to one of the few known realizations of one of the potential benefits of indirect theories that was cited in § 1 of this paper. It is known (see [20, p. 10] that any space which supports a WSC-structure has Čech homology (with rational coefficients) that is isomorphic to a subgroup of the homology of the nerve of a sufficiently fine cover. Hence this homology condition holds for all lc^* spaces — a fact that was known previously only in the case of finite dimensional lc^* spaces [27, p. 180]. This provides a case where a result having no connection with fixed point theory was discovered by work in a category devised for fixed point work.

Theorem (2.11) (in its strengthened form) is also the key to the proof of the following result which indicates that the theory of SC's can move in the direction of non-metric applications to analysis (see [24]).

THEOREM (2.12). *All finite unions of compact, convex subsets of locally convex topological linear spaces admit SC-structures.*

We can use WSC's to move into the opposite corner of fixed point theory and investigate the fixed point properties of continua. It was shown in [10, p. 667] that every chainable continuum is a quasi-complex — and hence has an SWSC-structure. This result is generalized in [23] where the notion of a regular set of covers for a tree-like continuum is introduced and the observation made that every chainable continuum is a tree-like continuum which admits a regular set of covers. The main result in this direction (see [23, p. 216]) is that regular covers give a complete characterization of the existence of WSC-structures.

THEOREM (2.13). *The following are equivalent conditions on a tree-like continuum T.*

(a) *T has a regular set of covers.*
(b) *Every cofinal collection of tree chains on T is regular.*
(c) *T admits a WSC-structure.*

To relate this to the fixed point property, we need only observe that, since a tree-like continuum is acyclic in the sense of Čech homology with rational coefficients, the existence of a WSC-structure on such a space implies the fixed point property for the space.

The delicate nature of fixed point questions for tree-like continua is illustrated by the following pair of spaces, each of which is formed by a "*T*" that is approached tangentially by a ray.

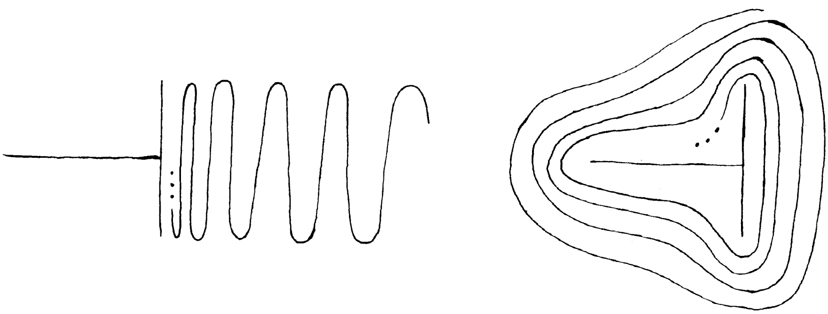

FIGURE (2.14)

T has a regular set of covers, and hence a WSC-structure, while *U* has a cofinal set of tree-chains that is not regular (see [9] and [23, p. 213]), and hence admits no WSC-structure.

It should be clear from the list of results given in this section that, while there is room for a great deal of further work, many of the problems of relating SC's with the topological categories normally studied can be (or have been) solved. Thus the most serious difficulty inherent in all direct fixed point theories does not appear to be an insurmountable hurdle in the case of SC's.

3. Open questions.

We will close our survey with several questions and problems, which, to the best of the author's knowledge, are now open and whose solutions would shed new light on the nature of semicomplexes. The first two of these questions are of a somewhat internal nature, in that they ask for information about the various categories SC's and their relationships.

As was outlined in § 2 of this paper, we may picture the implications between the different types of semicomplexes as shown in Figure (3.1). In this case SSC stands for simple semicomplex.

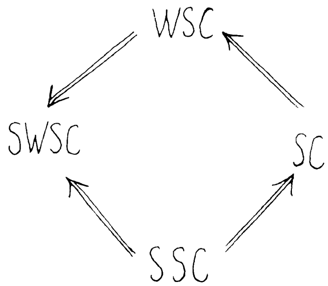

FIGURE (3.1)

There exist "cheap" examples to show that a space which admits a WSC-structure need not have an SC-structure. For example, the pseudo-arc is a chainable continuum and hence has a WSC-structure. However this space is not locally connected and hence can have no SC-structure. A more meaningful example would be a space which has a WSC-structure, violates none of the necessary conditions known for the underlying spaces of SC's but yet supports no SC-structure. The following question should be interpreted in this context.

QUESTION (3.2). Do there exist meaningful examples to show that implications in Figure (3.1) cannot be reversed?

In all of the cases where SC-structures have been constructed directly, it has happened that all maps between spaces have induced SC-morphisms between any of the SC-structures supported by their domains and ranges. In particular any two SC-structures on the same space have been found to be equivalent. It seems highly doubtful that this situation prevails in general.

PROBLEM (3.3). Find two SC's, $S(X)$ and $S(Y)$, and a map $f : X \to Y$ such that f is not an SC-morphism of $S(X)$ to $S(Y)$. As a special case of this, find two non-equivalent SC-structures on a single space.

The next two questions are aimed at clarifying the relation of two of the standard types of spaces to the different types of SC-structures. An example due to Borsuk [4] together with the remarks following

Theorem (2.11) of this paper show that a locally contractible, compact metric space need not admit an SC-structure. However, the corresponding answer is not known if we consider uniformly locally contractible (ULC) spaces (see [**26**]).

QUESTION (3.4). Does every compact ULC space admit an SC-structure?

As mentioned in § 1 of this paper, the question of whether or not all compact, metric ANR's admit SSC-structures (or even SWSC-structures) has been outstanding for many years. While its solution may be difficult, no list of problems about SC's would be complete without its inclusion. One possible approach to the question is outlined in [**22**, sections 3 and 4].

QUESTION (3.5). Does every compact metric ANR admit an SSC- or an SWSC-structure?

Note that, since every such space has SC- and WSC-structures, a negative answer to this question would provide a particularly satisfactory affirmative answer to part of Question (3.2).

As we have noted earlier, all of the indirect fixed point theories of the unrestricted mapping type, except the convexoids, have been shown to involve no more than special types of semicomplexes. Some work has already been done toward removing this last exception, but the question is still unresolved.

PROBLEM (3.6). Show that the all convexoid spaces support SC-structures and analyze the category of all SC's whose underlying spaces are convexoids.

One of the most promising directions for further research on semicomplexes is to extend this notion into a restricted mapping theory including some types of non-compact spaces. Such a program should be carried out in a manner that would allow one to obtain indirect proofs of some of the more useful results of the direct theories dealing with such spaces. It would also be desirable to establish the relationship between such a theory and the Q-simplicial spaces.

PROBLEM (3.7). Define SC-structures for a useful class of non-compact spaces and develop an indirect, restricted mapping type of fixed point theory in that setting.

Multi-valued functions offer another area into which one might expect to extend the methods of semicomplexes. This could either be done by considering such functions between spaces that support SC- or WSC-structures as presently defined, or in conjunction with the type of extension of the theory suggested in Problem (3.7).

PROBLEM (3.8). Establish fixed point results for multi-valued functions in the context of semicomplexes.

We have already observed that the category of spaces which support semicomplex structures offers an interesting topological setting in which to pursue questions outside the realm of fixed point theory. In this context, we can consider semicomplexes as a class of spaces, the nerves of whose covers form something like a direct system as well as the usual inverse system at the chain level. Such spaces might, for example, constitute a natural setting for Čech-type homology theories, allowing one to verify the exactness axiom for a wider class of spaces than is now possible. Considerations such as these would probably require the definition of semicomplex structures based on coefficients other than the rational numbers.

PROBLEM (3.9). Find topological applications having nothing to do with fixed point theory for the category of spaces that admit SC- or WSC-structures.

REFERENCES

1. W. Ayres, *Some generalizations of the Scherrer fixed point theorem*, Fund. Math., **16** (1930), 333-336.
2. A. Borel, *Seminar on transformation groups*, Princeton University Press, Princeton, New Jersey, 1960.
3. K. Borsuk, *Einige sätze über stetige streckenbilder*, Fund. Math., **18** (1932), 198-213.
4. ——, *Sur un espace compact localement contractible qui n'est pas un rétracte absolu de voisinage*, Fund. Math., **35** (1948), 175-180.
5. ——, *Theory of retracts*, Polish Scientific Publishers, Warsaw, Poland, 1967.
6. T. Brahana, *Products of quasi-complexes*, Proc. Amer. Math. Soc. **7** (1956), 954-958.
7. F. Browder, *On a generalization of the Schauder fixed point theorem*, Duke Math. J., **26** (1959), 291-303.
8. ——, *On the fixed point index for continuous mappings of locally connected spaces*, Summa Brasil. Math., **7** (1960), 253-293.
9. R. Chamberlin, *Tree-like continua and quasi-complexes*, Duke Math. J. **26** (1959), 511-517.
10. E. Dyer, *A fixed point theorem*, Proc. Amer. Math. Soc., **7** (1956), 662-672.
11. R. Knill, *Q-simplicial spaces*, Illinois J. Math., **14** (1970), 40-51.
12. S. Lefschetz, *Algebraic topology*, American Mathematical Society, Providence, R. I., 1942.
13. ——, *Topics in topology*, Princeton University Press, Princeton, N. J., 1942.
14. J. Leray, *Sur la forme des espaces topologiques et sur les points fixed des représentations*, J. Math. Pures Appl. **24** (1945), 95-167.
15. ——, *Sur la position d'un ensemble fermé de points d'un espace topologique*, J. Math. Pures Appl., **24** (1945), 169-199.
16. ——, *Sur les équations et les transformations*, J. Math. Pures Appl., **24** (1945), 201-248.

17. J. Schauder, *Der fixedpunktsatz in funktionalraum,* Studia Math., **2** (1930), 171-180.

18. J. Segal, *A fixed point theorem for the hyperspace of a snake-like continuum,* Fund. Math., **50** (1962), 237-248.

19. R. Thompson, *On the semicomplexes of F. Browder,* Bull. Amer. Math. Soc., 73 (1967), 531-536.

20. ——, *A unified approach to local and global fixed point indices,* Advances in Math., 3 (1969), 1-71.

21. ——, *A metatheorem for fixed point theories,* Comment. Math. Univ. Carolinae, 11 (1970), 813-815.

22. ——, *Retracts of semicomplexes,* Illinois J. of Math., **15** (1971), 258-272.

23. ——, *Weak semicomplexes and the fixed point theory of tree-like continua,* Duke Math. J., 38 (1971), 211-219.

24. ——, *Fixed point indices in locally convex spaces* (to appear in Advances in Math.).

25. T. van der Walt, *Fixed and almost fixed points,* Amsterdam Mathematisch Centrum, Amsterdam, Netherlands, 1963.

26. D. West, *Uniformly locally contractible spaces,* unpublished doctoral dissertation, University of Wisconsin, Madison, Wisconsin, 1965.

27. R. Wilder, *Topology of manifolds,* Revised edition, American Mathematical Society, Providence, R. I., 1949.

UNIVERSITY OF ARIZONA, TUCSON, ARIZONA 85721