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# COVERING COMPLEXES WITH APPLICATIONS TO ALGEBRA

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The Schreier-Nielsen theorem states that every subgroup of a free group is free. There are many proofs of this, but the most perspicacious, in my opinion, is the "topological" proof of Baer and Levi [**2**], which exploits the relationship between a topological space $X$ and the interplay between the subgroups of its fundamental group and certain "covering spaces" $\tilde{X}$ of $X$. It is well known (for example, see [**6**] or [**16**]) that if we concentrate on the space $X$ and its covering spaces $\tilde{X}$, various properties enjoyed by $X$ are inherited by each of its covering spaces $\tilde{X}$. In particular, if $X$ is a polyhedron, so is $\tilde{X}$. Since a polyhedron can be described without topology (as an *abstract simplicial complex*), and since an analog of the fundamental group (the *edge path group*) can be defined for abstract simplicial complexes, it has long been known that the usual interplay between spaces and fundamental groups can be described abstractly, without topology. In principle, then, there is nothing new in the exposition given here, so it is a reasonable question why this article should be written.

Before answering this question, let us remark that the interplay between covering spaces and fundamental groups is a Galois (rather a "co-Galois") correspondence; there thus appears to be a second inroad of this theory into algebra. Finally, from the other side, algebraists have been well aware of all these facts and, using certain graph-theoretical constructions, "Cayley diagrams", have given proofs of the Nielsen-Schreier subgroup theorem and other more difficult theorems of Kuroš and Gruško [5; 14; 18; 19]. See also Serre's notes [**15**] on "tree products" of groups. However, the algebraists have not been successful in abstracting the topological theory in such a way that it simultaneously gives the subgroup theorems and Galois theory. This is the program here. The only new idea appears to be the definition of "covering complex" (see below) which is merely a slavish imitation of the usual topological definition. There are three sections: 1. Complexes and Edgepath Groups; 2. Covering Complexes and Subgroup Theorems; 3. Galois Theory.

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## 1. Complexes and Edgepath Groups.

The material here is well known and is included for the reader's convenience.

DEFINITION. A *complex* $K$ is a set $V(K)$, called *vertices*, and a family of finite, nonempty subsets of $V(K)$, called *simplexes*, such that

(i) if $v \in V(K)$, then the singleton $\{v\}$ is a simplex;

(ii) if $s$ is a simplex, so is every nonempty subset of $s$.

A complex is usually called an "abstract simplicial complex" in the literature. Spelling is inconsistent here: the singular forms of the words just defined are: complex, vertex, simplex; the plural forms are: complexes, vertices, and simplexes.

DEFINITION. If $K$ and $K'$ are complexes, then a *map* $f: K \to K'$ is a function $f: V(K) \to V(K')$ such that whenever

$$s = \{v_0, \cdots, v_q\} \text{ is a simplex in } K,$$

$$\text{then } f(s) = \{fv_0, \cdots, fv_q\} \text{ is a simplex in } K'.$$

A map $f$ is called an *isomorphism* if it has an inverse, i.e., there is a map $g$ such that $f \circ g$ and $g \circ f$ are identities. A map is an isomorphism if it is one-one and onto.

Maps are usually called "simplicial maps" in the literature.

Of course, a map $f$ is allowed to identify distinct vertices of $K$. It is easy to see that the composite of maps (when defined) is again a map, and that all complexes and maps form a category $\mathcal{K}$.

We should remark on the geometric background so that the reader will understand complexes better and will not be shocked by the barrage of easy and, from a geometric viewpoint, natural definitions to come.

DEFINITION. A simplex $s = \{v_0, \cdots, v_q\}$ is called a $q$-simplex and is said to have *dimension* $q$ if all the displayed vertices are distinct; a complex $K$ has dimension $n$, denoted $\dim K = n$, if

$$n = \sup\{\dim s: s \text{ is a simplex of } K\}.$$

Of course, $n$ may be infinite; for completeness, let us define $\dim \emptyset = -1$ (note that $\emptyset$ is a complex: let $V(K) = \emptyset$). One also calls $K$ an *n-complex*; a 1-complex is also called a *graph*.

Here is the geometric picture: a 0-simplex is a point; a 1-simplex is a line segment (determined by its two endpoints $v_0, v_1$); a 2-simplex is a triangle (determined by its vertices $v_0, v_1, v_2$); a 3-simplex is a tetrahedron, and so forth. A complex is a space that is built of simplexes pasted together in a "nice" way.

Many familiar constructions exist in the category $\mathcal{K}$.

(a) *Subcomplex*

A complex $L$ is a *subcomplex* of a complex $K$ if $V(L) \subset V(K)$ and every simplex of $L$ is a simplex of $K$ (of course, this merely says that the hypothesized inclusion $i : V(L) \to V(K)$ is a map of complexes).

A subcomplex $L$ of $K$ is *full* if every simplex in $K$ having each of its vertices in $L$ is also a simplex in $L$.

As an example, for each integer $q \geqq 0$, the *q-skeleton* of $K$, denoted $K^{(q)}$, is defined as follows:

$$K^{(q)} = \{\text{simplexes } s \text{ in } K\text{: } \dim s \leqq q\}.$$

(b) *Union and Intersection*

If $\{L_i : i \in I\}$ is a family of subcomplexes of $K$, then

$$\bigcap L_i = \{\text{simplexes } s \text{ in } K : s \in L_i \text{ for all } i\}$$

and

$$\bigcup L_i = \{\text{simplexes } s \text{ in } K : s \in L_i \text{ for some } i\}.$$

One checks easily that $\bigcap L_i$ and $\bigcup L_i$ are subcomplexes of $K$. Note that two subcomplexes $L_1$ and $L_2$ are *disjoint*, $L_1 \cap L_2 = \emptyset$, if and only if $V(L_1) \cap V(L_2) = \emptyset$.

(c) *Quotient Complex*

Let $K$ be a complex and let $\{X_i : i \in I\}$ be a partition of $V(K)$. Alternatively, let $R$ be an equivalence relation on $V(K)$ with equivalence classes $\{X_i : i \in I\}$. Define a new complex $\bar{K} = K/R$ as follows:

(i) $V(\bar{K}) = \{X_i : i \in I\}$;

(ii) $\{X_{i_0}, \cdots, X_{i_q}\}$ is a simplex in $\bar{K}$ if there exist vertices $v_j \in X_{i_j}$ such that $\{v_0, \cdots, v_q\}$ is a simplex in $K$.

It is easy to see that $\bar{K}$ is a complex, the *quotient complex* modulo $R$, and that the *natural map* $V(K) \to V(\bar{K})$ assigning to each $v \in V(K)$ the unique $X_i$ containing it is a map of complexes.

(d) *Image and Inverse Image*

Assume $f : K \to K'$ is a map of complexes. Then

$$\operatorname{im} f = \{s' \in K' : s' = f(s) \text{ for some simplex } s \text{ in } K\}.$$

Note that $\operatorname{im} f$ is a subcomplex of $K'$. Moreover, the first isomorphism theorem holds: if $f : K \to K'$ is a map, then $R = \{X_{v'} = f^{-1}(\{v'\}) : v' \in V(K')\}$ is a partition of $V(K)$, and $K/R \cong \operatorname{im} f$.

If $f : K \to K'$ is a map and $L'$ is a subcomplex of $K'$, we define $f^{-1}(L')$ to be the full subcomplex of $K$ with vertices $f^{-1}(V(L'))$.

The most interesting complexes are the connected ones, whose definition is best understood with the geometric picture in mind.

DEFINITIONS. An *edge* $e$ in a complex $K$ is an ordered pair of (not necessarily distinct) vertices, $e = (u, v)$, where $\{u, v\}$ is a simplex in $K$ (of course $\{u, v\}$ is either a 0-simplex or a 1-simplex). One calls $u$ the *origin* of $e$ (and often writes orig $e = u$), and one calls $v$ the *end* of $e$.

A *path* $\alpha$ in $K$ is a finite sequence of edges

$$\alpha = e_1 \Box e_2 \Box \cdots \Box e_n,$$

where end $e_i$ = orig $e_{i+1}$ for all $i = 1, \cdots, n-1$. Define

$$\text{orig}\,\alpha = \text{orig}\,e_1$$

and

$$\text{end}\,\alpha = \text{end}\,e_n.$$

We say $\alpha$ is a path of *length n* from orig $\alpha$ to end $\alpha$.

A path $\alpha$ is *closed* at $u$ if orig $\alpha = u =$ end $\alpha$; a path $\alpha = e_1 \Box \cdots \Box e_n$ is *reduced* if no edge $(u, v)$ is adjacent to $(v, u)$.

With these preliminary definitions, we may now give an important one.

DEFINITION. A complex $K$ is *connected* if, for every pair of vertices $u, v$ in $K$, there is a path in $K$ from $u$ to $v$.

THEOREM 1.1. *Every complex K is the disjoint union of connected subcomplexes.*

PROOF. Define a relation on $V(K)$ by : $u \leftrightarrow v$ if there is a path in $K$ from $u$ to $v$. It is easy to check that this defines an equivalence relation on $V(K)$, say, with equivalence classes $\{V_i : i \in I\}$. If we let $K_i$ be the full subcomplex of $K$ with vertices $V_i$, then one can see that each $K_i$ is connected and that $K$ is their disjoint union.

DEFINITION. The subcomplexes $K_i$ in the above proof are called the *components* of $K$.

It is easy to see that if $f : K \to K'$ and $K$ is connected, then im $f$ is connected. Moreover, a subcomplex $L$ of $K$ is connected if and only if $L \cap K^{(1)}$ is connected, where $K^{(1)}$ is the 1-skeleton of $K$.

Paths may be multiplied if they match up: if $\alpha = e_1 \Box \cdots \Box e_n$ and $\alpha' = e_1' \Box \cdots \Box e_m'$ are paths in $K$ and if end $\alpha$ = orig $\alpha'$, then

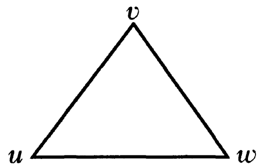
$$\alpha \Box \alpha' = e_1 \Box \cdots \Box e_n \Box e_1' \Box \cdots \Box e_m'.$$

Note that this multiplication of paths is associative, when defined, but that every other group axiom fails. We surmount many difficulties by imposing an equivalence relation on the paths in $K$.

DEFINITION. Two paths $\alpha$ and $\beta$ in $K$ are *equivalent*, denoted $\alpha \sim \beta$, if one can be obtained from the other by a finite number of "elementary moves" consisting of replacing one side of an equation

$$(u,v) \square (v,w) = (u,w)$$

by the other whenever $\{u, v, w\}$ is a simplex of $K$.




Thus, we regard the two paths $(u, v) \square (v, w)$ and $(u, w)$ as the same if $K$ contains the interior of the triangle above.

It is easy to check that $\alpha \sim b$ defines an equivalence relation on the set of all paths in $K$; we denote the equivalence class of a path $\alpha$ by $[\alpha]$, and we denote the family of all $[\alpha]$ by $\pi(K)$.

The following properties are easily verified.

(1) Every path is equivalent to a reduced path.
(2) Equivalent paths have the same origin and the same end.
(3) Equivalence is compatible with multiplication of paths:
if $\alpha \sim \alpha'$, $\beta \sim \beta'$, and end $\alpha =$ orig $\beta$,
then $\alpha \square \beta \sim \alpha' \square \beta'$.

It follows from this that we may multiply path classes:

$$\text{if end } \alpha = \text{orig } \beta, \text{ then } [\alpha]\ [\beta] = [\alpha \square \beta]$$

is independent of the choice of representative path.

We almost have a group (indeed, we have a "groupoid"). First, a little notation. If $v \in V(K)$, let $i_v = (v, v)$; if $e = (u, v)$ is an edge, let $e^{-1} = (v, u)$; if $\alpha = e_1 \square \cdots \square e_n$ is a path, let $\alpha^{-1} = e_n^{-1} \square \cdots \square e_1^{-1}$. The proof of the next theorem is mechanical.

THEOREM 1.2. *If $K$ is a complex, then $\pi(K)$ satisfies the following properties*:

(i) *each $[\alpha] \in \pi(K)$ has an origin $u$ and an end $v$; moreover,*

$$[i_u]\ [\alpha] = [\alpha] = [\alpha]\ [i_v];$$

(ii) *the associative law holds when defined*;

(iii) *if* $\text{orig}[\alpha] = u$ *and* $\text{end}[\alpha] = v$, *then*

$$[\alpha]\ [\alpha^{-1}] = [i_u] \text{ and } [\alpha^{-1}]\ [\alpha] = [i_v].$$

The only obstacle preventing $\pi(K)$ from being a group is that multiplication is not always defined (some authors do not regard this as a drawback, however, and are quite content with groupoids, e.g., Higgins [5]). We force $\pi(K)$ to be a group in the most naive way possible. Arbitrarily choose a vertex $v_* \in V(K)$, which we call a *basepoint*. With each such choice, we define

$$\pi(K, v_*) = \{[\alpha] \in \pi(K) : \alpha \text{ is a closed path at } v_*\}.$$

THEOREM 1.3. *$\pi(K, v_*)$ is a group (for each choice of basepoint $v_*$).*

PROOF. Immediate from Theorem 1.2, for we have restricted attention to a subset of $\pi(K)$ in which multiplication is always defined.

DEFINITION. $\pi(K, v_*)$ is called the *edgepath group* of the complex $K$ with basepoint $v_*$.

THEOREM 1.4. *If $K$ is a connected complex and $u_*, v_* \in V(K)$, then*

$$\pi(K, u_*) \cong \pi(K, v_*).$$

PROOF. Since $K$ is connected, there is a path $\beta$ in $K$ from $u_*$ to $v_*$. Define $f : \pi(K, u_*) \to \pi(K, v_*)$ by

$$[\alpha] \mapsto [\beta^{-1} \square \alpha \square \beta]$$

(note that the multiplication inside the second bracket occurs in $\pi(K)$, but that $[\beta^{-1}\square\alpha\square\beta] \in \pi(K, v_*)$). Using Theorem 1.2, it is easy to see that $f$ is a homomorphism with inverse $[\alpha'] \mapsto [\beta\square\alpha'\square\beta^{-1}]$.

Thus, the isomorphism class of $\pi(K, v_*)$ is independent of the base point when $K$ is connected. In order to construct groups, we have been forced to modify our original subject; instead of considering only complexes $K$, we must also consider *pointed complexes* $(K, v_*)$, i.e., a complex $K$ with some vertex $v_*$ chosen as basepoint. (Often, we shall not be so pedantic and let $K$ denote a pointed complex). All other definitions must be modified accordingly: e.g., a map $f : K \to K'$ is a map of pointed complexes only if it preserves the basepoint. In short, we have a second category $\mathcal{K}_*$, the category of pointed complexes and their maps.

THEOREM 1.5. *$\pi : \mathcal{K}_* \to$ Groups is a covariant functor.*

PROOF. It is enough to define $\pi$ on a map $f : (K, v_*) \to (L, u_*)$. If $[\alpha] \in \pi(K, v_*)$, then $\alpha = (v_*, v_1) \square (v_1, v_2) \square \cdots \square (v_n, v_*)$; define

$$\pi(f) : \pi(K, v_*) \to \pi(L, u_*)$$

by

$$[\alpha] \mapsto [(u_*, fv_1)\square(fv_1, fv_2)\square \cdots \square(fv_n, u_*)].$$

The check of the axioms is easy.

We shall usually write $f_{\#}$ instead of $\pi(f)$.

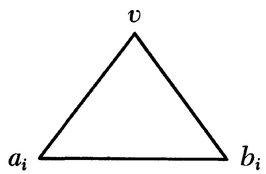
REMARK. Let $\alpha = (v_0, v_1)\square(v_1, v_2)\square \cdots \square(v_{n-1}, v_n)$ be a path in $K$ of length $n$. Define a 1-complex $I_n$ by setting $V(I_n) = \{t_0, t_1, \cdots, t_n\}$ and by decreeing that the only 1-simplexes have the form $\{t_i, t_{i+1}\}$, where $0 \leqq i < n$. (Picture $I_n$ as the unit interval chopped into $n$ intervals). Regard $I_n$ as a pointed complex by choosing $t_0$ as basepoint. Now we may identify a path $\alpha$ in $K$ with the map $\alpha : I_n \to K$ defined by $\alpha(t_i) = v_i$, where $0 \leqq i \leqq n$. Under this identification, the homomorphism $\pi(f) : \pi(K, v_*) \to \pi(L, u_*)$ is succinctly described by $[\alpha] \mapsto [f \circ \alpha]$.

In a pointed complex $(K, v_*)$, every (pointed) subcomplex $(L, v_*)$ must have the same basepoint (for the inclusion map, as every map, is now required to preserve basepoints). Thus, pointed subcomplexes can not be disjoint. If a family of pointed complexes $\{L_i : i \in I\}$ is as disjoint as possible, i.e.,

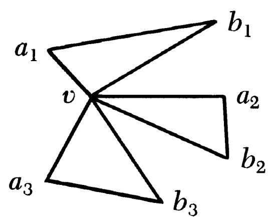
$$L_i \cap L_j = \text{basepoint}, \qquad i \neq j,$$

then one writes $\bigvee L_i$ instead of $\bigcup L_i$, and calls $\bigvee L_i$ the *wedge* of the $L_i$.

For example, let $L_i$ be a "circle": the 1-complex having vertices $v, a_i, b_i$, and 1-simplexes $\{a_i, b_i\}$, $\{a_i, v\}$, and $\{b_i, v\}$.




Then the wedge $\bigvee L_i$ is often called a *bouquet of circles*; its picture is:



Let us compute edgepath groups of low dimensional connected complexes. If dim $K = 0$, then $(K, v_*) = (\{v_*\}, v_*)$ and $\pi(K, v_*) = \{1\}$. If dim $K = 1$, then $K$ is, by definition, a connected graph; we shall see that $\pi(K, v_*)$ is free. Two-dimensional complexes are as bad as possible, for if $(K, v_*)$ is any connected complex, then there is a 2-complex $L$ with $\pi(K, v_*) \cong \pi(L, u_*)$.

DEFINITION. A *circuit* is a reduced, closed path in $K$; a *tree* in $K$ is a connected subcomplex of $K$ of dimension $\leqq 1$ that contains no circuits; a *maximal tree* is a tree contained in no larger tree.

For example, a maximal tree in the bouquet of circles pictured above is obtained by deleting all edges of the form $(a_i, b_i)$. The following fact is easy to prove. If $K$ is a finite complex, let $n_0(K) = |V(K)|$ and $n_1(K)$ equal the number of 1-simplexes in $K$; if $K$ is a tree, then

$$n_0(K) - n_1(K) = 1.$$

(If dim $K = 1$, this last number is called the *Euler-Poincaré characteristic* of $K$).

Let us note two simple facts about maximal trees. First of all, they exist; indeed, every tree is contained in a maximal tree (use Zorn's lemma, for an ascending union of trees is a tree); second, if $T$ is a tree in $K$ and $u, v \in V(T)$, then there is a unique reduced path in $T$ from $u$ to $v$ (a path exists, since $T$ is connected; we may assume it is reduced by cancelling all adjacent edges (if any) of the form $e \square e^{-1}$; it is unique lest one followed by the inverse of the second produces a circuit in $T$.)

The following criterion is so well-known, we omit its proof (it is a straightforward exercise; see [12] or [16], for example).

THEOREM 1.6. *If $K$ is a connected complex, then a tree $T$ in $K$ is a maximal tree if and only if* $V(T) = V(K)$.

The next result enables one to give a presentation of $\pi(K, v_*)$; for an elementary proof, we refer the reader to [12] or [16].

If $K$ is a connected complex and $T$ is a maximal tree in $K$, define a group $G_{K,T}$ having the following presentation:

*generators*: all edges $(u, v)$ in $K$;

*relations*: (a) $(u, v)(v, w) = (u, w)$ if $\{u, v, w\}$ is a simplex of $K$;

(b) $(u, v) = 1$ if $(u, v)$ is an edge of $T$.

**Theorem 1.7.** *If $K$ is a connected complex with maximal tree $T$, then*

$$\pi(K, v_*) \cong G_{K,T}.$$

It follows from Theorem 1.7 that, as far as edgepath groups are concerned, 2-complexes are arbitrarily bad. If $K$ is a connected complex, then Theorem 1.7 shows that $\pi(K, v_*) \cong \pi(K^{(2)}, v_*)$, where $K^{(2)}$ is the 2-skeleton of $K$. A more striking corollary is the next result.

**Theorem 1.8.** *If $K$ is a connected graph, then $\pi(K, v_*)$ is a free group. Moreover, if $T$ is a maximal tree in $K$, then*

$$\operatorname{rank} \pi(K, v_*) = \operatorname{card}\{\text{1-simplexes } s : s \text{ in } K - T\}.$$

**Proof.** Theorem 1.7 allows us to examine $G_{K,T}$. Surely $G_{K,T}$ is generated by all edges $(u, v)$ not in $T$. If $e = (u, v)$, the type (a) relation $ee^{-1} = 1$ allows us to discard one of the two edges $(u, v)$ and $(v, u)$ determined by the 1-simplex $\{u, v\}$ in $K - T$. We claim the remaining edges freely generate $G_{K,T}$. If $\{u, v, w\}$ is a simplex of $K$, then two of the vertices must be the same, for $\dim K = 1$. Thus, the relations of type (a) have the form:

$$(u, u)(u, v) = (u, v);$$

$$(u, v)(v, v) = (u, v);$$

$$(u, v)(v, u) = (u, u).$$

All of these are trivial; since $(v, v) = 1$ and $(u, v) = (v, u)^{-1}$, the subgroup of relations is $\{1\}$. Thus, $G_{K,T}$ is free.

**Theorem 1.9.** *Given a set $I$, there exists a connected complex $K$ with $\pi(K, v_*)$ free of rank* $=$ card $I$.

**Proof.** Let $K$ be a bouquet of circles ($K = \bigvee L_i$ in the notation of our example); the complement of the maximal tree exhibited there consists of the simplexes $\{\{a_i, b_i\} : i \in I\}$.

In particular, $\pi(K, v_*) \cong Z$ if $K$ is a "circle" and $\pi(K, v_*)$ is free of rank 2 if $K$ is a "figure 8".

We end this section with a useful technical lemma.

**Theorem 1.10.** *Let $K$ be a connected complex containing a subcomplex $L$ that is a disjoint union of trees. Then $L$ is contained in a maximal tree of $K$.*

**Proof.** Assume $L = \bigcup T_i$ exhibits $L$ as a disjoint union of trees. There is no loss in generality in assuming $V(L) = V(K)$; should there be a vertex $v \notin V(L)$, call the singleton $\{v\}$ a tree as well. In the quotient complex $\bar{K}$ of $K$ modulo $\bigcup T_i$, the vertices are $\bar{T}_i$, each original tree identified to a point. Now $\bar{K}$ is a connected complex, being the image of the connected complex $K$ under the natural map $\nu$. If $\bar{M}$ is a maximal tree in $\bar{K}$, then $V(\bar{M}) = V(\bar{K})$, by Theorem 1.6.

Consider now the subcomplex $\nu^{-1}(\bar{M})$ of $K$. First of all, $L = \bigcup T_i \subset \nu^{-1}(\bar{M})$, for $\nu(T_i) = \bar{T}_i \in V(\bar{K})$, all $i$. Second, we claim that $\nu^{-1}(\bar{M})$ is connected. Let $u, v \in V(\nu^{-1}(\bar{M}))$, so that $\nu(u) = \bar{T}, \nu(v) = \bar{T}'$ in $\bar{M} \subset \bar{K}$. Assume $\bar{E}_1 \square \cdots \square \bar{E}_n$ is a path in $\bar{M}$ from $\bar{T}$ to $\bar{T}'$. If $\bar{E}_j = (\bar{T}_j, \bar{T}_{j+1})$, say, then by definition of quotient complex, there are vertices $v_j \in T_j$, $v_{j+1} \in T_{j+1}$ with $\{v_j, v_{j+1}\}$ a simplex of $K$. In particular, $\bar{E}_1 = (\bar{T}, \bar{T}_2)$ provides vertices $v_1, v_2 \in V(K)$ with $\{v_1, v_2\}$ a simplex in $K$. Now both $u$ and $v_1$ lie in $\bar{T}$ which is, by hypothesis, connected. There is thus a path in $T$ from $u$ to $v_1$ in $K$, hence a path in $\nu^{-1}(\bar{M})$ from $u$ to $v_2$. This argument can be completed by induction on $n$, using the connectedness of each $T_i$ at each step.

Finally, let $K^{(1)}$ be the 1-skeleton of $K$, and set $M = K^{(1)} \cap \nu^{-1}(\bar{M})$. Since $\dim T_i \leqq 1$, all $i$, we have $L \subset M$, and clearly $\dim M \leqq 1$. $M$ is connected since $\nu^{-1}(\bar{M})$ is. Thus, $M$ is a tree with all desiderata with the possible exception of $M$ being maximal. But observe that $M$ contains every vertex of $K$, so it is indeed a maximal tree, by Theorem 1.6 (an alternative to using Theorem 1.6 is an application of Zorn's lemma, which allows us to imbed any tree in a maximal tree).

## 2. Covering Complexes and the Subgroup Theorems.

If the reader has seen the usual definition of a covering space of a topological space, he will not be astounded by the next definition.

DEFINITION. Let $K$ be a complex. A pair $(\tilde{K}, p)$ is a *covering complex* of $K$ if

(i) $p : \tilde{K} \to K$ is a map;

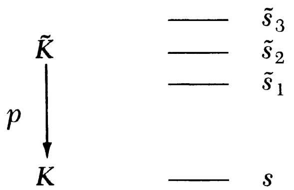
(ii) $\tilde{K}$ is connected;

(iii) for every simplex $s$ in $K$, $p^{-1}(s)$ is a union of pairwise disjoint simplexes,

$$p^{-1}(s) = \bigcup \tilde{s}_i,$$

with $p \mid \tilde{s}_i : \tilde{s}_i \to s$ a one-one correspondence for each $i$.

The map $p$ is called the *projection* and the simplexes $\tilde{s}_i$ are called the *sheets* over $s$. The picture to keep in mind is



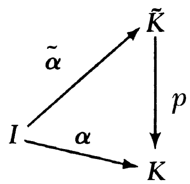

A trivial example of a covering complex is provided by $(\tilde{K}, p)$, where $\tilde{K} = K$ is a connected complex and $p = 1_K$, the identity map. A more interesting example is provided by a "circle" $K : V(K) = \{v_0, v_1, v_2\}$, and the 1-simplexes are $\{v_0, v_1\}$, $\{v_1, v_2\}$ and $\{v_2, v_0\}$. Define $\tilde{K}$ as the 1-complex having vertices $\{t_i : i \in \mathrm{Z}\}$ and 1-simplexes $\{t_i, t_{i+1}\}$, all $i \in \mathrm{Z}$. If we define $p : \tilde{K} \to K$ by $p(t_i) = v_j$, where $i \equiv j$ (mod 3), then $(\tilde{K}, p)$ is a covering complex of $K$.

There are some very elementary observations about covering complexes. If $(\tilde{K}, p)$ is a covering complex, then $K = \operatorname{im} p$; it follows that $K$ is connected (because $\tilde{K}$ is). Also, if $s$ is a $q$-simplex in $K$, each sheet $\tilde{s}_i$ over $s$ is also a $q$-simplex, for $p \mid \tilde{s}_i$ is a one-one correspondence. Since $\tilde{K} = p^{-1}(K) = \bigcup_s p^{-1}(s)$, it follows that $\dim \tilde{K} = \dim K$. In particular, a covering complex of a graph is itself a graph, as is illustrated in our example.

We shall eventually need the following result, whose straightforward proof is left to the reader. Let $p : \tilde{K} \to K$ be a covering complex and let $L$ be a connected subcomplex of $K$. If $\tilde{L}$ is a component of $p^{-1}(L)$, then $p \mid \tilde{L} : \tilde{L} \to L$ is a covering complex.

THEOREM 2.1. (Lifting Lemma) *Let $p : \tilde{K} \to K$ be a covering complex. Assume $v_*$ is a basepoint in $K$ and $p(\tilde{v}_*) = v_*$. Given a path $\alpha$ in $K$ with origin $v_*$, there exists a unique path $\tilde{\alpha}$ in $\tilde{K}$ with origin $\tilde{\alpha} = \tilde{v}_*$ and $p\tilde{\alpha} = \alpha$.*

REMARK. One calls $\tilde{\alpha}$ a *lifting* of $\alpha$ because of the picture




PROOF. Let $\alpha$ have length $n$ and origin $v_*$. We prove by induction on $n$ that there exists a unique $\tilde{\alpha}$ with origin $\tilde{v}_*$ and $p\tilde{\alpha} = \alpha$.

If $n = 1$, then $\alpha = (v_*, v)$, where $\{v_*, v\}$ is a simplex $s$ of $K$. We may assume $v_* \neq v$, so that $s$ is a 1-simplex. If $\tilde{s}_i$ is the sheet over $s$ containing $\tilde{v}_*$, then $\tilde{s}_i$ is a 1-simplex, as we remarked earlier. Hence $\tilde{s}_i = \{\tilde{v}_*, \tilde{v}\}$ for some vertex $\tilde{v}$ in $\tilde{K}$. Thus $\tilde{\alpha} = (\tilde{v}_*, \tilde{v})$ is an edge in $\tilde{K}$, it has origin $\tilde{v}_*$, and $p\tilde{\alpha} = \alpha$. To prove uniqueness of $\tilde{\alpha}$, suppose that $(\tilde{v}_*, \tilde{u})$ is a second lifting of $\alpha$. If $s = \{v_*, v\}$, then $\tilde{v}_*$, $\tilde{v}$, and $\tilde{u}$ all lie in $p^{-1}(s)$. Since $p^{-1}(s)$ is the full subcomplex of $\tilde{K}$ on $p^{-1}(\{v_*, v\})$, both edges $(\tilde{v}_*, \tilde{v})$ and $(\tilde{v}_*, \tilde{u})$ lie in $p^{-1}(s)$. Visibly the sheet containing $\tilde{v}$ and the sheet containing $\tilde{u}$ are not disjoint. It follows that $\tilde{v} = \tilde{u}$. The inductive step is routine and is left to the reader.

Henceforth, we shall write the first two sentences in the statement of Theorem 2.1 as "Let $p : (\tilde{K}, \tilde{v}_*) \to (K, v_*)$ be a covering complex".

LEMMA 2.2. *Let $p : (\tilde{K}, \tilde{v}_*) \to (K, v_*)$ be a covering complex. If $\alpha$ and $\beta$ are equivalent paths in $K$ with origin $v_*$, then their liftings $\tilde{\alpha}$ and $\tilde{\beta}$ having origin $\tilde{v}_*$ are also equivalent.*

PROOF. Recall that equivalence means that $\alpha$ can be transformed into $\beta$ by a finite number of elementary moves that replace one side of an equation

$$(u, v) \,\square\, (v, w) = (u, w)$$

by the other when $\{u, v, w\}$ is a simplex in $K$. Thus, it suffices to prove that if $(\tilde{u}, \tilde{v}) \,\square\, (\tilde{v}, \tilde{w})$ is a lifting of $(u, v) \,\square\, (v, w)$ and if $s = \{u, v, w\}$ is a simplex in $K$, then $\{\tilde{u}, \tilde{v}, \tilde{w}\}$ is a simplex in $\tilde{K}$. Let $\tilde{s}$ be the sheet over $s$ containing $\tilde{v}$ and let $\tilde{s}_1$ be the sheet over $s$ containing $\tilde{u}$. Thus, $\tilde{s}_1 = \{\tilde{u}, \tilde{v}_1, \tilde{w}_1\}$, where $p\tilde{v}_1 = v$ and $p\tilde{w}_1 = w$. Now $(\tilde{u}, \tilde{v}_1)$ and $(\tilde{u}, \tilde{v})$ are both liftings of the path $(u, v)$ having origin $\tilde{u}$. By the uniqueness in Theorem 2.1, $\tilde{v}_1 = \tilde{v}$. Hence $\tilde{s}$ and $\tilde{s}_1$ are not disjoint, and so $\tilde{u} \in \tilde{s}$. Similarly $\tilde{w} \in \tilde{s}$, so that $\tilde{s} = \{\tilde{u}, \tilde{v}, \tilde{w}\}$ is a simplex of $\tilde{K}$.

THEOREM 2.3. *If $p : (\tilde{K}, \tilde{v}_*) \to (K, v_*)$ is a covering complex, then $p_\# : \pi(\tilde{K}, \tilde{v}_*) \to \pi(K, v_*)$ is one-to-one.*

PROOF. Assume $[\tilde{\alpha}]$ and $[\tilde{\beta}] \in \pi(\tilde{K}, \tilde{v}_*)$ are such that $p_\#[\tilde{\alpha}] = p_\#[\tilde{\beta}]$ in $\pi(K, v_*)$. Then $[p\tilde{\alpha}] = [p\tilde{\beta}]$ and $p\tilde{\alpha} \sim p\tilde{\beta}$. Visibly $\tilde{\alpha}$ and $\tilde{\beta}$ are the (unique) liftings of $p\tilde{\alpha}$ and $p\tilde{\beta}$ having origin $\tilde{v}_*$. By Theorem 2.2, $\tilde{\alpha} \sim \tilde{\beta}$ and $[\tilde{\alpha}] = [\tilde{\beta}]$. Therefore $p_\#$ is one-to-one.

What happens to $p_\#\pi(\tilde{K}, \tilde{v})$ as the basepoint $\tilde{v}$ is changed?

THEOREM 2.4. *Let $p : \tilde{K} \to K$ be a covering complex, and assume $p(\tilde{v}) = v_* = p(\tilde{u})$; then $p_\#\pi(\tilde{K}, \tilde{v})$ and $p_\#\pi(\tilde{K}, \tilde{u})$ are conjugate subgroups of $\pi(K, v_*)$. Conversely, if $H$ is conjugate to $p_\#\pi(\tilde{K}, \tilde{v})$, then $H = p_\#\pi(\tilde{K}, \tilde{u})$ for some $\tilde{u}$ with $p(\tilde{u}) = v_*$.*

Proof. If $\tilde{\beta}$ is a path in $\tilde{K}$ from $\tilde{u}$ to $\tilde{v}$ and if $\beta = p\tilde{\beta}$, then $[\beta] \in \pi(K, v_*)$ and

$$[\beta] p_\# \pi(\tilde{K}, \tilde{v}) [\beta]^{-1} = p_\# \pi(\tilde{K}, \tilde{u}).$$

Conversely, assume $H = [\alpha] p_\# \pi(\tilde{K}, \tilde{v}) [\alpha]^{-1}$. Let $\tilde{\beta}$ be the lifting of $\alpha^{-1}$ having origin $\tilde{v}$. If end $\tilde{\beta} = \tilde{u}$, then $p(\tilde{u}) = v_*$. Moreover

$$\pi(\tilde{K}, \tilde{u}) = [\tilde{\beta}^{-1}] \pi(\tilde{K}, \tilde{v}) [\tilde{\beta}],$$

so that

$$p_\# \pi(\tilde{K}, \tilde{u}) = [\alpha] p_\# \pi(\tilde{K}, \tilde{v}) [\alpha^{-1}] = H.$$

Let us now consider the existence of covering complexes (this construction is called the "loop space" in topology). To motivate what we do, suppose we have a covering complex $p : (\tilde{K}, \tilde{v}_*) \to (K, v_*)$. Each vertex $\tilde{v}$ of $\tilde{K}$ can be described by a path $\alpha$ in $K$ having origin $v_*$ as follows: choose a path $\tilde{\alpha}$ in $\tilde{K}$ from $\tilde{v}_*$ to $\tilde{v}$ and define $\alpha = p\tilde{\alpha}$. Had we chosen a second such path in $\tilde{K}$, say $\tilde{\beta}$, then the path $\beta = p\tilde{\beta}$ is also a path in $K$ from $v_*$ to $v$; moreover $[\alpha \Box \beta^{-1}] = p_\# [\tilde{\alpha} \Box \tilde{\beta}^{-1}] \in p_\# \pi(\tilde{K}, \tilde{v}_*)$.

Definition. Let $K$ be a complex with basepoint $v_*$, and let $\pi$ be a subgroup of $\pi(K, v_*)$. If $\alpha$ and $\beta$ are paths in $K$ with origin $v_*$, then

$$\alpha \equiv \beta (\operatorname{mod} \pi)$$

if end $\alpha =$ end $\beta$ and $[\alpha \Box \beta^{-1}] \in \pi$.

It is easy to see that $\alpha \equiv \beta$ (mod $\pi$) defines an equivalence relation on the set of all paths in $K$ having origin $v_*$; denote the equivalence class of such a path $\alpha$ by $\bar{\alpha}$, and denote the family of all $\bar{\alpha}$ by $K_\pi$. We shall make $K_\pi$ into a complex.

Let $s$ be a simplex in $K$ and let $\alpha$ be a path with origin $v_*$ and end in $s$. A *continuation* of $\alpha$ in $s$ is a path $\alpha \Box \alpha'$, where $\alpha'$ is a path lying wholly in $s$. Let

$$[s, \bar{\alpha}] = \{\bar{\beta} \in K_\pi : \beta \text{ is a continuation of } \alpha \text{ in } s\}.$$

Define the simplexes in $K_\pi$ to be all $[s, \bar{\alpha}]$, where $s$ is a simplex in $K$ and $\bar{\alpha} \in K_\pi$ is such that end $\bar{\alpha}$ is in $s$.

Theorem 2.5. *Let $K$ be a connected complex, $v_*$ a vertex of $K$, and $\pi$ a subgroup of $\pi(K, v_*)$. Then $K_\pi$ is a complex and the function $V(K_\pi) \to V(K)$ given by $\bar{\alpha} \mapsto$ end $\bar{\alpha}$ defines a map $p: K_\pi \to K$.*

Proof. Straightforward.

There is an obvious choice of a basepoint in $K_\pi$: let $\bar{v}_* = \bar{\alpha}$, where $\alpha = (v_*, v_*)$.

LEMMA 2.6. *If $K$ is a connected complex, then every path $\alpha$ in $K$ with origin $v_*$ can be lifted to a path $A$ in $K_\pi$ from $\bar{v}_* = \overline{(v_*, v_*)}$ to $\bar{\alpha}$.*

PROOF. Let $(v_*, v_1) \square (v_1, v_2) \square \cdots \square (v_{n-1}, v_n)$ be a path in $K$ from $v_*$ to $v_n = \text{end}\,\alpha$ (which exists because $K$ is connected). Define paths $\alpha_i$ by $\alpha_i = (v_*, v_1) \square (v_1, v_2) \square \cdots \square (v_{i-1}, v_i)$. Observe that if $s$ is the simplex $\{v_i, v_{i+1}\}$, then $\bar{\alpha}_i$ and $\bar{\alpha}_{i+1}$ both lie in $[s, \bar{\alpha}_i]$; hence $(\bar{\alpha}_i, \bar{\alpha}_{i+1})$ is an edge in $K_\pi$. Therefore $A = (\bar{v}_*, \bar{\alpha}_1) \square (\bar{\alpha}_1, \bar{\alpha}_2) \square \cdots \square (\bar{\alpha}_{n-1}, \bar{\alpha}_n)$ is a path in $K_\pi$ from $\bar{v}_*$ to $\bar{\alpha}_n = \bar{\alpha}$ which lifts $\alpha$.

COROLLARY 2.7. *If $K$ is a connected complex, then $K_\pi$ is connected.*

PROOF. There is a path in $K_\pi$ from $\bar{v}_*$ to every vertex $\bar{\alpha}$ of $K_\pi$.

THEOREM 2.8. *Let $K$ be a connected complex, and let $\pi$ be a subgroup of $\pi(K, v_*)$. Then $p: K_\pi \to K$ is a covering complex and $p_* \pi(K_\pi, \bar{v}_*) = \pi$.*

PROOF. Let us first show that $p: K_\pi \to K$ is a covering complex; only condition (iii) of the definition remains to be checked.

We claim that $p \mid [s, \bar{\alpha}] : [s, \bar{\alpha}] \to s$ is a one-to-one correspondence. Suppose $\bar{\beta}$ and $\bar{\gamma} \in [s, \bar{\alpha}]$ and $p(\bar{\beta}) = p(\bar{\gamma})$. Then $\beta = \alpha \square \beta_1$ and $\gamma = \alpha \square \gamma_1$, where $\beta_1, \gamma_1$ lie wholly in $s$. Moreover, $\beta \square \gamma^{-1}$ is defined and $\beta \square \gamma^{-1} = \alpha \square \beta_1 \square {\gamma_1}^{-1} \square \alpha^{-1} \sim \alpha \square \alpha^{-1} \sim 1$, since $\beta_1 \square {\gamma_1}^{-1} \sim 1$, being a path lying wholly in a simplex, so that $[\beta \square \gamma^{-1}] = 1$ in $\pi(K, v_*)$. Since $1 \in \pi$, $\beta \equiv \gamma \pmod{\pi}$ and $\bar{\beta} = \bar{\gamma}$. Hence $p \mid [s, \bar{\alpha}]$ is one-to-one. To see that $p \mid [s, \bar{\alpha}]$ is onto, let $v$ be a vertex in $s$. If $\alpha'$ is a path in $s$ from end $\alpha$ to $v$, then

$$\overline{\alpha \square \alpha'} \in [s, \bar{\alpha}] \text{ and } p(\overline{\alpha \square \alpha'}) = v.$$

Let $s$ be a simplex in $K$ and let $w$ be a vertex in $s$. It is easy to check that $p^{-1}(s) = \bigcup [s, \bar{\alpha}]$, where the union ranges over all $\bar{\alpha} \in K_\pi$ with end $\bar{\alpha} = w$. To prove that $p: K_\pi \to K$ is a covering complex, it suffices to prove the sheets $[s, \bar{\alpha}]$ are pairwise disjoint. Assume $\bar{\gamma} \in [s, \bar{\alpha}] \cap [s, \bar{\beta}]$. Then $\gamma \equiv \alpha \square \alpha_1 \pmod{\pi}$ and $\gamma \equiv \beta \square \beta_1 \pmod{\pi}$, where $\alpha_1$ and $\beta_1$ are paths lying wholly in $s$. The definition of equivalence gives end $\alpha_1 =$ end $\beta_1$, so that $\alpha_1 \square {\beta_1}^{-1}$ is a path lying wholly in $s$; moreover, $\alpha_1 \square {\beta_1}^{-1}$ is a closed path at $w$. Hence

$$1 = [\gamma \square \gamma^{-1}] = [\alpha \square \alpha_1 \square {\beta_1}^{-1} \square \beta^{-1}] = [\alpha \square \beta^{-1}] \in \pi.$$

It follows that $\alpha \equiv \beta \pmod{\pi}$, i.e., that $\bar{\alpha} = \bar{\beta}$ and $[s, \bar{\alpha}] = [s, \bar{\beta}]$.

We claim that $\pi = p_{\#}\pi(K_\pi, \bar{v}_*)$. Let $[\alpha] \in \pi(K, v_*)$. Since $p : K_\pi \to K$ is a covering complex, there is a unique lifting $\tilde{\alpha}$ of $\alpha$ with origin $\bar{v}_*$. But we constructed such a lifting $A$ in Lemma 2.6; therefore $A$ must be $\tilde{\alpha}$ and so end $\tilde{\alpha} =$ end $A = \bar{\alpha}$. The following statements are equivalent: $[\alpha] \in p_{\#}\pi(K_\pi, \bar{v}_*)$; $[\alpha] = [pA]$, where $[A] \in \pi(K_\pi, \bar{v}_*)$; end $A =$ orig $A = \bar{v}_*$; $\bar{\alpha} = \bar{v}_*$; $[\alpha \,\square\, (v_*, v_*)^{-1}] \in \pi$; $[\alpha] \in \pi$.

We may now give the first application.

**Theorem** 2.9 (Schreier-Nielsen). *Every subgroup $H$ of a free group $F$ is itself free.*

**Proof.** Let $F$ be free of rank $|I|$; let $K$ be the bouquet of $|I|$ circles constructed in Theorem 1.9, so that $\pi(K, v_*)$ may be identified with $F$. By Theorem 2.8, there is a covering complex $p : K_H \to K$ with $p_{\#}\pi(K_H, \bar{v}_*) = H$. Since $p_{\#}$ is one-one, we have $\pi(K_H, \bar{v}_*) \cong H$. Since $\dim K = 1$, however, we remarked earlier that $\dim K_H = 1$. Therefore $H$ is free, by Theorem 1.8.

The usual proofs of the Schreier-Nielsen theorem yield more information about the free subgroup $H$: they actually exhibit a basis for $H$; in particular, one may compute rank $H$. In order to give one such computation, we first give a "geometric" interpretation of the index of a subgroup.

**Definition.** If $p : \tilde{K} \to K$ is a map and $v \in V(K)$, then $p^{-1}(v)$ is called the *fiber* over $v$.

**Theorem** 2.10. *Let $p : (\tilde{K}, \tilde{v}_*) \to (K, v_*)$ be a covering complex. Then*

$$|p^{-1}(v_*)| = [\pi(K, v_*) : p_{\#}\pi(\tilde{K}, \tilde{v}_*)].$$

**Proof.** Denote $p_{\#}\pi(\tilde{K}, \tilde{v}_*)$ by $\tilde{\pi}$. Define a function $\theta$ from the family of all right cosets of $\tilde{\pi}$ into the fiber $p^{-1}(v_*)$ by

$$\theta : \tilde{\pi}[\alpha] \mapsto \text{end}\, \tilde{\alpha},$$

where $\tilde{\alpha}$ is the lifting of $\alpha$ having origin $\tilde{v}_*$. We claim that $\theta$ is a (well defined) one-one correspondence.

(i) $\theta$ is well defined.

Suppose $[\tilde{\beta}] \in \pi(\tilde{K}, \tilde{v}_*)$, so that $\tilde{\beta}$ is a path from $\tilde{v}_*$ to $\tilde{v}_*$. Then $p_{\#}[\tilde{\beta}] \,\square\, [\alpha] = [p\tilde{\beta} \,\square\, \alpha]$. A lifting of $p\tilde{\beta} \,\square\, \alpha$ is $\tilde{\beta} \,\square\, \tilde{\alpha}$, which is, in fact, *the* lifting of this path, by uniqueness. Hence end $\tilde{\beta} \,\square\, \tilde{\alpha} =$ end $\tilde{\alpha}$.

(ii) $\theta$ is onto.

Let $\tilde{u} \in p^{-1}(v_*)$ and let $\tilde{\beta}$ be a path in $\tilde{K}$ from $\tilde{v}_*$ to $\tilde{u}$ (for the first time we use the connectedness of $\tilde{K}$). Then $\alpha = p\tilde{\beta}$ is a path from $v_*$ to itself, so $[\alpha] \in \pi(K, v_*)$. Visibly $\tilde{\alpha} = \tilde{\beta}$, so that $\theta(\tilde{\pi}[\alpha]) = \text{end}\,\tilde{\alpha} = \text{end}\,\tilde{\beta} = \tilde{u}$.

(iii) $\theta$ is one-to-one.

If $\theta(\tilde{\pi}[\alpha]) = \theta(\tilde{\pi}[\beta])$, then $\text{end}\,\tilde{\alpha} = \text{end}\,\tilde{\beta}$. Hence $\tilde{\alpha} \square \tilde{\beta}^{-1}$ is defined and is a path in $\tilde{K}$ from $\tilde{v}_*$ to $\tilde{v}_*$, i.e., $[\tilde{\alpha} \square \tilde{\beta}^{-1}] \in \pi(\tilde{K}, \tilde{v}_*)$. Therefore

$$[\alpha] \square [\beta]^{-1} = [p\tilde{\alpha}] \square [p\tilde{\beta}^{-1}] = p_\# [\tilde{\alpha} \square \tilde{\beta}^{-1}] \in \tilde{\pi},$$

and so $\tilde{\pi}[\alpha] = \tilde{\pi}[\beta]$.

Corollary 2.11. *Let $p : \tilde{K} \to K$ be a covering complex. If $v_1, v_2 \in V(K)$, then*

$$|p^{-1}(v_1)| = |p^{-1}(v_2)|,$$

*i.e., all fibers have the same cardinal.*

Proof. Let $\tilde{v}_i \in p^{-1}(v_i)$, $i = 1, 2$, let $\tilde{\beta}$ be a path in $\tilde{K}$ from $\tilde{v}_1$ to $\tilde{v}_2$, and let $\beta = p\tilde{\beta}$. One quickly checks the following diagram commutes

$$\begin{array}{ccc} \pi(\tilde{K}, \tilde{v}_1) & \xrightarrow{\quad G \quad} & \pi(\tilde{K}, \tilde{v}_2) \\ {\scriptstyle p_\#}\downarrow & & \downarrow{\scriptstyle p_\#} \\ \pi(K, v_1) & \xrightarrow{\quad g \quad} & \pi(K, v_2), \end{array}$$

where $G[\tilde{\alpha}] = [\tilde{\beta}^{-1} \square \tilde{\alpha} \square \tilde{\beta}]$ and $g[\alpha] = [\beta^{-1} \square \alpha \square \beta]$. Of course, $G$ and $g$ are isomorphisms (Theorem 2.4). Since $p_\#$ is one-one, it follows that the index on the left is equal to the index on the right. The result follows from Theorem 2.10.

It is an easy induction on dimension that if $p : \tilde{K} \to K$ is a covering complex and if there are $j$ points in each fiber, then there are $j$ sheets over each simplex in $K$.

Theorem 2.12. *Let $F$ be free of finite rank $n$ and let $H$ be a subgroup of $F$ having finite index $j$. Then $H$ is free of rank $jn - j + 1$.*

Proof. For a finite graph $K$, let $n_0(K)$ be the number of its vertices and $n_1(K)$ the number of its 1-simplexes. If $T$ is a maximal tree in $K$, our earlier remark about Euler-Poincaré characteristic gives $n_0(T) - n_1(T) = 1$. Therefore, the number of 1-simplexes in $K - T$ is $n_1(K) - n_1(T) = n_1(K) - n_0(T) + 1$. Since $T$ is a maximal tree, Theorem 1.6 gives $n_0(T) = n_0(K)$. Therefore, Theorem 1.8 shows, for a finite graph $K$, that $\pi(K, v_*)$ is free of rank $n_1(K) - n_0(K) + 1$.

Assume now that $K$ is a bouquet of $n$ circles: $n_0(K) = 2n + 1$ and $n_1(K) = 3n$. Let $p : K_H \to K$ be the covering complex corresponding to $H$. By Theorem 2.10 and Corollary 2.11, there are exactly $j = [F : H]$ points in each fiber: $n_0(K_H) = jn_0(K)$. Moreover, our remark after Corollary 2.11 gives $n_1(K_H) = jn_1(K)$. Substitution into the general formula derived in the first paragraph gives

$$n_1(K_H) - n_0(K_H) + 1 = jn - j + 1.$$

A construction of a basis for an arbitrary subgroup $H$ of a free group $F$ in terms of "Schreier transversals" can also be given by these methods (see [10; 12; 14]). In essence, bases of $H$ arise from choosing maximal trees in $K$ and lifting them, using Theorem 1.10, to maximal trees of $K_H$. One consequence of this further squeezing is the surprising fact that the commutator subgroup of a free group of rank 2 is a free group of infinite rank.

As a second application of covering complexes, we prove the theorem of Kuroš characterizing subgroups of free products (we assume the reader is familiar with the definition of free product). First of all, we relate free products to complexes.

Theorem 2.13. *Let $K$ be a connected complex with connected subcomplexes $\{K_i : i \in I\}$. Assume $\bigcup K_i = K$ and that there is a tree $T$ in $K$ with $K_i \cap K_j = T$, for all $i \neq j$. Then*

$$\pi(K, v) \cong \mathop{*}_{i \in I} \pi(K_i, v_i)$$

*for suitable vertices $v$ in $K$ and $v_i$ in $K_i$.*

Proof. We claim there are maximal trees $T_i$ of $K_i$ that contain $T$ and such that $T' = \bigcup T_i$ is a maximal tree of $K$. Define $T_i$ to be some maximal tree of $K_i$ containing $T$. Now the family $\{T, T_i - T, i \in I\}$ is a disjoint union of trees, so that Theorem 1.10 asserts that their union $T'$ is contained in a maximal tree $M$ of $K$. Since $V(T') = V(K)$, however, we have $T' = M$, as desired.

By Theorem 1.7, there is a presentation $(E_i \mid R_i)$ of $\pi(K_i, v_i)$, where $E_i$ is the set of edges in $K_i$ and $R_i$ is the set of relations of the form: (a) $(u, v)(v, w) = (u, w)$ when $\{u, v, w\}$ is a simplex of $K_i$; (b) suppress all edges in $T_i$. Now a presentation for $*_{i \in I}\pi(K_i, v_i)$ is $(\bigcup E_i \mid \bigcup R_i)$. But we claim that this is also a presentation for $\pi(K, v) : \bigcup E_i$ consists of all the edges of $K$ (for $K = \bigcup K_i$); since $T' = \bigcup T_i$, an edge in $K$ lies in $T'$ if and only if it lies in some $T_i$; moreover $\{u, v, w\}$ lies in a simplex of $K$ if and only if it lies in a simplex of some $K_i$. By Theorem 1.7, we have $\pi(K, v) \cong * \pi(K_i, v_i)$.

Which groups arise as edgepath groups?

THEOREM 2.14. *Given a group $G$, there exists a connected 2-complex $K^\circ$ with $\pi(K^\circ, v^\circ) \cong G$ for some vertex $v^\circ$ in $K^\circ$.*

PROOF. A proof may be found in [12]; it is essentially a construction of an "Eilenberg-Mac Lane space" $K(G, 1)$. One begins with a presentation of $G$, say $G \cong F/N$, where $F$ is free, i.e., a covering complex $p: K_N \to K$, where $K$ is a bouquet of $r$ circles ($r = \operatorname{rank} F$) and $K_N$ is the covering complex corresponding to $N$. It is easy to adjoin certain 2-simplexes to the bouquet $K$ to force the relations of $N$ to hold in this augmented complex, which we denote $K^\circ$. We thus have the diagram

$$\begin{array}{ccc} K_N & & \\ {\scriptstyle p}\big\downarrow & & \\ K & \longrightarrow & K^\circ. \end{array}$$

In order to prove that $\pi(K^\circ, v^\circ) \cong G$, it is only a matter of completing the diagram above with a covering complex of $K^\circ$ that makes the resulting picture commute.

Several remarks are in order. First, the complex $K^\circ$ is not unique (at the very least, it depends on a presentation of $G$). Second, if $G$ is finite, one may choose $K^\circ$ so that it, too, is finite (i.e., $V(K^\circ)$ is finite). Finally, a group $G$ is finitely presented if and only if there is a finite connected complex $K^\circ$ with $G \cong \pi(K^\circ, v^\circ)$.

THEOREM 2.15 (KUROŠ). *If $H$ is a subgroup of $*_{i \in I} G_i$, then $H = F * ( * H_\alpha)$, where $F$ is free and each $H_\alpha$ is isomorphic to a subgroup of some $G_i$.*

PROOF. By Theorem 2.14, there exist connected complexes $K_i$ with $\pi(K_i, v_i) \cong G_i$. Define a new complex $K$ by adjoining a new vertex $v_*$ to the disjoint union $\cup K_i$ and new 1-simplexes $\{v_*, v_i\}$, all $i \in I$. Then $\pi(K, v_*) \cong * G_i$, by Theorem 2.13. Let $p: K_H \to K$ be the covering complex and $\tilde{v} \in p^{-1}(v_*)$ so that $p_\# \pi(K_H, \tilde{v}) = H$ (having identified $*G_i$ with $\pi(K, v_*)$). For each $i$, $p^{-1}(K_i)$ is the disjoint union of its components $\tilde{K}_{ij}$; choose a maximal tree $\tilde{T}_{ij}$ in $\tilde{K}_{ij}$. Let $\tilde{L}$ be the 1-subcomplex consisting of $\cup T_{ij}$ and $p^{-1}\{v_*, v_i\}$, all $i \in I$. Finally, let $\tilde{T}$ be a maximal tree in $\tilde{L}$ containing $\cup \tilde{T}_{ij}$ (which exists courtesy of Theorem 1.10). Observe that $\tilde{T}$ contains no edges in $\tilde{K}_{ij}$ aside from those in $\tilde{T}_{ij}$ lest we violate the maximality of $\tilde{T}_{ij}$ in $\tilde{K}_{ij}$.

Consider the subcomplexes $\tilde{L}$ and $\tilde{K}_{ij} \cup \tilde{T}$ in $K_H$. Clearly $K_H$ is the union of these. Further, the intersection of any two of these is the tree $\tilde{T}$. Therefore, Theorem 2.13 gives

$$\pi(K_H, \tilde{v}) \cong \pi(\tilde{L}, \tilde{v}) * ( * \pi(\tilde{K}_{ij} \cup \tilde{T}, \tilde{v})).$$

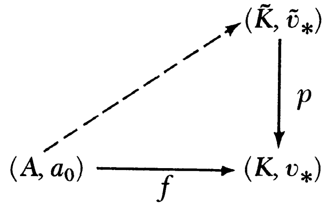
Now $\pi(\tilde{L}, \tilde{v})$ is free because $\dim \tilde{L} = 1$. Since $\tilde{T}$ is a maximal tree in $\tilde{K}_{ij} \cup \tilde{T}$, Theorem 1.7 gives $\pi(\tilde{K}_{ij} \cup \tilde{T}, \tilde{v}) \cong \pi(\tilde{K}_{ij}, \tilde{v}_{ij})$ for some vertex $\tilde{v}_{ij}$ in $\tilde{K}_{ij}$. We remarked, just after defining covering complexes, that $p \mid \tilde{K}_{ij} : \tilde{K}_{ij} \to K_i$ is also a covering complex. Hence $\pi(\tilde{K}_{ij}, \tilde{v}_{ij})$ is isomorphic to a subgroup of $\pi(K_i, v_i) \cong G_i$. Therefore, $H = p_{\#}\, \pi(K_H, \tilde{v})$ is a free product as described in the theorem, for $p_{\#}$ is one-one.

There are stronger versions of the Kuroš theorem (see [7; **9**; **10**]), but we believe we have made our point that covering complexes do give "conceptual" proofs of various subgroup theorems.

## 3. **Galois Theory.**

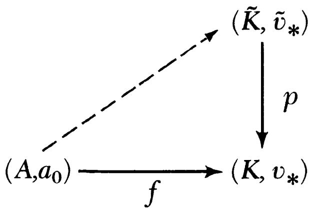
In this section we survey all the covering complexes of a connected complex; we shall see an analog of classical Galois theory emerge.

THEOREM 3.1 (UNIQUENESS OF LIFTINGS). *Let $p : (\tilde{K}, \tilde{v}_*) \to (K, v_*)$ be a covering complex, and let $f : (A, a_0) \to (K, v_*)$ be a map, where $A$ is a connected complex. Then there exists at most one lifting $\tilde{f} : (A, a_0) \to (\tilde{K}, \tilde{v}_*)$, i.e., at most one $\tilde{f}($ with $p\tilde{f} = f$ and $\tilde{f}(a_0) = \tilde{v}_*$.*




PROOF. Suppose $\tilde{f}$ and $f' : (A, a_0) \to (\tilde{K}, \tilde{v}_*)$ are liftings of $f$. Assume there is a vertex $a' \in A$ for which $\tilde{f}(a') \neq f'(a')$. There is a path in $A$ from $a_0$ to $a'$ because $A$ is connected. Since $\tilde{f}(a_0) = f'(a_0)$, there is an edge $(a_1, a_2)$ with $\tilde{f}$ and $f'$ agreeing on $a_1$ but disagreeing on $a_2$. By Theorem 2.1, the path $f \circ \alpha = (fa_1, fa_2)$ has a unique lifting to a path in $\tilde{K}$ with origin $\tilde{f} a_1 = f' a_1$. But two such liftings are $(\tilde{f} a_1, \tilde{f} a_2)$ and $(f' a_1, f' a_2)$. We conclude that $\tilde{f} a_2 = f' a_2$, which is a contradiction. Therefore $\tilde{f} = f'$.

THEOREM 3.2 (LIFTING CRITERION). *Let $p : (\tilde{K}, \tilde{v}_*) \to (K, v_*)$ be a covering complex and let $f : (A, a_0) \to (K, v_*)$ be a map, where $A$ is a connected complex. Then there exists a unique lifting $\tilde{f}$ of $f$ if and only if $f_{\#}\, \pi(A, a_0) \subset p_{\#}\, \pi(\tilde{K}, v_*)$.*




REMARKS. (i) If we regard a path $\alpha$ in $K$ as a map $\alpha : I_n \to K$ (for some $n$), then Theorem 3.2 generalizes Theorem 2.1, for $I_n$ is a connected complex with $\pi(I_n, t_0) = \{1\}$.

(ii) In view of Theorem 3.1, we need only consider existence of $\tilde{f}$.

PROOF. Assume first that a lifting $\tilde{f}$ exists. Then $f_\# = (p\tilde{f})_\# = p_\# \tilde{f}_\#$ and

$$f_\# \pi(A, a_0) = p_\# \tilde{f}_\# \pi(A, a_0) \subset p_\# \pi(\tilde{K}, \tilde{v}_*).$$

To prove the converse, let us first define a function $V(A) \to V(\tilde{K})$. If $a \in V(A)$, choose a path $\ell$ in $A$ from $a_0$ to $a$ ($A$ is connected). Then $f \circ \ell$ is a path in $K$ from $v_*$ to $f(a)$; let $\tilde{\lambda}$ be the (unique) lifting of $f \circ \ell$ with origin $\tilde{v}_*$ (the existence of $\tilde{\lambda}$ is guaranteed by Theorem 2.1). We claim that end $\tilde{\lambda}$ is independent of the choice of path $\ell$. Suppose $\ell_1$ is a second path in $A$ from $a_0$ to $a$, and let $\tilde{\lambda}_1$ be the unique lifting of $f \circ \ell_1$ having origin $\tilde{v}_*$. Thus, $p\tilde{\lambda} = f \circ \ell$, $p\tilde{\lambda}_1 = f \circ \ell_1$. Now $[\ell \,\square\, \ell_1^{-1}] \in \pi(A, a_0)$ so that

$$[(f \circ \ell) \,\square\, (f \circ \ell_1^{-1})] = f_\# [\ell \,\square\, \ell_1^{-1}] \in f_\# \pi(A, a_0) \subset p_\# \pi(\tilde{K}, \tilde{v}_*).$$

There thus exists a closed path $\tilde{g}$ in $\tilde{K}$ at $\tilde{v}_*$ *with*

$$[(f \circ \ell) \,\square\, (f \circ \ell_1^{-1})] = [p \circ \tilde{g}].$$

It follows that

$$(f \circ \ell) \,\square\, (f \circ \ell_1^{-1}) \,\square\, (p\tilde{\lambda}_1) \sim (p\tilde{g}) \,\square\, (p\tilde{\lambda}_1).$$

Since $p\tilde{\lambda}_1 = f\ell_1$, we have

$$f \circ \ell \sim p(\tilde{g} \,\square\, \tilde{\lambda}_1).$$

By Theorem 2.2, equivalent paths have equivalent liftings:

$$\tilde{\lambda} \sim \tilde{g} \,\square\, \tilde{\lambda}_1.$$

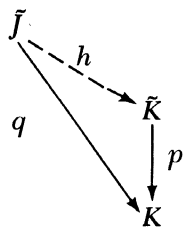
Thus end $\tilde{\lambda}$ = end $\tilde{g} \,\square\, \tilde{\lambda}_1$ = end $\tilde{\lambda}_1$, for $\tilde{g}$ is a closed path at $\tilde{v}_*$. We have shown that end $\tilde{\lambda}$ (where $\tilde{\lambda}$ is the lifting of $f \circ \ell$ with origin

$\tilde{v}_*$) is independent of the choice of path $\ell$ in $A$ from $a_0$ to $a$. Therefore, the function $\tilde{f} : V(A) \to V(\tilde{K})$ given by $a \mapsto \text{end}\,\tilde{\lambda}$ is well defined. Note that $\tilde{f}(a_0) = \tilde{v}_*$. It remains to prove that $\tilde{f}$ is a map (it is obvious that $p\tilde{f} = f$).

Let $s = \{b_0, \cdots, b_q\}$ be a simplex in $A$; we must show that $\tilde{f}(s) = \{\tilde{f}b_0, \cdots, \tilde{f}b_q\}$ is a simplex in $\tilde{K}$. Let $\ell_0$ be a path in $A$ from $a_0$ to $b_0$, and, for each $i$ with $0 < i \leqq q$, choose a path $\ell_i$ in $s$ from $b_0$ to $b_i$. Since $f$ is a map, we know that $t = \{fb_0, \cdots, fb_q\}$ is a simplex of $K$; let $\tilde{t}$ be the sheet over $t$ containing $\tilde{f}b_0$. We claim that $\tilde{f}(s) \subset \tilde{t}$. Indeed, let $\tilde{\lambda}_0$ lift $f \circ \ell_0$ (so that $\tilde{\lambda}_0$ is a path in $\tilde{K}$ from $\tilde{v}_*$ to $\tilde{f}b_0$). Consider the path $\tilde{\mu}_i = \tilde{\lambda}_0 \,\square\, \{(p\,|\tilde{t})^{-1} \circ f \circ \ell_i\}$, where $0 < i \leqq q$. Note that $p\tilde{\mu}_i = p\tilde{\lambda}_0 \,\square\, f \circ \ell_i = \ell_0 \,\square\, \ell_i$ and that $\tilde{\mu}_i$ has origin $\tilde{v}_*$. It follows that $\tilde{f}(b_i) = \text{end}\,\tilde{\mu}_i$, and, visibly, end $\tilde{\mu}_i \in \text{im}(p\,|\tilde{t})^{-1} = \tilde{t}$. Therefore $\tilde{f}$ is a map of $(A, a_0)$ to $(\tilde{K}, \tilde{v}_*)$ that lifts $f$.

We are now ready to compare different covering complexes of $K$.

**Theorem** 3.3. *Assume $p : (\tilde{K}, \tilde{v}_*) \to (K, v_*)$ and $q : (\tilde{J}, \tilde{w}_*) \to (K, v_*)$ are covering complexes. If $q_\# \pi(\tilde{J}, \tilde{w}_*) \subset p_\# \pi(\tilde{K}, \tilde{v}_*)$, then there exists a unique map $h : (\tilde{J}, \tilde{w}_*) \to (\tilde{K}, \tilde{v}_*)$ with $ph = q$. Moreover, $h : \tilde{J} \to \tilde{K}$ is a covering complex.*




**Proof.** Since $\tilde{J}$ is a connected complex (as is every covering complex), the lifting criterion just proved provides a unique map $h : (\tilde{J}, \tilde{w}_*) \to (\tilde{K}, \tilde{v}_*)$ with $ph = q$. It remains to prove that $h : \tilde{J} \to \tilde{K}$ is a covering complex.

Let $\tilde{t}$ be a simplex in $\tilde{K}$, and let $s = p(\tilde{t})$. As $s$ is a simplex in $K$, we have $q^{-1}(s) = \bigcup \sigma_j$, where the $\sigma_j$ are pairwise disjoint simplexes for which $q\,|\sigma_j : \sigma_j \to s$ is a one-one correspondence. Let $p^{-1}(s) = \bigcup \tilde{t}_i$, where the $\tilde{t}_i$ are pairwise disjoint simplexes for which each $p\,|\,\tilde{t}_i$ is a one-one correspondence (note that $\tilde{t} = \tilde{t}_i$ for one $i$). Now $ph = q$ implies $h^{-1}p^{-1}(s) = q^{-1}(s)$, so that

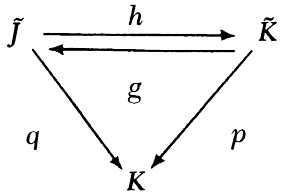
$$h^{-1}(\bigcup \tilde{t}_i) = \bigcup \sigma_j.$$

In particular, $h^{-1}(\tilde{t}) \subset \bigcup \sigma_j$. Consider only those $\sigma_j$ (call them $\sigma_{ji}$)

with $h(\sigma_j) = \tilde t$. Visibly these $\sigma_{ji}$ are pairwise disjoint simplexes in $\tilde J$ (all the $\sigma_j$ are), and $h|\sigma_{ji} : \sigma_{ji} \to \tilde t$ is a one-one correspondence (since $ph = q$ and the corresponding restrictions of $p$ and $q$ are). It only remains to show $h^{-1}(\tilde t) = \bigcup_i \sigma_{ji}$, and we have only to prove $\bigcup \sigma_{ji} \subset h^{-1}(\tilde t)$. This follows from the observation that if one vertex of $\sigma_{ji}$, say $w$, satisfies $h(w) \in \tilde t$, then $h(\sigma_{ji}) = \tilde t$ ($h(\sigma_{ji})$ is connected and the components of $\bigcup \tilde t_i$ are precisely the $\tilde t_i$). We have verified that $h : \tilde J \to \tilde K$ is a covering complex.

COROLLARY 3.4. *With the same notation as in Theorem* 3.3, *if* $p_{\#}\pi(\tilde K, \tilde v_*) = q_{\#}\pi(\tilde J, \tilde w_*)$, *then the map* $h : \tilde J \to \tilde K$ *is an isomorphism.*

PROOF. By Theorem 3.3, there is a unique map $g : (\tilde K, \tilde v_*) \to (|\tilde J, \tilde w_*)$ with $qg = p$. Now the composite $gh$ and the identity $1_{\tilde K}$ both complete the diagram




Uniqueness gives $gh = 1_{\tilde K}$. A similar argument gives $hg = 1_{\tilde J}$.

NOTATION. If $p : (\tilde K, \tilde v_*) \to (K, v_*)$ is a covering complex, let $f(\tilde K/K)$ denote $|p^{-1}(v_*)|$, the cardinal of the fiber over $v_*$.

By Corollary 2.11, $f(\tilde K/K)$ is independent of the choice of basepoint $v_*$ in $K$.

COROLLARY 3.5 (FIBER FORMULA). *Assume* $p : (\tilde K, \tilde v_*) \to (K, v_*)$ *and* $q : (\tilde J, \tilde w_*) \to (K, v_*)$ *are covering complexes with* $q_{\#}\pi(\tilde J, \tilde w_*) \subset p_{\#}\pi(\tilde K, \tilde v_*)$. *Then there exists a unique* $h : \tilde J \to \tilde K$ *such that* $ph = q$ *and* $h : \tilde J \to \tilde K$ *is a covering complex.*

*Moreover, if* $f(\tilde J/K)$ *is finite, then*

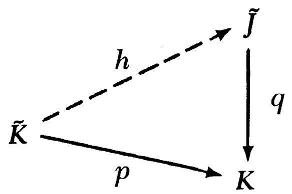
$$f(\tilde J/K) = f(\tilde J/\tilde K) f(\tilde K/K).$$

PROOF. Theorem 3.3 provides the map $h : \tilde J \to \tilde K$ exhibiting $(\tilde J, h)$ as a covering complex of $\tilde K$. The formula follows from Corollary 2.11 and the observation that if $v_* \in V(K)$, then $p^{-1}(v_*) = (qh)^{-1}(v_*) = h^{-1}(q^{-1}v_*)$.

To summarize, each covering complex $(\tilde K, \tilde v_*)$ of $(K, v_*)$ determines a subgroup of $\pi(K, v_*)$, namely, $p_{\#}\pi(\tilde K, \tilde v_*)$; moreover, two covering complexes determining the same subgroup are isomorphic. Finally, if two covering complexes determine subgroups ordered by inclusion, then one is an "intermediate" covering complex. Galois theory is beginning to emerge.

The next definition gives an analog of algebraic closure.

DEFINITION. A *universal covering complex* of $K$ is a covering complex $p : \tilde{K} \to K$ such that, for every covering complex $q : \tilde{J} \to K$ there exists a unique map $h : \tilde{K} \to \tilde{J}$ making the following diagram commute:



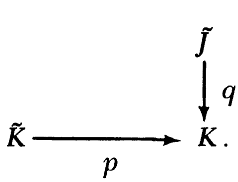

As any solution to a universal mapping problem, a universal covering complex of $K$ is unique to isomorphism if it exists.

DEFINITION. A complex $K$ is *simply connected* if it is connected and if $\pi(K, v) = \{1\}$ for some vertex $v$.

THEOREM 3.6. *Let $K$ be a connected complex. Then $K$ has a universal covering complex $p : \tilde{K} \to K$. Moreover, a covering complex $p : \tilde{K} \to K$ is universal if and only if $\tilde{K}$ is simply connected.*

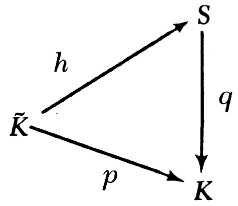
PROOF. Let $\pi = \{1\}$ be the trivial subgroup of $\pi(K, v_*)$; set $\tilde{K} = K_\pi$ and let $p : \tilde{K} \to K$ be the covering complex as constructed in Theorem 2.8. Since $p_\# \pi(\tilde{K}, \tilde{v}_*) = \{1\}$ and $p_\#$ is one-one, it follows that $\tilde{K}$ is simply connected.

Assume now that $q : \tilde{J} \to K$ is a covering complex; we thus have the diagram




Since $p_\# \pi(\tilde{K}, \tilde{v}_*) = \{1\} \subset q_\# \pi(\tilde{J}, \tilde{w}_*)$, the lifting criterion provides a unique map $h : \tilde{K} \to \tilde{J}$ with $qh = p$. Thus, $p : \tilde{K} \to K$ is a universal covering complex of $K$.

For the converse, assume $p : \tilde{K} \to K$ is a universal covering complex of $K$. As we saw above, there exists a simply connected covering complex $q : S \to K$. By the universal property, we have a commutative diagram

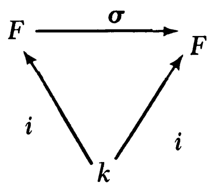



Therefore $p_{\#} = (qh)_{\#} = q_{\#}h_{\#}$ is trivial because $q_{\#}$ is. Thus $\pi(\tilde{K}, \tilde{v}_*) = \{1\}$, for $p_{\#}$ is one-one, and $\tilde{K}$ is simply connected.

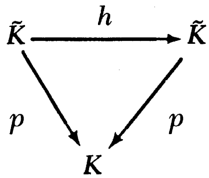
We remark that the analog of Theorem 3.6 is not true if we replace complexes by topological spaces: there may exist universal covering spaces that are not simply connected. The analog is true for polyhedra, however.

If $K$ is a connected complex, we seek a one-one correspondence between "intermediate" covering complexes of $K$ and subgroups of $\pi(K, v_*)$. There are two difficulties: base points; set theory. If we do choose base points, i.e., if we work within $\mathcal{K}_*$, then we can pass back and forth: to the covering complex $p : (\tilde{K}, \tilde{v}_*) \to (K, v_*)$, assign the subgroup $p_{\#}\pi(\tilde{K}, \tilde{v}_*)$; to the subgroup $\pi$ of $\pi(K, v_*)$, assign the covering complex $p : (K_\pi, \bar{v}) \to (K, v)$ as constructed in Theorem 2.8. Unfortunately, the composite of these is not the identity. For example, if we begin with $p : (K_\pi, \bar{w}) \to (K, v)$, where $\bar{w} \neq \bar{v}$ lies in the fiber over $v$, then the composite yields the complex $K_\pi$, but with base point $\bar{v}$. If we do not choose base points, then the assignment $q_{\#}\pi(\tilde{K}, \tilde{v}_*)$ is not well defined (we know that changing basepoint replaces a subgroup by a conjugate; it is well defined, of course, if $\pi(K, v_*)$ is abelian or hamiltonian). Somehow, we must avoid choosing basepoints. The second difficulty is set-theoretical: is the totality of all covering complexes of $K$ a set?

We deal with the problem of basepoints by adapting the definition of Galois group. Recall that if $F$ is a field extension of a field $k$, then the Galois group $\mathrm{Gal}(F/k)$ is defined to be the group of all automorphisms of $F$ fixing $k$ pointwise (the group operation, of course, is composition of functions). If $i : k \to F$ is the inclusion map, then $\mathrm{Gal}(F/k)$ consists of all automorphisms $\sigma$ of $F$ making the following diagram commute:



Definition. Let $p : \tilde{K} \to K$ be a covering complex. A *covering map* (or *deck map*) is an isomorphism $h$ making the following diagram commute:

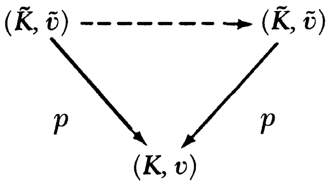



Definition. If $p : \tilde{K} \to K$ is a covering complex, then $\mathrm{Cov}(\tilde{K}/K)$ is the group of all covering maps (under composition).

Our notation is sloppy, for it does not exhibit the projection map $p$.

Theorem 3.7. *Let $p : \tilde{K} \to K$ be a covering complex and let $h \in \mathrm{Cov}(\tilde{K}/K)$. Then either $h = 1_{\tilde{K}}$ or $h$ has no fixed points.*

Proof. Assume there is a vertex $\tilde{v}$ with $h(\tilde{v}) = \tilde{v}$. Then both $h$ and $1_{\tilde{K}}$ complete the commutative diagram




where $v = p(\tilde{v})$. By Theorem 3.1, $h = 1_{\tilde{K}}$.

Corollary 3.8. *Let $h_1, h_2 \in \mathrm{Cov}(\tilde{K}/K)$. If $h_1$ and $h_2$ agree on a vertex, then $h_1 = h_2$.*

Proof. The covering map $h_1 h_2{}^{-1}$ fixes a vertex.

Of course, the fundamental theorem of Galois theory holds only if we impose certain conditions (normality, separability) on the field extensions.

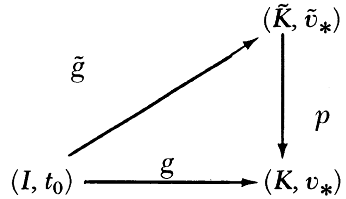
Definition. A *regular covering complex* of $K$ is a covering complex $p : (\tilde{K}, \tilde{v}_*) \to (K, v_*)$ for which $p_\# \pi(\tilde{K}, \tilde{v}_*) \lhd \pi(K, v_*)$.

Theorem 3.9. *Let $p : (\tilde{K}, \tilde{v}_*) \to (K, v_*)$ be a regular covering complex. Then*

$$\mathrm{Cov}(\tilde{K}/K) \cong \pi(K, v_*)/p_\# \pi(\tilde{K}, \tilde{v}_*).$$

REMARK. If $p : \tilde{K} \to K$ is a universal covering complex, it is clearly regular. In this case, Theorem 3.9 gives $\text{Cov}(\tilde{K}/K) \cong \pi(K, v_*)$. This indicates we are on the right track, for $\text{Cov}(\tilde{K}/K)$ allows us to recapture the edgepath group without forcing us to choose a basepoint.

PROOF. Let us define a function $\varphi : \pi(K, v_*) \to \text{Cov}(\tilde{K}/K)$. If $[g] \in \pi(K, v_*)$, then we may regard $g$ as a map $I_n \to K$ (some $n$) with $g(t_0) = v_* = g(t_n)$. By Theorem 2.1, there is a unique lifting of $g$ to a path $\tilde{g}$ in $\tilde{K}$ with origin $\tilde{v}_*$; let $\tilde{v}_1 = \text{end}\,\tilde{g}$.




Note that $\tilde{v}_1$ is in the fiber over $v_*$ (i.e., $p\tilde{v}_1 = v_*$) and, by Lemma 2.2, that $\tilde{v}_1$ is independent of the choice of path $g$ in $[g]$.

Theorem 2.4 says that $p_\# \pi(\tilde{K}, v_*)$ and $p_\# \pi(\tilde{K}, \tilde{v}_1)$ are conjugate subgroups of $\pi(K, v_*)$; they are thus equal, for we are assuming that $p_\# \pi(\tilde{K}, \tilde{v}_*)$ is a normal subgroup. It follows from Corollary 3.4 (essentially the lifting criterion) that there exists a unique isomorphism $h : (\tilde{K}, \tilde{v}_*) \to (\tilde{K}, \tilde{v}_1)$ with $ph = p$, i.e., $h \in \text{Cov}(\tilde{K}/K)$. We define $\varphi([g]) = h$. To summarize, $\varphi[g]$ is the unique $h \in \text{Cov}(\tilde{K}/K)$ with $h(\tilde{v}_*) = \tilde{v}_1 = \text{end}\,\tilde{g}$, where $\tilde{g}$ is the lifting of $g$ with origin $\tilde{v}_*$.

(i) $\varphi : \pi(K, v_*) \to \text{Cov}(\tilde{K}/K)$ is a homomorphism.

To prove this, we first remind the reader about one of the ingredients in the recipe defining $\varphi([g]) = h$. We know that $h(\tilde{v}_*) = \tilde{v}_1$; what is $h(\tilde{v})$ for another vertex $\tilde{v}$ of $\tilde{K}$? According to the construction in Theorem 3.2, taking $(A, a_0) = (\tilde{K}, \tilde{v}_*)$ and $f = p$ and the covering complex $p : (\tilde{K}, \tilde{v}_1) \to (K, v_*)$, choose a path $\tilde{\ell}$ in $\tilde{K}$ from $\tilde{v}_*$ to $\tilde{v}$; let $\tilde{\lambda}$ be the lifting of $p\tilde{\ell}$ with origin $\tilde{v}_1$; then $h(\tilde{v}) = \text{end}\,\tilde{\lambda}$.

If $[g_i] \in \pi(K, v_*)$ for $i = 1, 2$, then $\varphi([g_i]) = h_i \in \text{Cov}(\tilde{K}/K)$, where $h_i(\tilde{v}_*) = \tilde{v}_i = \text{end}\,\tilde{g}_i$ and $\tilde{g}_i$ is the lifting of $g_i$ with origin $\tilde{v}_*$. Let us first evaluate $h_1 \circ h_2$ on the vertex $\tilde{v}_*$:

$$h_1 h_2(\tilde{v}_*) = h_1(\tilde{v}_2).$$

The recipe above instructs us to choose a path $\tilde{\ell}$ from $\tilde{v}_*$ to $\tilde{v}_2$ (but $\tilde{g}_2$ is such a path!), drop to the (closed) path $p\tilde{\ell} = p\tilde{g}_2 = g_2$, and lift $g_2$ to the path $\tilde{\lambda}$ with origin $\tilde{v}_1$; the result is $h_1(\tilde{v}_2) = \text{end}\,\tilde{\lambda}$.

If $[g] = [g_1 \square g_2]$ and $h = \varphi([g])$, let us evaluate $h(\tilde{v}_*)$. Consider the path $\tilde{g}_1 \square \tilde{\lambda}$ in $\tilde{K}$; it is defined, for $\text{end}\,\tilde{g}_1 = \tilde{v}_1 = \text{orig}\,\tilde{\lambda}$. One checks easily that $\tilde{g}_1 \square \tilde{\lambda}$ is a lifting of $g_1 \square g_2$ having origin $\tilde{v}_*$. Therefore, our recipe gives $h(\tilde{v}_*) = \text{end}(\tilde{g}_1 \square \tilde{\lambda}) = \text{end}\,\tilde{\lambda}$.

We have shown that $h_1 \circ h_2$ $(= \varphi([g_1])\varphi([g_2]))$ and $h(= \varphi([g_1 \square g_2]))$ agree on the vertex $\tilde{v}_*$. By Corollary 3.8, $h_1 \circ h_2 = h$; we have proved that $\varphi$ is a homomorphism.

(ii) $\varphi$ is onto.

If $h' \in \text{Cov}(\tilde{K}/K)$, then $h'(\tilde{v}_*) = \tilde{v}'$, where $\tilde{v}'$ is in the fiber over $v_*$ (for $ph = p$). Choose a path $\tilde{\ell}$ in $\tilde{K}$ from $\tilde{v}_*$ to $\tilde{v}'$; then $p\tilde{\ell} = g$ is a closed path in $K$ at $v_*$, i.e., $[g] \in \pi(K, v_*)$. Let $\tilde{\lambda}$ be the lifting of $g$ having origin $\tilde{v}_*$ (uniqueness says $\tilde{\lambda} = \tilde{\ell}$). The definition gives $\varphi([g])) = h$, where $h(\tilde{v}_*) = \text{end}\,\tilde{\lambda} = \text{end}\,\tilde{\ell} = \tilde{v}'$. Therefore $h$ and $h'$ agree on the vertex $\tilde{v}_*$; it follows from Corollary 3.8 that $h = h'$ and $\varphi$ is onto.

(iii) $\ker \varphi = p_{\#}\pi(\tilde{K}, \tilde{v}_*)$.

If $[g] \in \pi(K, v_*)$, then the following are equivalent: $\varphi([g]) = 1$; $\tilde{g}$ is a closed path in $\tilde{K}$ at $\tilde{v}_*$; $[\tilde{g}] \in \pi(\tilde{K}, \tilde{v}_*)$; $[g] \in p_{\#}\pi(\tilde{K}, \tilde{v}_*)$. The proof is completed by the first isomorphism theorem.

Corollary 3.10. *If $p : \tilde{K} \to K$ is a universal covering complex, then*

$$\pi(K, v_*) \cong \text{Cov}(\tilde{K}/K)$$

*for every choice of vertex $v_*$ in $K$.*

Corollary 3.11. *Let $p : \tilde{K} \to K$ be a regular covering complex, and let $p^{-1}(v)$ be the fiber over a vertex $v$ in $K$. Then* $\text{Cov}(\tilde{K}/K)$ *acts transitively on $p^{-1}(v)$.*

Proof. This is precisely what we proved in step (ii) of the proof of Theorem 3.9.

Using groups of covering maps instead of edgepath groups will be our way of avoiding basepoint difficulties. The set-theoretical difficulty we mentioned earlier is treated by defining an appropriate equivalence relation.

Definition. Two covering complexes $p_i : \tilde{K}_i \to K_i$, $i = 1, 2$, are *equivalent* if there are isomorphisms $F$ and $f$ making the following diagram commute:

$$\begin{array}{ccc} \tilde{K}_1 & \xrightarrow{F} & \tilde{K}_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ K_1 & \xrightarrow{f} & K_2 \end{array}$$

The reader may verify that we have, indeed, defined an equivalence relation on all covering complexes.

LEMMA 3.12. *Assume we have a commutative diagram*

$$\begin{array}{ccc} \tilde{K}_1 & \xrightarrow{F} & \tilde{K}_2 \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\ K_1 & \xrightarrow{f} & K_2 \end{array},$$

*where $p_1 : \tilde{K}_1 \to K_1$ is a covering complex and $F$ and $f$ are isomorphisms; then $p_2 : \tilde{K}_2 \to K_2$ is a covering complex.*

PROOF. Straightforward.

DEFINITION. Let $K$ be a complex and let $G$ be a group of automorphisms of $K$ (we say $G$ *acts* on $K$). If $v, v' \in V(K)$, write $v \equiv v'$ if $v' = h(v)$ for some $h \in G$. This defines an equivalence relation on $V(K)$; the equivalence class of a vertex $v$ is denoted $\bar{v}$ and is called the *orbit* of $v$ (more precisely, the *G-orbit* of $v$).

The orbit of $v$ is $\{h(v) : h \in G\}$. Since we have a partition of $V(K)$, we may form the corresponding quotient complex, which we denote $K/G$.

DEFINITION. If $G$ is a group of automorphisms of a complex $K$, then the quotient complex $K/G$ is called the *orbit complex*.

LEMMA 3.13. *Let $q : L \to K$ be a regular covering space, and let $G = \mathrm{Cov}(L/K)$. Then there is an isomorphism $f : K \to L/G$ making the following diagram commute*

$$\begin{array}{ccccc} & & L & & \\ & {\scriptstyle q}\swarrow & & \searrow{\scriptstyle p} & \\ K & & \xrightarrow{\quad f \quad} & & L/G \end{array}$$

*where $p : L \to L/G$ is the natural map. Moreover, $p : L \to L/G$ is a covering complex.*

PROOF. If $v \in V(K)$, there is a vertex $w$ in $L$ with $q(w) = v$; define $f : V(K) \to V(L/G)$ by $f(v) = p(w) = \bar{w}$; in words, $f(v)$ is the fiber over $v$.

(i) $f$ is well defined.

If $q(w_1) = v$, then $w_1$ is in the fiber over $v$. Since $q : L \to K$ is regular, there exists $h \in \text{Cov}(L/K)$ with $h(w) = w_1$, by Corollary 3.11. Hence $w \equiv w_1$ and $\bar{w} = \bar{w}_1$, as desired.

(ii) That $f$ is onto is a trivial verification. If $f(v) = f(v')$, then there are vertices $w, w' \in L$ with $q(w) = v$ and $q(w') = v'$. Since $f(v) = f(v')$, there exists $h \in \text{Cov}(L/K)$ with $hw = w'$. Therefore

$$v = g(w) = qh(w) = q(w') = v'.$$

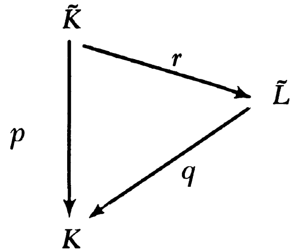
(iii) $f : K \to L/G$ is a map.

Let $s = \{v_0, \cdots, v_t\}$ be a simplex in $K$. Then $g^{-1}(s) = \cup \tilde{s}_i$; choose some fixed $i$, and let $\tilde{s}_i = \{w_0, \cdots, w_t\}$. Thus, $\tilde{s}_i$ is a simplex in $L$ and each $w_j$ is in the fiber over $v_j$. Therefore $f(s) = \{\bar{w}_o, \cdots, \bar{w}_t\}$ is a simplex in $L/G$; indeed $f(s) = p(\tilde{s}_i)$.

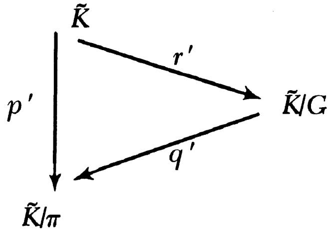
(iv) That $p : L \to L/G$ is a covering complex follows from Lemma 3.12.

The next theorem essentially shows that we may choose orbit complexes as canonical representatives of equivalence classes of covering complexes.

Lemma 3.14. *Consider the commutative diagram of covering complexes*




*where* $p : \tilde{K} \to K$ *and* $r : \tilde{K} \to \tilde{L}$ *are regular. If* $\pi = \text{Cov}(\tilde{K}/K)$ *and* $G = \text{Cov}(\tilde{K}/\tilde{L})$ *then there is a commutative diagram of covering complexes*




*each of which is equivalent to the corresponding original covering complex.*

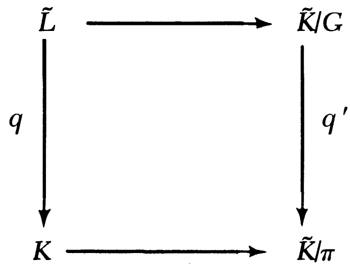
PROOF. By Lemma 3.13, $p' : \tilde{K} \to \tilde{K}/\pi$ is a covering complex equivalent to $p : \tilde{K} \to K$ and $r' : \tilde{K} \to \tilde{K}/G$ is a covering complex equivalent to $r : \tilde{K} \to \tilde{L}$. It remains to define a map $q'$ making the diagram commute and so that $q' : \tilde{K}/G \to \tilde{K}/\pi$ is a covering complex equivalent to $q : \tilde{L} \to K$.

We claim that $G \subset \pi$: if $h \in G = \mathrm{Cov}(\tilde{K}/\tilde{L})$, then $h$ is an automorphism of $\tilde{K}$ satisfying $rh = r$; it follows that $qrh = qr$, i.e., that $ph = p$, and so $h \in \pi = \mathrm{Cov}(\tilde{K}/K)$.

If $\tilde{v} \in V(\tilde{K})$, its $G$-orbit =

$$\{h\tilde{v} : h \in G\} \subset \{h\tilde{v} : h \in \pi\} = \text{the } \pi\text{-orbit of } \tilde{v}.$$

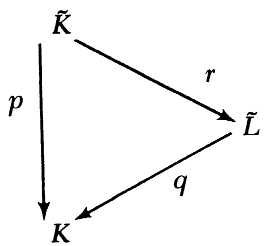
Thus, the function $q' : \tilde{K}/G \to \tilde{K}/\pi$ defined by sending the $G$-orbit of $\tilde{v}$ to the $\pi$-orbit of $\tilde{v}$ is a well defined function, which is easily seen to be a map and which satisfies $q'r' = p'$. Finally, the commutative diagram




in which the horizontal maps are isomorphisms as in Lemma 3.13 shows (Lemma 3.12) that the right side is a covering complex and is equivalent to the left side.

We are ready to give the fundamental theorem of what we call co-Galois theory, as the arrows go the opposite way.

THEOREM 3.15 (Fundamental Theorem of co-Galois Theory). *Let $p : \tilde{K} \to K$ be a regular covering complex; let $\mathcal{C}$ denote the set of all equivalence classes of intermediate covering complexes $q : \tilde{L} \to K$, i.e.,*



*let $\mathcal{S}$ be the set of all intermediate groups $p_* \pi(\tilde{K}, \tilde{v}_*) \subset G \subset \pi(K, v_*)$. Then there is a one-one correspondence between $\mathcal{C}$ and $\mathcal{S}$ implemented by*

$$\Phi : \begin{array}{c} \tilde{L} \\ \big\downarrow \\ K \end{array} \mapsto \operatorname{Cov}(\tilde{K}/\tilde{L}) \text{ and } \Psi : G \mapsto \begin{array}{c} \tilde{K}/G \\ \big\downarrow \\ K \end{array}$$

Proof. In virtue of Theorem 3.9, we identify $\mathcal{S}$ with the set of all subgroups of $\operatorname{Cov}(\tilde{K}/K)$.

First, observe that the functions $\Phi : \mathcal{C} \to \mathcal{S}$ and $\Psi : \mathcal{S} \to \mathcal{C}$ are well defined. If $q : \tilde{L} \to K$ is an intermediate covering complex, then $\operatorname{Cov}(\tilde{K}/\tilde{L})$ is certainly defined; it does lie in $\mathcal{S}$, as we saw in the proof of Lemma 3.14 (in the notation of that proof, $G \subset \pi$). To show that $\Psi : \mathcal{S} \to \mathcal{C}$ is well defined, recall that we saw in Lemma 3.14 that $\tilde{K}/G \to \tilde{K}/\pi$ is equivalent to an intermediate covering complex of $K$; thus, $\Psi(G) \in \mathcal{C}$.

It remains to prove the composites $\Phi\Psi$ and $\Psi\Phi$ are identities. Suppose $G \subset \operatorname{Cov}(\tilde{K}/K)$. The composite $\Phi\Psi$ sends

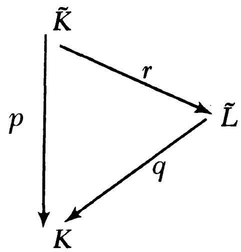
$$G \mapsto \tilde{K}/G \mapsto \operatorname{Cov}(\tilde{K}/(\tilde{K}/G));$$

call this last subgroup $\pi'$. Is $\pi' = G$? If $h \in G$, then $h$ is an automorphism of $\tilde{K}$ making the following diagram commute

$$\begin{array}{ccc} \tilde{K} & \xrightarrow{\;h\;} & \tilde{K} \\ & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle r} & \\ & \tilde{K}/G & \end{array}$$

where $r$ is the natural map (we are merely asserting $\tilde{v} \equiv h(\tilde{v})$ for each vertex $\tilde{v}$ of $\tilde{K}$). Hence $G \subset \pi'$. For the reverse inclusion, let $h \in \pi'$, so that $rh = r$ as in the diagram above. If $\tilde{v} \in V(\tilde{K})$, then $rh(\tilde{v}) = r(\tilde{v}) =$ the $G$-orbit of $\tilde{v}$; therefore $h(\tilde{v}) \equiv \tilde{v}$. By definition of orbit, there is a map $g \in G$ with $g(h\tilde{v}) = \tilde{v}$. Since $G \subset \pi'$ (as we have just proved), $gh \in \pi'$. It follows that $gh = 1$, for it is a covering map having a fixed point. Therefore $h = g^{-1} \in G$ and $\pi' \subset G$, as desired.

Finally, suppose $q : \tilde{L} \to K$ is an intermediate covering complex, i.e., there is a commutative diagram of covering complexes




By Lemma 3.14, the covering complex $r : \tilde{K} \to \tilde{L}$ is equivalent to the covering complex $\tilde{K} \to \tilde{K}/G$, where $G = \mathrm{Cov}(\tilde{K}/\tilde{L})$. Hence $\tilde{L} \to K$ is equivalent to $\tilde{K}/G \to K$, and $\Phi(\tilde{K}/G \to K) = \mathrm{Cov}(\tilde{K}/(\tilde{K}/G))$. But we have seen in the paragraph above that this latter group is, in fact, $G$. Therefore

$$\Psi\Phi(\tilde{K}/G \to K) = \Psi(G) = \tilde{K}/G \to K,$$

as desired.

Corollary 3.16. *An intermediate covering complex* $q : \tilde{L} \to K$ *of* $p : \tilde{K} \to K$ *is regular if and only if* $\mathrm{Cov}(\tilde{K}/\tilde{L}) \lhd \mathrm{Cov}(\tilde{K}/K)$.

The ostensible purpose of this rather long but straightforward exercise has been to convince the reader that one can look at algebraic problems through geometric spectacles, but the "geometric" spectacles are really algebraic after all. Though amusing, this exercise would be rather witless if all it did was replace open sets by simplexes and continuous functions by maps; after all, the point set topology in standard expositions of covering spaces is not difficult (though the proofs are a bit longer than ours).

The real purpose of this article, however, is propaganda for category theory! We now know two very similar theories: co-Galois theory of covering complexes; classical Galois theory of field extensions. Let us mention only two other theories: Galois theory of commutative algebras (initiated by Chase-Harrison-Rosenberg [3]); Galois theory of simple rings (initiated by Artin-Whaples [1]). We suggest that each of these fits into a general framework: a theorem about categories, giving axioms for a Galois correspondence. One such set of axioms is given by Grothendieck [4].

The best finale for this paper would be a proof of classical Galois theory using covering complexes. Such a proof is impossible at this stage. Beginning with a finite Galois extension of a field $k$ by a field $L$, one forms the Galois group $G = \mathrm{Gal}(L/k)$, and then constructs the Eilenberg-Mac Lane complex $K = K(G, 1)$. Analysis of the covering complexes of $K$ cannot yield information about $L/k$ because we have thrown away too much information; all that survives is the Galois group. One possibility is to impose some algebraic structure of the complex $K(G, 1)$; for example, one might try to imitate sheaf theory and endow the fibers with more structure. A more promising approach may be to abandon Eilenberg-Mac Lane complexes in favor of a complex $K'$ which intrinsically reflects information about $L/k$ (recall that the only property of $K(G, 1)$ we have used is that it is a connected complex having edgepath group $G$). Once this is done (and I confess I haven't done it yet), the classical fundamental theorem of Galois theory should look as follows.

Let $L/k$ be a finite Galois extension of fields, and let $\mathcal{A}$ be the category of all $k$-subalgebras of $L$; let $\mathcal{C}$ be the category of all covering complexes of $K$, where $\pi(K, v_*) \cong \mathrm{Gal}(L/k)$ (and where covering complexes are suitably algebraicized). Then there are contravariant functors $\Gamma : \mathcal{A} \to \mathcal{C}$ and $\Delta : \mathcal{C} \to \mathcal{A}$ that are inverse to one another. If $k \subset F \subset L$, then $\Gamma(F) = \tilde{K}/\mathrm{Gal}(L/F)$; if $H \subset G$, then $\Delta(\tilde{K}/H) = L^H = \{a \in L : \sigma(a) = a \text{ for all } \sigma \in H\}$.

Observe that such a theorem gives the classical theorem: there exists an order-reversing bijection $\mathcal{A} \to \mathcal{S}$ (the subgroups of $\mathrm{Gal}(L/k)$), for we may take the composite of $\Gamma : \mathcal{A} \to \mathcal{C}$ with the function $\Phi : \mathcal{C} \to \mathcal{S}$ of Theorem 3.15. The result above would give more, for it would also be defined on morphisms, hence would exhibit every finite Galois group $G = \mathrm{Gal}(L/k)$ as the fundamental group of a suitable complex $K = \tilde{K}/G$. Thus, the techniques of algebraic topology would apply.

Let us illustrate this by giving a too sophisticated proof of a very simple result. The homology (and cohomology) groups of the complex $K$ are well-known:

$$H_n(K) \cong H_n(G, Z),$$

where the right side is the $n$th homology group of the group $G$ with coefficients in the additive group of integers $Z$ made into a trivial $G$-module ($gn = n$ for all $g \in G$ and $n \in Z$). The Hurewicz theorem gives

$$\pi(K, v)/[\pi(K, v), \pi(K, v)] \cong H_1(K),$$

i.e.,

$$G/G' \cong H_1(G, Z),$$

a well-known and easy result.

I wish to thank Leon McCulloh for several stimulating conversations

which led me to simplify the definition of covering complex I had originally used.

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