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STRONGLY RIGID RELATIONS

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ABSTRACT. Vopěnka, Pultr and Hedrlín proved in 1965 that on any set A there exists a binary rigid relation ρ, i.e. a relation such that the identity transformation is the single homomorphism (compatible mapping) of ρ into ρ. We prove the existence of a strongly rigid binary relation on any set with at least three elements. It is a relation such that all homomorphisms of ρ^n into ρ are projections for all $n = 1, 2, \cdots$. We characterize all strongly rigid relations on a set with two elements. Our result can be also stated as follows: There exists a binary (if $|A| > 2$) or ternary (if $|A| = 2$) relation ρ on A such that the trivial universal algebra $\langle A; \phi \rangle$ is equivalent to $\langle A; A_\rho \rangle$ where A_ρ is the set of all operations on A preserving ρ.

1. Let A and I be sets such that $|A| > 1$, $|I| > 0$. Let A^I be the set of all mappings from I to A. Any subset ρ of A^I will be called an *I-relation* or *$|I|$-ary relation* on A. If $|I| = k < \aleph_0$ we will identify A^I with A^k and, in particular, for $|I| = 1, 2, 3$ any I-relation is simply a unary, binary or ternary relation on A. Let ρ_i be I-relations on A_i $(i = 1, 2)$. A mapping $f : A_1 \to A_2$ is a *homomorphism* of ρ_1 into ρ_2 (or $\rho_1\rho_2$ *compatible mapping* [13]) if $g \in \rho_1$ implies $f \circ g \in \rho_2$. A homomorphism $f : A \to A$ of ρ into ρ is called an *endomorphism*. A relation ρ is *rigid* [13] if the identity transformation is the single endomorphism of ρ. The existence of a binary rigid relation on any set is proved in [13].

Given an I-relation ρ on A and $0 < n < \aleph_0$ we define the I-relation ρ^n on A^n as follows: $f \in \rho^n$ if there exist $f_i \in \rho$ $(i = 1, \cdots, n)$ such that $fx = \langle f_1x, \cdots, f_nx \rangle$ for all $x \in I$. For $1 \leqq i \leqq n < \aleph_0$ define the projections [4] (called sometimes *selective* or *trivial operations*) $e_i^n : A^n \to A$ by $e_i^n x_1 \cdots x_n = x_i$ for all $x_1, \cdots, x_n \in A$. Finally set $J = \{e_i^n \mid 1 \leqq i \leqq n < \aleph_0\}$.

DEFINITION. *Let ρ be an I-relation on A. The set of all homomorphisms of ρ^n into ρ $(1 \leqq n < \aleph_0)$ will be denoted by A_ρ. The relation ρ will be called a strongly rigid relation if $A_\rho = J$.*

The sets A_ρ were introduced in [3] for $|I| \leqq |A| < \aleph_0$ and used in [1], [2], [14] and [7] – [12]. Obviously $f \in A_\rho$ if and only if ρ is a subalgebra of $\langle A^I, \{f\} \rangle$. A relation ρ is strongly rigid if and

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only if for any $n = 1, 2, \cdots$, any homomorphism f of ρ^n into ρ is a projection, i.e. $f = e_i^n$ for a suitable $1 \leqq i \leqq n$. If ρ is strongly rigid, then for $n = 1$ any homomorphism $f : \rho \to \rho$ is the identity and ρ is rigid. The converse is not true; and, in particular, the rigid relation of [13] is not strongly rigid.

EXAMPLE. *Let ρ be a rigid binary relation with a minimal element* 0 *and a maximal element* 1 *(i.e. $\langle a, 0 \rangle \in \rho$ and $\langle 1, a \rangle \in \rho$ for no $a \in A$). Then ρ is not strongly rigid.*

Indeed it is sufficient to define $f : A^2 \to A$ as follows: Let $f01 = 1$ and $fx_1x_2 = x_1$ for all $\langle x_1, x_2 \rangle \in A^2 \setminus \{\langle 0, 1 \rangle\}$. If $\langle x_i', x_i'' \rangle \in \rho$ $(i = 1, 2)$, then $x_2' \neq 1$ and $x_1'' \neq 0$ so that $\langle fx_1'x_2', fx_1''x_2'' \rangle = \langle x_1', x_1'' \rangle \in \rho$. Hence $f \in A_\rho$ and this shows that ρ is not strongly rigid.

Universal algebra provides a motivation for the study of strongly rigid relations.

Let $A = \langle A; F \rangle$ be a universal algebra [4]. It is proved in [12] that there exist a set I and an I-relation ρ on A such that $A \simeq \langle A; A_\rho \rangle$. The *relational degree* of A is the least cardinality of such a set I. In this paper we will in fact prove that the relational degree of $\langle A; \emptyset \rangle$ is 2 if $|A| > 2$ and 3 if $|A| = 2$.

In §2, using Post's results, we characterize all strongly rigid relations on a set with two elements. In §§3 — 4 using the rigid relation from [13] we prove the existence of a strongly rigid relation on any set with at least three elements. The problem to characterize all strongly rigid relations on a set with more than two elements remains open. It seems that this problem is more difficult than the following opposite problem which was solved in [11]: Characterize all relations ρ on I such that A_ρ is the set of all operations on A.

2. Let A be a set with two elements which for convenience will be denoted by 0 and 1. The operations on A are simply Boolean functions. We will need the following operations:

(1) The zero operations (constants) **0** and **1**.

(2) The unary operation $\neg$ (*negation*) defined by $\neg 0 = 1$ and $\neg 1 = 0$.

(3) The binary operation $\vee$ (*disjunction* or *alternative*) defined by $0 \vee 0 = 0$ and $0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1$.

(4) The binary operation $\wedge$ (*conjunction*) defined by $0 \wedge 0 = 1 \wedge 0 = 0 \wedge 1 = 0$ and $1 \wedge 1 = 1$.

(5) The binary operation $\dotplus$ (*the sum* mod 2) defined by $0 \dotplus 0 = 1 \dotplus 1 = 0$ and $0 \dotplus 1 = 1 \dotplus 0 = 1$.

(6) The ternary operation l defined by $lx_1x_2x_3 = x_1 \dotplus x_2 \dotplus x_3$ for any $x_1, x_2, x_3 \in A$.

(7) The ternary operation maj (*majority function*) defined by $\operatorname{maj} x_1x_2x_3 = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3)$ for all $x_1, x_2, x_3 \in A$. Further let 0_I and 1_I be the mappings $I \to \{0\}$ and $I \to \{1\}$, respectively.

If $f_1, \cdots, f_n \in A^I$ and $g : A^n \to A$ then $h = gf_1 \cdots f_n$ is the element of A^I defined by $hi = g(f_1 i) \cdots (f_n i)$ for each $i \in I$. Now we can state

THEOREM 1. *An I-relation ρ on $\{0, 1\}$ is strongly rigid if and only if $0_I \notin \rho$, $1_I \notin \rho$ and there exist $f_i \in \rho$ $(i = 1, \cdots, 11)$ such that $\neg f_1 \notin \rho$, $f_2 \vee f_3 \notin \rho$, $f_4 \wedge f_5 \notin \rho$, $f_6 + f_7 + f_8 \notin \rho$ and $\operatorname{maj} f_9 f_{10} f_{11} \notin \rho$.*

PROOF. Necessity follows from the fact that none of $0, 1, \neg, \vee, \wedge, l$, maj belongs to A_ρ if ρ is strongly rigid.

Sufficiency. Assume that $A_\rho \supset J$. It is proved in [6] that A_ρ then contains at least one of R_8, R_6, R_4, S_2, P_2, L_4, and D_2. But these sets contain functions $0, 1, \neg, \vee, \wedge, l$ and maj so that at least one of the conditions of the theorem is not met. A direct check shows:

COROLLARY 1. *There is no strongly rigid binary relation on* $\{0, 1\}$.

Let $V_i^3 = \{\langle a_1, a_2, a_3\rangle \in A^3 \mid a_1 + a_2 + a_3 = i\}$ $(i = 1, 2)$. A more detailed analysis using permutations of places and transition to dual relations, can be used to show

COROLLARY 2. *The ternary strongly rigid relations on $\{0, 1\}$ have the form $V_1^3 \cup \lambda$ $(\lambda \subset V_2^3)$ or $V_2^3 \cup \mu$ $(\mu \subset V_1^3)$.*

Let $s(n)$ be the number of strongly rigid n-ary relations on $\{0, 1\}$. We know that $s(2) = 0$, $s(3) = 14$. We will not investigate $s(n)$ here and we will give only the following crude lower bound.

COROLLARY 3. *For every n, $S(n) \geqq 2^{2^n - 7}$.*

PROOF. If ρ has the property (1) $\langle 0, \cdots, 0, 1\rangle \in \rho$, $\langle 0, \cdots, 0, 1, 0\rangle \in \rho$, $\langle 0, \cdots, 0, 1, 0, 0\rangle \in \rho$ and (2) $\langle 0, \cdots, 0\rangle \notin \rho$, $\langle 0, \cdots, 0, 1, 1, 1\rangle \notin \rho$, $\langle 1, \cdots, 1, 0\rangle \notin \rho$, and $\langle 1, \cdots, 1\rangle \notin \rho$, then it can be easily checked that ρ satisfies the conditions of the proposition. Obviously we may include in ρ any of the remaining $2^n - 7$ n-tuples.

3. The example given in §1 shows that a strongly rigid relation cannot have a minimal and a maximal element. Since the rigid relation given in [13] has both elements we will correct the situation by adjoining two elements and extending the relation so that the new relation does not possess maximal or minimal elements.

Let D be a nonempty set with a rigid binary relation $\sqsubset$. Assume that $\langle D; \sqsubset \rangle$ has no closed paths of length 2 or 3, i.e. for no x, y, and z in D we have $x \sqsubset y \sqsubset x$ or $x \sqsubset y \sqsubset z \sqsubset x$. In order to get a strongly rigid relation we will add to D two elements, which for convenience will be denoted by 0 and 1. Let $B = \{0, 1\}$ and let $B \cap D = \emptyset$. On the set $A = B \cup D$ define a binary relation $\prec$ as follows:

(1) set $0 \prec d \prec 1$ for all $d \in D$,
(2) set $0 \prec 1 \prec 0$, and
(3) for any $x, y \in D$ set $x \prec y$ if and only if $x \sqsubset y$.

Thus $\prec$ is an extension of $\sqsubset$ with 0 below and 1 above all elements of D and a full edge between 0 and 1. We will prove that under certain restrictions $\prec$ is a strongly rigid relation. Let $f\colon A^n \to A$ be a homomorphism of $\prec^n$ into $\prec$. First we will prove that there exists $1 \leqq p \leqq n$ such that $x_p \in B$ implies $fx_1 \cdots x_n = x_p$ and $x_p \in D$ implies $fx_1 \cdots x_n \in D$. We will need the following lemmas.

Lemma 1. *The relation $\prec$ is rigid.*

Proof. Let $h : A \to A$ be an endomorphism of $\prec$. From the definition we get $h0 \prec h1 \prec h0$ and $h0 \prec hd \prec h1$ for all $d \in D$. Since $\sqsubset$ has no cycles of length 2 we get $h0 = 0$, $h1 = 1$ and $hd \in D$. Since $\prec$ agrees with $\sqsubset$ on D we see that h restricted to D is an endomorphism of $\sqsubset$, so that $hd = d$ for any $d \in D$.

Elements of A^n will be denoted by $\tilde{x} = \langle x_1, \cdots, x_n \rangle$ and for $a \in A$ the element $\langle a, \cdots, a \rangle \in A^n$ will be denoted by $\tilde{a}$.

Lemma 2. *We have $f\tilde{a} = a$ for any $a \in A$.*

Proof. Let $ha = f\tilde{a}$ for any $a \in A$. It is easily verified that h is an endomorphism of $\prec$ and therefore by Lemma 1 the identity.

For any $\tilde{x} \in B^n$ set $\neg \tilde{x} = \langle \neg x_1, \cdots, \neg x_n \rangle$. A function $h : B^n \to B$ will be called a Boolean function. We will say that h is *self-dual* if $\neg h\tilde{x} = h \neg \tilde{x}$ for any $x \in B^n$.

Let g be the restriction of f to B^n.

Lemma 3. *The function g is a Boolean self-dual function.*

Proof. Let $\tilde{x} \in B^n$. Then $\neg \tilde{x} \in B^n$ and $\tilde{x} \prec \neg \tilde{x} \prec \tilde{x}$ (here and in the sequel we will write $\prec$ instead of $\prec^n$ whenever possible). Hence $f\tilde{x} \prec f \neg \tilde{x} \prec f\tilde{x}$ and from the definition of $\prec$ we can conclude $\{f\tilde{x}, f \neg \tilde{x}\} = B$ and $\neg f\tilde{x} = f \neg \tilde{x}$.

If $\tilde{x}, \tilde{y} \in B^n$ and $x_i \leqq y_i$ for all $i = 1, 2, \cdots, n$ (i.e. $\langle x_i, y_i \rangle \in \{\langle 0,0 \rangle, \langle 0,1 \rangle, \langle 1,1 \rangle\}$) we will write $\tilde{x} \leqq \tilde{y}$. Boolean function is *monotonic* if $\tilde{x} \leqq \tilde{y}$ implies $f\tilde{x} \leqq f\tilde{y}$.

Lemma 4. *The function g is monotonic.*

PROOF. Let $\tilde{x}, \tilde{y} \in B^n$, $\tilde{x} \leqq \tilde{y}$ and suppose $f\tilde{x} = 1$, $f\tilde{y} = 0$. Let $z_i = \neg x_i$ if $x_i = y_i$ and $z_i \in D$ if $\langle x_i, y_i \rangle = \langle 0, 1 \rangle$. Then clearly $\tilde{x} \prec \tilde{z} \prec \tilde{y}$ which leads to the contradiction $1 \prec f\tilde{z} \prec 0$.

Let $V_i = \{\tilde{x} \in B^n \mid x_1 + \cdots + x_n = i\}$ $(i = 0, \cdots, n)$.

LEMMA 5. *There exists* $\tilde{z} \in V_{n-1}$ *such that* $g\tilde{z} = 0$.

PROOF. The function g being self-dual is not constant. Using this and monotonicity of G it is not difficult to prove that there exists $0 \leqq i < n$, $\tilde{x} \in V_i$, $\tilde{y} \in V_{i+1}$, $\tilde{x} \leqq \tilde{y}$ such that $f\tilde{x} = 0$ and $f\tilde{y} = 1$ (see e.g. [5]). Then there exists $1 \leqq p \leqq n$ such that $x_p = 0$, $y_p = 1$ and $x_m = y_m$ for all $1 \leqq m \leqq n$, $m \neq p$. Choose $d \in D$ and set (1) $u_p = d$, $v_p = 1$, and $w_p = 0$, (2) $u_m = 1$, $v_m = 0$, $w_m = d$ if $x_m = 0$, and (3) $u_m = 0$, $v_m = d$, $w_m = 1$ if $x_m = 1$. It is easy to check $\tilde{x} \prec \tilde{u} \prec \tilde{y}$. This gives $0 \prec f\tilde{u} \prec 1$ so that $f\tilde{u} \in D$. Direct check shows that $\tilde{u} \prec \tilde{v} \prec \tilde{w} \prec \tilde{u}$ so that $f\tilde{u} \prec f\tilde{v} \prec f\tilde{w} \prec f\tilde{u}$. We have no closed path of length 3 in D and therefore $f\tilde{v} = 1$ and $f\tilde{w} = 0$. Let $z_p = 0$ and $z_m = 1$ for all $1 \leqq m \leqq n$, $m \neq p$. Then $\tilde{z} \in V_{n-1}$ and $\tilde{v} \prec \tilde{z}$. Hence $1 = f\tilde{v} \prec f\tilde{z}$ and from $\tilde{z} \in V_{n-1} \subset B^n$ we conclude $f\tilde{z} \in B$ and $f\tilde{z} = 0$.

LEMMA 6. *There exists* $1 \leqq p \leqq n$ *such that* $g = e_p{}^n$.

PROOF. By Lemma 5 there exists $\tilde{z} \in V_{n-1}$ with $g\tilde{z} = 0$. If $z_p = 0$ then for any $\tilde{y} \in B^n$ with $y_p = 0$ we get $\tilde{y} \leqq \tilde{z}$ and applying Lemma 4 we obtain $g\tilde{y} \leqq g\tilde{z} = 0$, i.e. $g\tilde{y} = 0$. For any $\tilde{t} \in B^n$ with $t_p = 1$, this and self-duality of g yields $\neg g\tilde{t} = g \neg \tilde{t} = 0$. Thus we get the required result.

LEMMA 7. *If* $x_p \in B$ *then* $f\tilde{x} = x_p$.

PROOF. Let $\tilde{x} \in A^n$ and $x_p \in B$. For each $i = 1, \cdots, n$ set $y_i = z_i = \neg x_i$ if $x_i \in B$ and $y_i = 0$, $z_i = 1$ if $x_i \in D$. Immediate check shows $\tilde{y} \prec \tilde{x} \prec \tilde{z}$ so that $f\tilde{y} \prec f\tilde{x} \prec f\tilde{z}$. Since $\tilde{y}, \tilde{z} \in B$, applying Lemma 6 we get $f\tilde{y} = y_p = \neg x_p$ and $f\tilde{z} = z_p = \neg x_p$. Thus $\neg x_p \prec f\tilde{x} \prec \neg x_p$ and this implies $f\tilde{x} \in B$ and $f\tilde{x} = x_p$.

LEMMA 8. *If* $x_p \in D$ *then* $f\tilde{x} \in D$.

PROOF. Let $\tilde{x} \in A^n$ and $x_p \in D$. For each $i = 1, \cdots, n$ set (1) $y_i = 0$, $z_i = 1$ if $x_i \in D$ and (2) $y_i = z_i = \neg x_i$ if $x_i \in B$. Then $\tilde{y} \prec \tilde{x} \prec \tilde{z}$ so that $f\tilde{y} \prec f\tilde{x}$. $f\tilde{z}$. Since $x_p \in D$ we have $y_p = 0$ and $z_p = 1$. Applying Lemma 7 we get $f\tilde{y} = y_p = 0$ and $f\tilde{z} = z_p = 1$. Thus $0 \prec f\tilde{x} \prec 1$ and this yields $f\tilde{x} \in D$.

4. For strong rigidity we will make further assumptions on $\sqsubset$. The first type will be used for finite sets D.

We say that a binary relation $\lhd$ on S is a *spanning forest* if given any $u, v \in D$ there exists at most one finite sequence $y_0, \cdots, y_m$ in D such that $u = y_0$, $v = y_m$ and $y_i \lhd y_{i+1}$ or $y_{i+1} \lhd y_i$ for each $i = 0, 1, \cdots, m-1$, and $\{x \in D \mid x \lhd d \text{ or } d \lhd x\} \neq \emptyset$ for each $d \in D$.

Theorem 2. *Let $\sqsubset$ be a rigid relation on D. Let $\lhd$ be a spanning forest which is a subrelation of the relation $\sqsubset$ such that the identity transformation is the only homomorphism of $\lhd$ into $\sqsubset$.*

Then the relation $\prec$ on $D \cup \{0, 1\}$ defined in §3 is strongly rigid.

Proof. Let f be a homomorphism from $\prec^n$ into $\prec$. Since we can permute the variables, using Lemmas 7 and 8 we can assume without loss of generality that (1) $f\tilde{x} = x_1$ for all $\tilde{x} \in A^n$, $x_1 \in B$ and (2) $f\tilde{x} \in D$ for all $\tilde{x} \in A^n$, $x_1 \in D$. Fix $\tilde{x} \in A^n$ with $x_1 \in D$. Let G be the connected component of the graph $\langle D; \lhd \rangle$ containing x_1 and let $t \in G$. We will define $\phi t \in A^n$ in the following way: (1) Let $\phi x_1 = \tilde{x}$. (2) Let $y_0, \cdots, y_m$ be the unique finite sequence in D such that $x_1 = y_0$, $t = y_m$ and $y_i \lhd y_{i+1}$ or $y_{i+1} \lhd y_i$ for every $i = 0, \cdots, m-1$. Assume that $\tilde{u} = \phi y_{m-1}$ has been already defined. We set $z_1 = y_m$ and for $i = 2, \cdots, n$ we set (1) $z_i = \neg u_i$ if $u_i \in B$, (2) $z_i = 1$ if $u_i \in D$ and $y_{m-1} \lhd t$, and (3) $z_i = 0$ if $u_i \in D$ and $t \lhd y_{m-1}$. This defines a mapping $\phi : G \to A^n$. Let $hx = f\phi x$ for $x \in G$ and $hx = x$ for $x \in D \backslash G$. Hence h is a mapping from D into A. We claim that h maps D into D. This is obvious for any $x \in D\backslash G$. Let $x \in G$ and $\tilde{u} = \phi x$. We can see from the definition that $u_1 \in D$ so that by Lemma 8 we get $hx = f\tilde{u} \in D$. Since $\lhd$ is a subrelation of $\sqsubset$, it is easy to verify from the definition of ϕ that for $u, v \in G$, $u \lhd v$ implies $\phi u \prec \phi v$. Hence h is a homomorphism of $\lhd$ into $\prec$. By assumption h is the identity transformation. In particular, $f\tilde{x} = f\phi x_1 = hx_1 = x_1$ and this completes the proof.

Corollary 4. *A strongly rigid binary relation exists on any finite set with at least* 3 *elements.*

Proof. Let $D = \{d_0, \cdots, d_k\}$ and let $d_i \sqsubset d_j$ if and only if $i < k$ and $j = i + 1$. It is sufficient to set $\lhd$ equal to $\sqsubset$ and verify all assumptions of the Theorem 2.

For infinite sets we will have the following sufficient condition.

Theorem 3. *Let $\sqsubset$ be a rigid relation on D. Suppose that there exists $E \subseteq D$ satisfying*

(1) *To any* $x \in E$ *there exist* $a_1, \cdots, a_4 \in D$ *and* $y \in D$ *such that*

$$x \sqsubset a_1 \sqsubset \cdots \sqsubset a_4 \sqsubset y,$$

and

$$z \sqsubset b_1 \sqsubset \cdots \sqsubset b_4 \sqsubset y \Rightarrow z = x$$

for any $z, b_1, \cdots, b_4 \in D$.

(2) *If* $y_1 \in D \backslash E$, $y_2 \in D$, *and for any* $e \in E$

$$e \sqsubset y_1 \Rightarrow e \sqsubset y_2, \qquad y_1 \sqsubset e \Rightarrow y_2 \sqsubset e,$$

then $y_1 = y_2$.

Then the relation $\prec$ *on* $D \cup \{0, 1\}$ *defined in* §3 *is strongly rigid.*

Proof. Let $\tilde{x} \in A^n$ and let $x_1 \in E$. By assumption there exist $a_1, \cdots, a_4, y$ in D such that $x_1 \sqsubset a_1 \sqsubset \cdots \sqsubset a_4 \sqsubset y$. Fix an element $d \in D$ and define $\tilde{t}^{(i)} \in A^n$ $(i = 1, \cdots, 4)$ as follows: Let $t_1{}^{(i)} = a_i$. For $j = 2, \cdots, n$ set (1) $t_j{}^{(i)} = 1/2[1 - (-1)^i]$ if $x_j \in D \cup \{0\}$, and (2) $t_j{}^{(1)} = 0$, $t_j{}^{(2)} = d$, $t_j{}^{(3)} = 1$, and $t_j{}^{(4)} = 0$ if $x_j = 1$. It is easy to check that $\tilde{x} \prec \tilde{t}^{(1)} \prec \cdots \prec \tilde{t}^{(4)} \prec \tilde{y}$. Hence $f\tilde{x} \prec f\tilde{t}^{(1)} \prec \cdots \prec f\tilde{t}^{(4)} \prec f\tilde{y}$. Since by Lemma 2 we have $f\tilde{y} = y$, the assumption yields the required equality $f\tilde{x} = x_1$.

Let $\tilde{x} \in A^n$ and let $x_1 \in D \backslash E$. Let $y_1 = x_1$ and $y_2 = f\tilde{x}$. Assume $e \in E$ and $e \sqsubset y_1$. Let $t_1 = e$ and for $i = 2, \cdots, n$ set (1) $t_i = 0$ if $x_i \in D$ and (2) $t_i = \neg x_i$ if $x_i \in B$. Then $\tilde{t} \prec \tilde{x}$ and therefore $f\tilde{t} \prec f\tilde{x}$. But in the first part of the proof we have shown that $t_1 = e \in E$ implies $f\tilde{t} = t_1 = e$. Thus $e \prec f\tilde{x}$. We know from Lemma 7 that $f\tilde{x} \in D$ and therefore $e \sqsubset f\tilde{x}$, i.e. $e \sqsubset y_2$. A similar argument shows $y_1 \sqsubset e \Rightarrow y_2 \sqsubset e$. Hence using the second part of the assumptions we get $y_1 = y_2$, i.e. $f\tilde{x} = x_1$.

We conclude with

Theorem 4. *A strongly rigid binary relation exists on any set with at least three elements.*

Proof. For finite sets the statement was proved in Corollary 4. Let A be infinite. We can choose two elements $0', 1' \in A$ and set $D = A \backslash \{0', 1'\}$. The existence of a rigid relation on D was proved in [13]. Using the denotation of [13] we set $E = D \backslash \{1, \omega_\xi, \omega_\xi + 1\}$. For the condition 1 of Theorem 3 it suffices to set $\alpha_i = x + i$ $(i = 1, \cdots, 4)$ and $y = x + 5$.

We will check the condition (2). Let $y_1 = 1$. Choosing $e = 0$, from $0 \sqsubset 1$ we get $0 \sqsubset y_2$. Similarly, for $e = 2$, from $1 \sqsubset 2$ we get $y_2 \sqsubset 2$. Therefore $y_2 = 1 = y_1$. Consider now $y_1 = \omega_\xi$. Choosing

$e = \omega_\xi + 1$ we get $y_2 \sqsubset \omega_\xi + 1$. If $\omega_\xi \in D_1$, then from $\omega_0 \sqsubset \omega_\xi$ we get $\omega_0 \sqsubset y_2$ and this together with $y_2 \sqsubset \omega_\xi + 1$ yields $y_2 = \omega_\xi = y_1$. If $\omega_\xi \in D_0$, then there exist an increasing sequence $\{\alpha_n\}$ such that $\alpha_n \sqsubset \omega_\xi$ and $\sup \alpha_n = \omega_\xi$. Then again $\alpha_n \sqsubset y_2$ for all n and therefore $y_2 = \omega_\xi = y_1$. Finally, let $y_1 = \omega_\xi + 1$. Then from $\omega_\xi \sqsubset \omega_\xi + 1$ we get $\omega_\xi \sqsubset y_2$ and therefore $y_2 = \omega_\xi + 1 = y_1$. Thus (2) holds and the relation $\prec$ is a strongly rigid relation on A.

CONCLUDING REMARKS. (1) The referee has pointed out that it is natural to ask the following question. Let α be a cardinal. A relation ρ is α-strongly rigid if for any set J with $|J| < \alpha$ the only homomorphisms $\rho^J \to \rho$ are the projections. Here we have proved only the existence of an $\aleph_0$-strongly rigid relation.

(2) The number of strongly rigid n-ary relations on a set is also of interest. We conjecture that for a finite set with k elements the number $s(n)$ of n-ary strongly rigid relations satisfies

$$\lim_{n \to \infty} 2^{-k^n} s(n) = 1,$$

i.e. for a big n almost all n-ary relations are strongly rigid.

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