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FUNCTIONS ANALOGOUS TO COMPLETELY CONVEX FUNCTIONS[1]

S. P. PETHE AND A. SHARMA

ABSTRACT. We consider the problem of expanding a function $f \in C^{\infty}[0, 1]$ in a $L_{3,0,0}$ series, which is an analogue of Lidstone series. To this end we are led to consider the class $W_{3,0,0}$ of functions and the class of minimal $W_{3,0,0}$ functions. Following Widder's method for completely convex functions, we show that a function f has an absolutely convergent $L_{3,0,0}$ series expansion if and only if $f = g - h$, where g, h belong to the class of minimal $W_{3,0,0}$ functions. The existence of five more similar results is pointed out.

1. **Introduction.** Recently Leeming and Sharma [3] have given a generalization of completely convex functions by considering an analogue of Lidstone series. In 1942, Widder [9] showed the close connection of completely convex functions with Lidstone series, similar to the one that exists between the completely monotonic functions of Bernstein and the Taylor series. For details we refer the reader to [10]. In 1942 many deep studies were made in connection with "the influence of the sign of the derivatives of a function on its analytic character", which seem to have deep connections with Widder's class of completely convex functions. In particular Boas and Pólya [1] showed, roughly speaking, that if $\{n_k\} \uparrow$ and $\{q_k\}$ are two sequences of nonnegative integers and if

$$(1.1) \qquad f^{(n_k)}(x) f^{(n_k + 2q_k)}(x) \leqq 0 \quad \text{on a given interval } I,$$

then f must coincide on I with an entire function of order 1 and finite type. Although this result is of a very general nature, and although it extends one of the results of Widder considerably, the other interesting results of Widder on completely convex functions went almost unnoticed.

Our object here is to follow the point of view of Widder [9] and to consider a two point interpolation problem analogous to Lidstone interpolation. We shall use the notation of an incidence matrix for an

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interpolation problem (Schoenberg [7]). We shall consider the incidence matrix

$$(1.2)\qquad \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix}$$

which is obtained by repeating $M_1 = \left(\begin{smallmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right)$ infinitely many times. I: the nodes of interpolation are 0 and 1, the matrix corresponds to the problem of finding a function $f \in C^{\infty}[0, 1]$ such that

$$f^{(3k)}(1), \quad f^{(3k+1)}(0), \quad f^{(3k+2)}(0), \qquad k = 0, 1, 2, \cdots,$$

are prescribed. An expansion of $f(x)$ using only these derivative: shall be called $L_{3,0,0}$ series. The reason for this notation will become clear later. Here L stands for Lidstone who considered the interpolation problem given by the incidence matrix

$$(1.3)\qquad \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 1 & 0 & \cdots \end{pmatrix}.$$

Leeming and Sharma [3] considered a class of functions which they called 'completely W_p-convex' which reduces to completely convex functions for $p = 2$. For $p = 3$, their results are related to the incidence matrix

$$(1.4)\qquad \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & \cdots \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix}.$$

Our results complement the results of Leeming and Sharma. It turn out that for $p = 3$, there are essentially six different classes of function which are analogues of completely convex functions. We give later a table of these classes and the corresponding incidence matrices. Fo $p > 3$, it follows similarly that there are at least $p(p + 1)/2$ such classes of functions.

In §2 we give some properties of generalized trigonometric functions which we shall need. §3 deals with a representation theorem and the definition of $L_{3,0,0}$ series. §§4 and 5 are devoted to some propertie and estimates of the fundamental polynomials which occur in the $L_{3,0,0}$ series expansion. In §6 we define the class of $W_{3,0,0}$ function and their properties. This leads in §7 to the class of minimal $W_{3,0,}$ functions and the main result of this paper in Theorem 6.5, which gives a necessary and sufficient condition for a function to have an absolutely convergent $L_{3,0,0}$ representation. We also give three table:

Table I gives the six classes of functions and the six interpolation matrices related to them. In Table II, we give the comparative study of completely monotonic, completely convex, $c - c^*$ and $c - c^{**}$ functions with respect to the interpolation matrices associated with them. Table III deals with the generating functions of the fundamental polynomials and the kernels associated with the six interpolation problems in Table I. In the end we give a schematic diagram for writing down the six extremal functions for the six classes of functions. If the extremal function is $M_{3,0}(\lambda_{01}^{(3)}x)$ the corresponding class of functions is termed $W_{3,0,0}$ and the corresponding series representation is termed $L_{3,0,0}$. Similarly one can see the rationale for the classes $W_{3,1,0}$, $W_{3,1,1}$, $W_{3,2,0}$, $W_{3,2,1}$ and $W_{3,2,2}$ respectively.

Preliminaries. We shall need the generalized sine and hyperbolic functions of order p. For a given integer $p \geqq 0$, we set

$$(2.1)\qquad \begin{aligned} M_{p,j}(t) &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{np+j}}{(np+j)!}, && j = 0, 1, \cdots, p-1, \\ N_{p,j}(t) &= \sum_{n=0}^{\infty} \frac{t^{np+j}}{(np+j)!}, && j = 0, 1, \cdots, p-1. \end{aligned}$$

Thus $M_{1,0}(t) = e^{-t}$, $M_{2,1}(t) = \sin t$, $M_{2,0}(t) = \cos t$, $N_{1,0}(t) = e^t$, $N_{2,1}(t) = \sinh t$, $N_{2,0}(t) = \cosh t$. Thus

$$(2.2)\qquad \begin{aligned} M_{p,j}^{(r)}(t) &= M_{p,j-r}(t), && 0 \leqq r \leqq j \\ &= -M_{p,p+j-r}(t), && j < r \leqq p. \end{aligned}$$

Mikusinski [4] gave the following addition formula:

$$(2.3)\qquad M_{p,j}(x+y) = \sum_{k=0}^{j} M_{p,k}(x) M_{p,j-k}(y) - \sum_{k=j+1}^{p-1} M_{p,k}(x) M_{p,p+j-k}(y).$$

Also

$$N_{p,j}(t) = \omega^{j/2} M_{p,j}(t\omega^{-1/2}), \qquad \omega = e^{2\pi i/p}.$$

If we arrange the real zeros ($\neq 0$) of $M_{p,j}(t)$ in increasing order of magnitude, we denote the ith zero of $M_{p,j}(t)$ by $\lambda_{j,i}^{(p)}$. Thus

$$0 < \lambda_{j,1}^{(p)} < \lambda_{j,2}^{(p)} < \cdots \qquad (j = 0, 1, \cdots, p-1).$$

Mikusinski [4] proved that the zeros of $M_{p,j}(t)$ are simple and if $0 \leqq j < k < p$, then no zeros of $M_{p,j}(t)$ and $M_{p,k}(t)$ coincide. Further if $0 < \lambda_{j,m}^{(p)} < \lambda_{j,m+1}^{(p)}$ are two consecutive zeros of $M_{p,j}(t)$

there exists exactly one zero of $M_{p,k}(t)$ between them. Besides this interlacing property of the real zeros of $M_{p,j}(t)$, we shall also need the following inequalities, due to Mikusinski [4]:

$$\{(p+1)!/j!\}^{1/p} < \lambda_{j,1}^{(p)} < \{2\cdot(p+j)!/j!\}^{1/p}. \tag{2.4}$$

Also it is easy to verify that

$$\lambda_{j,k}^{(p)} < \lambda_{p-1,k}^{(p)}, \qquad (j = 0, 1, \cdots, p-2; k = 1, 2, \cdots$$

$$(-1)^k M_{p,j}(\lambda_{p-1,k}^{(p)}) > 0, \qquad (j = 0, 1, \cdots, p-2; k = 1, 2, \cdots$$

Lastly we mention the biorthogonal property [3]:

$$\begin{aligned}\int_0^1 & M_{p,p-1}(\lambda_{p-1,k}^{(p)}x)M_{p,p-1}(\lambda_{p-1,j}^{(p)}(1-x))\,dx \\ &= 0, \qquad\qquad j \neq k, \\ &= -\frac{1}{p}M_{p,p-2}(\lambda_{p-1,k}^{(p)}), \qquad j = k.\end{aligned}$$

3. $L_{3,0,0}$ **series.** We shall be concerned throughout with the inter polation problem (1.2) which is the iteration of the incidence matri $M_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0\end{pmatrix}$. Now there are five more such matrices:

$$M_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0\end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0\end{pmatrix},$$

$$M_5 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0\end{pmatrix}, \quad M_6 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0\end{pmatrix}.$$

Each of these matrices when iterated infinitely often will lead to matrix similar to (1.2). The list of these interpolation problems, th generating functions of their fundamental polynomials and the kerne are given in Table III. There we denote the fundamental polynomia of the six interpolation problems at the nodes 0 and 1 by $A^{(i)}(x)$ an $B^{(i)}(x)$ with suitable suffixes. We shall prove the results only for (1.2 We shall however write, for the sake of simplicity, $B_{3k}(x)$, $A_{3k+1}(x)$ an $A_{3k+2}(x)$ for $B_{3k}^{(1)}(x)$, $A_{3k+1}^{(1)}(x)$ and $A_{3k+2}^{(1)}(x)$ respectively.

THEOREM 3.1. *The following representation holds for every enti function $f(z)$ of exponential type $\tau < \lambda_{0,1}^{(3)}$ where $\lambda_{0,1}^{(3)}$ is the first re zero ($\neq 0$) of $M_{3,1}(t)$:*

$$f(z) = \sum_{n=0}^{\infty} \{f^{(3n)}(1)B_{3n}(z) + f^{(3n+1)}(0)A_{3n+1}(z) + f^{(2n+2)}(0)A_{3n+2}(z)\}, \tag{3.1}$$

where $\{B_{3n}(z)\}_0^\infty$, $\{A_{3n+1}(z)\}_0^\infty$ *and* $\{A_{3n+2}(z)\}_0^\infty$ *are the polynomials defined by the generating functions*:

$$\sum_{k=0}^{\infty} t^{3k}B_{3k}(z) = \frac{N_{3,0}(tz)}{N_{3,0}(t)} = \psi_0(z, t^3), \tag{3.2}$$

$$\sum_{k=0}^{\infty} t^{3k+1}A_{3k+1}(z) = N_{3,1}(tz) - \frac{N_{3,1}(t)N_{3,0}(tz)}{N_{3,0}(t)} = t\psi_1(z, t^3), \tag{3.3}$$

and

$$\sum_{k=0}^{\infty} t^{3k+2}A_{3k+2}(z) = N_{3,2}(tz) - \frac{N_{3,2}(t)N_{3,0}(tz)}{N_{3,0}(t)} = t^2\psi_2(z, t^3). \tag{3.4}$$

The series on the right of (3.1) *converges to* $f(z)$ *for all* z *and the convergence is uniform on all compact subsets of the plane.*

We shall say that the series on the right of (3.1) is a $L_{3,0,0}$ series expansion of f and that $\{B_{3n}(z)\}_0^\infty$, $\{A_{3n+1}(z)\}_0^\infty$ and $\{A_{3n+2}(z)\}_0^\infty$ are its fundamental polynomials.

PROOF. Setting $f(z) = e^{zt}$ in (3.1) and writing ψ_i for $\psi_i(z, t^3)$, $i = 0$, 1, 2, we get a formal $W_{3,0,0}(L)$ representation of e^{zt} as

$$e^{zt} = e^t\psi_0 + t\psi_1 + t^2\psi_2. \tag{3.5}$$

Let $\omega = e^{2\pi i/3}$. Then replacing t by ωt and $\omega^2 t$ in (3.5), we get

$$e^{z\omega t} = e^{\omega t}\psi_0 + \omega t\psi_1 + \omega^2 t^2\psi_2, \tag{3.6}$$

$$e^{z\omega^2 t} = e^{\omega^2 t}\psi_0 + \omega^2 t\psi_1 + \omega t^2\psi_2. \tag{3.7}$$

Bearing in mind that $1 + \omega + \omega^2 = 0$ and that

$$\sum_{m=0}^{2} \omega^{-mj}e^{\omega^m t} = 3N_{3,j}(t) \qquad (j = 0, 1, 2)$$

we obtain ψ_0 by adding (3.5), (3.6) and (3.7). To obtain ψ_1 we multiply (3.5), (3.6) and (3.7) by 1, ω^{-1} and ω^{-2} respectively and add.

ψ_2 is obtained by multiplying these equations by 1, ω^{-2} and ω^{-1} respectively. Now the Pólya representation of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n/n!$ of finite type is given by

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma} e^{zt}F(t)\,dt,$$

where $F(z) = \sum_{n=0}^{\infty} a_n/z^{n+1}$ is the Borel transform of $f(z)$ and Γ is a contour surrounding the convex hull of the set of singularities of $F(t)$, called the conjugate diagram $D(f)$ of f. From (3.2), (3.3) and (3.4) we see that the right side of (3.5) is regular in all circles $|t| < \lambda_{0,1}^{(3)}$. Therefore the series given by ψ_0, ψ_1 and ψ_2 converge uniformly in any compact subset of the disk $|t| < \lambda_{0,1}^{(3)}$.

It is well-known [2] that if f is of exponential type τ, then $D(f)$ lies inside the disk $|t| \leqq \tau$. Therefore, Γ can be taken to be any circle $|t| < \lambda_{0,1}^{(3)}$. The proof of Theorem 3.1 is then completed by applying the kernel expansion method [2, p. 10] with e^{zt} as kernel.

Remark. The generating functions of the fundamental polynomials in the other interpolation problems are given in Table III.

Note. Throughout the rest of this paper, we will write $\lambda_{j,i}$ for $\lambda_{j,i}^{(3)}$ $(j = 0, 1, 2;\ i = 1, 2, \cdots)$.

4. **Properties of the fundamental polynomials.** Let L be a linear operator on $C^{(3)}[0, 1]$ defined by

$$L(f) = f(x) - \{f(1)\,B_0(x) + f'(0)A_1(x) + f''(0)A_2(x)\}. \tag{4.1}$$

It can be easily seen that $L(p) = 0$ for any polynomial of degree $\leqq 2$. Therefore by Peano's theorem [8] we have

$$L(f) = \int_0^1 K_1(x, t)f'''(t)\,dt, \tag{4.2}$$

where $K_1(x, t) = \frac{1}{2}L_x[(x - t)_+^2]$. It is easy to verify that

$$\begin{aligned} 2K_1(x, t) &= (x - t)^2 - (1 - t)^2 && (0 \leqq t < x \leqq 1), \\ &= -(1 - t)^2 && (0 \leqq x \leqq t \leqq 1). \end{aligned} \tag{4.3}$$

Putting $f(x) = 1, x, \cdots, x^6$ in succession in (3.1) we find that

$$B_0(x) = 1, \qquad B_3(x) = \frac{x^3 - 1}{6}, \qquad B_6(x) = \frac{x^6 - 20x^3 + 19}{720};$$

$$A_1(x) = x - 1, \qquad A_4(x) = \frac{x^4 - 4x^3 + 3}{24}; \tag{4.4}$$

$$A_2(x) = \frac{x^2 - 1}{2}, \qquad A_5(x) = \frac{x^5 - 10x^3 + 9}{120}.$$

Further, it can be easily seen from (3.2), (3.3) and (3.4) that

$$\begin{aligned}(4.5)\qquad & B_{3n}'''(x) = B_{3(n-1)}(x), \qquad A_{3n+1}'''(x) = A_{3(n-1)+1}(x),\\ & A_{3n+2}'''(x) = A_{3(n-1)+2}(x)\end{aligned}$$

and therefore in general

$$\begin{aligned}(4.6)\qquad & B_{3n}^{(3k)}(x) = B_{3(n-k)}(x), \qquad A_{3n+1}^{(3k)}(x) = A_{3(n-k)+1}(x),\\ & A_{3n+2}^{(3k)}(x) = A_{3(n-k)+2}(x).\end{aligned}$$

Again from (3.2), (3.3) and (3.4), we see that

$$\begin{aligned}& A_{3n+j}(1) = 0 \qquad (j = 1, 2;\ n = 0, 1, \cdots),\\ (4.7)\ & A_{3n+2}'(0) = A_{3n+1}''(0) = B_{3n}'(0) = B_{3n}''(0) = 0 \qquad (n = 0, 1, \cdots),\\ & A_{3n+1}'(0) = A_{3n+2}''(0) = B_{3n}(1) = 0 \qquad (n = 1, 2, \cdots),\end{aligned}$$

while $A_1'(0) = 1$ and $A_2''(0) = 1$.

$K_1(x, t)$ is also seen to be the Green's function for the differential system

$$(4.8)\quad y'''(x) = \phi(x), \qquad y(1) = 0, \qquad y'(0) = 0, \qquad y''(0) = 0$$

where $\phi(x)$ is any function continuous on $0 \leqq x \leqq 1$, so that

$$y(x) = \int_0^1 K_1(x, t)\phi(t)\, dt$$

is the unique solution of the system (4.8). Since from (4.5) and (4.7) we see that $B_{3n}(x)$ $(n = 1, 2, \cdots)$ satisfies the system (4.8) with $\phi(x) = B_{3(n-1)}(x)$ we have

$$(4.9)\qquad B_{3n}(x) = \int_0^1 K_1(x, t)B_{3(n-1)}(t)\, dt \qquad (n = 1, 2, \cdots).$$

Setting

$$(4.10)\qquad K_n(x, t) = \int_0^1 K_1(x, u)K_{n-1}(u, t)\, du \qquad (n = 2, 3, \cdots),$$

and observing that $B_0(x) = 1$, we can write (4.9) as

$$(4.11)\qquad B_{3n}(x) = \int_0^1 K_n(x, t)\, dt \qquad (n = 1, 2, \cdots).$$

Similarly we have

$$A_{3n+1}(x) = \int_0^1 K_n(x, t)A_1(t)\,dt \qquad (n = 1, 2, \cdots) \tag{4.12}$$

and

$$A_{3n+2}(x) = \int_0^1 K_n(x, t)A_2(t)\,dt \qquad (n = 1, 2, \cdots). \tag{4.13}$$

LEMMA 4.1. *If* $f(x) \in C^{(3n)}[0, 1]$, *then*

$$\begin{aligned} f(x) = \sum_{k=0}^{n-1} f^{(3k)}(1)B_{3k}(x) &+ \sum_{k=0}^{n-1} f^{(3k+1)}(0)A_{3k+1}(x) \\ &+ \sum_{k=0}^{n-1} f^{(3k+2)}(0)A_{3k+2}(x) + R_n(f; x), \end{aligned} \tag{4.14}$$

where

$$R_n(f; x) = \int_0^1 K_n(x, t)f^{(3n)}(t)\,dt, \tag{4.15}$$

with $K_n(x, t)$ *given by* (4.10).

PROOF. For $n = 1$, (4.14) is given by (4.1) and (4.2). The proof is completed by induction on n.

LEMMA 4.2. *For* $0 \leqq x \leqq 1$ *the following inequalities hold*:

$$(-1)^n K_n(x, t) \geqq 0 \qquad (0 \leqq t \leqq 1;\ n = 1, 2, \cdots), \tag{4.16}$$

$$(-1)^n B_{3n}(x) \geqq 0 \qquad (n = 0, 1, \cdots), \tag{4.17}$$

$$(-1)^{n+1} A_{3n+j}(x) \geqq 0 \qquad (j = 1, 2;\ n = 0, 1, \cdots). \tag{4.18}$$

PROOF. For $n = 1$, (4.16) is clear from (4.3). For $n > 1$, (4.16) is proved from (4.10). (4.17) is immediate from (4.11). Observing that $A_1(x) = x - 1$ and $A_2(x) = (x_2 - 1)/2$, we prove (4.18) from (4.12) and (4.13).

5. **Estimates for the fundamental polynomials.** Throughout the rest of this chapter we shall denote positive constants by $C_0, C_1, \cdots$.

LEMMA 5.1. *For* $0 \leqq x \leqq 1$, $n = 0, 1, \cdots$, *we have*

$$\begin{aligned} \left| (-1)^{n+1} A_{3n+j}(x) - \frac{3M_{3,j}(\lambda_{0,1})M_{3,0}(\lambda_{0,1}x)}{\lambda_0{}^{3n+j+1} M_{3,2}(\lambda_{0,1})} \right| \\ \leqq \frac{C_0}{(\lambda_{0,1})^{3n+1}} \qquad (j = 1,2) \end{aligned} \tag{5.1}$$

and

$$\left| (-1)^n B_{3n}(x) - \frac{3M_{3,0}(\lambda_{0,1}x)}{\lambda_{0,1}^{3n+1}M_{3,2}(\lambda_{0,1})} \right| < \frac{C_1}{\lambda_{0,1}^{3n+1}}. \tag{5.2}$$

PROOF. We will prove (5.1) for $j = 1$. (5.1) for $j = 2$ and (5.2) can be dealt with similarly. Let Γ_0 and Γ_1 be the circles:

$$\Gamma_0 : |t| = \frac{\lambda_{0,1}}{2} = r_0 \text{ and } \Gamma_1 : |t| = \frac{\lambda_{0,1} + \lambda_{0,2}}{2} = r_1.$$

Define

$$A_{3n+1,m}(x) = \frac{1}{2\pi i}\int_{\Gamma_m} t^{-3n-1}\psi_1(x, t^3)\, dt \qquad (0 \leqq x \leqq 1;\ m = 0,1) \tag{5.3}$$

where $\psi_1(x, t^3)$ is defined by (3.3). Then it is easy to see that $A_{3n+1,0}(x) = A_{3n+1}(x)$. Now

$$\begin{aligned} A_{3n+1,1}(x) - A_{3n+1}(x) &= \frac{1}{2\pi i}\int_{\Gamma_1-\Gamma_0} t^{-3n-1}\psi_1(x, t^3)\, dt \\ &= \frac{1}{2\pi i}\int_{\Gamma_1-\Gamma_0} \frac{t^{-3n-2}[N_{3,1}(tx)N_{3,0}(t) - N_{3,1}(t)N_{3,0}(tx)]}{N_{3,0}(t)}\, dt. \end{aligned} \tag{5.4}$$

From (5.4), it is clear that the only poles of $t^{-3n-1}\psi_1(x, t^3)$ are the simple poles at $t = \omega^{\nu+1/2}\lambda_{0,1}$, $\nu = 0, 1, 2$. By simple calculations we find that the residue of $t^{-3n-1}\psi_1(x, t^3)$ at each of these poles is

$$(-1)^n M_{3,1}(\lambda_{0,1})M_{3,0}(\lambda_{0,1}x)/\lambda_{0,1}^{3n+2}M_{3,2}(\lambda_{0,1}).$$

Therefore

$$A_{3n+1,1}(x) - A_{3n+1}(x) = \frac{(-1)^n 3M_{3,1}(\lambda_{0,1})M_{3,0}(\lambda_{0,1}x)}{\lambda_{0,1}^{3n+2}M_{3,2}(\lambda_{0,1})}. \tag{5.5}$$

Further from (5.3) we get,

$$|A_{3n+1}(x)| = \left| \frac{1}{2\pi_i}\int_{\Gamma_1} \frac{t^{-3n-2}N_{3,1}(tx)N_{3,0}(t) - N_{3,1}(t)N_{3,0}(tx)}{N_{3,0}(t)}\, dt \right|$$

$$\begin{aligned} &\leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{(1+|x|)r_1}\, d\theta}{r_1{}^{3n+2}|N_{3,0}(r_1e^{i\theta})|} \\ &\leqq \frac{C_0}{r_1{}^{3n+1}} < \frac{C_0}{\lambda_{0,1}^{3n+1}}, \quad \text{as } \frac{\lambda_{0,1}}{r_1} < 1. \end{aligned} \tag{5.6}$$

From (5.5) and (5.6), we get (5.1).

LEMMA 5.2. *For* $0 \leqq x \leqq 1, n = 0, 1, 2, \cdots,$

$$0 \leqq (-1)^{n+1} A_{3n+j}(x) \leqq C_2/\lambda_{0,1}^{3n} \qquad (j = 1,2) \tag{5.7}$$

and

$$0 \leqq (-1)^n B_{3n}(x) \leqq C_3/\lambda_{0,1}^{3n}. \tag{5.8}$$

PROOF. The results can be easily deduced from Lemma 5.1.

LEMMA 5.3. *For any fixed* x_0, $0 < x_0 < 1$, *there exist constants* C_4 *and* C_5 *such that*

$$(-1)^{n+1} A_{3n+j}(x_0) \geqq C_4/\lambda_{0,1}^{3n} \qquad (j = 1, 2; n = 0, 1, \cdots) \tag{5.9}$$

and

$$(-1)^n B_{3n}(x_0) \geqq C_5/\lambda_{0,1}^{3n} \qquad (n = 0, 1, \cdots). \tag{5.10}$$

PROOF. From (5.1) we have

$$\lim_{n\to\infty} \frac{(-1)^{n+1} A_{3n+j}(x_0) M_{3,2}(\lambda_{0,1}) \lambda_{0,1}^{3n}}{M_{3,j}(\lambda_{0,1}) M_{3,0}(\lambda_{0,1} x_0)} = \lambda_{0,1}^{j+1} \qquad (j = 1, 2)$$

which proves (5.9). (5.10) is similarly proved.

LEMMA 5.4. *For* $0 \leqq x \leqq 1, n = 1, 2, \cdots,$

$$0 \leqq (-1)^n \int_0^1 K_n(x, t)\, dt \leqq \frac{C_3}{\lambda_{0,1}^{3n}}.$$

PROOF. The proof follows immediately from (4.11) and (5.8).

6. The class $W_{3,0,0}$ of functions.

DEFINITION 6.1. A real valued function f defined on $[a, b]$ is said to belong to the class $W_{3,0,0}$ of functions ($f \in W_{3,0,0}$) if

(i) $f \in C^\infty[a, b]$,
(ii) $(-1)^k f^{(3k)}(x) \geqq 0$ $(a \leqq x \leqq b; k = 0, 1, 2, \cdots)$,
(iii) $(-1)^{k+1} f^{(3k+j)}(a) \geqq 0$ $(j = 1, 2; k = 0, 1, 2, \cdots)$.

We now prove some basic properties of functions of the class $W_{3,0,0}$.

LEMMA 6.2. *If* $f \in W_{3,0,0}$ *on* $0 \leqq x \leqq 1$ *then for sufficiently large* k *we have*

$$(-1)^k f^{(3k)}_{(1)} \leqq C_6 \lambda_{0,1}^{3k} \tag{6.1}$$

and

$$(-1)^{k+1}f^{(3k+j)}(0) \leqq C_7\lambda_{0,1}^{3k} \qquad (j = 1, 2). \tag{6.2}$$

PROOF. From Definition 6.1 and (4.17) and (4.18) we find that every term on the right of (4.14) is nonnegative. Therefore

$$0 \leqq f^{(3k)}(1)B_{3k}(x) \leqq f(x), \tag{6.3}$$

$$0 \leqq f^{(3k+j)}(0)A_{3k+j}(x) \leqq f(x) \qquad (j = 1, 2). \tag{6.4}$$

Choosing $x = \frac{1}{2}$ and applying Lemma 5.3 to (6.3) and (6.4) we get (6.1) and (6.2).

LEMMA 6.3. *If* (i) $f(x) \geqq 0$, $f'''(x) \leqq 0$, $0 \leqq x \leqq 1$, *and* (ii) $f'(0) \leqq 0, f''(0) \leqq 0$ *then*

$$f(x) \geqq f(x_0) \qquad (0 \leqq x \leqq x_0), \tag{6.5}$$

$$f(x) \geqq \frac{1-x}{1-x_0}f(x_0) \geqq (1-x)f(x_0) \qquad (x_0 \leqq x \leqq 1). \tag{6.6}$$

PROOF. Setting $n = 1$ in (4.14) and replacing the node 1 by x_0 $(0 < x_0 \leqq 1)$, we get

$$\begin{aligned} f(x) = {} & f(x_0)B_0\left(\frac{x}{x_0}\right) + f'(0)A_1\left(\frac{x}{x_0}\right) \\ & + f''(0)A_2\left(\frac{x}{x_0}\right) + R_1\left(f; \frac{x}{x_0}\right), \end{aligned} \tag{6.7}$$

where $R_1(f; x) = \int_0^1 K_1(x, t)f'''(t)\,dt$. On account of conditions (i) and (ii) and Lemma 4.2, all the terms on the right of (6.7) are nonnegative. Therefore $f(x) \geqq f(x_0)$ $(0 \leqq x \leqq x_0)$. To prove (6.6), consider

$$\begin{aligned} L^*(f) &= f(x) - \left\{ f(1)\frac{x-x_0}{1-x_0} + f(x_0)\frac{1-x}{1-x_0} + f''(0)\frac{(x-x_0)(x-1)}{2} \right\} \\ &= R^*(f; x_0) \qquad (x_0 \leqq x \leqq 1). \end{aligned} \tag{6.8}$$

It can be easily verified that $L^*(P) = 0$ for all polynomials of degree $\leqq 2$. Therefore, by Peano's theorem

$$R^*(f; x_0) = \int_0^1 K^*(x, x_0, t)f'''(t)\,dt,$$

where

$$2K^*(x, x_0, t) = (x-t)^2 - (1-t)^2 \frac{x-x_0}{1-x_0} \quad (x_0 \leqq t \leqq x \leqq 1),$$

$$= -(1-t)^2 \frac{x-x_0}{1-x_0} \quad (x_0 \leqq x \leqq t < 1),$$

$$= (x-t)^2 - (1-t)^2 \frac{x-x_0}{1-x_0} - \frac{(x_0-t)^2(1-x)}{1-x_0},$$

$$(0 \leqq t \leqq x_0 \leqq x < 1).$$

It is easy to see that $K^*(x, x_0, t) \leqq 0$ for $x_0 \leqq x \leqq 1$ and $0 \leqq t \leqq 1$. Thus $R^*(f : x_0) \geqq 0$, whence we have (6.6).

LEMMA 6.4. *Under the same conditions as in Lemma* 6.3, *for* $0 < b < 1$,

$$f(x) \leqq \frac{2}{b} \int_0^b f(t)\, dt \qquad (0 < x < 1). \tag{6.9}$$

PROOF. By Lemma 6.3 for any x_0 with $0 \leqq x_0 < b < 1$ we have

$$\begin{aligned} \int_0^b f(x)\, dx &\geqq \int_0^{x_0} f(x_0)\, dx + \int_{x_0}^b (1-x) f(x_0)\, dx \\ &= f(x_0)[-b^2 + 2b + x_0^2]/2 \geqq b f(x_0)/2. \end{aligned} \tag{6.10}$$

It can be easily seen that (6.9) is also true for $0 \leqq b \leqq x_0 < 1$. Since (6.10) is true for any x_0, $0 \leqq x_0 < 1$, we get (6.9).

We shall need the following lemma due to Hadamard.

LEMMA 6.5 (HADAMARD [10]). *If* $g(x) \in C^{(3)}(I)$ *where* I *is a closed interval of length* α *and if*

$$|g(x)| \leqq M_0; \qquad |g'''(x)| \leqq M_1, \qquad x \in I,$$

then throughout the interval I

$$|g^{(j)}(x)| \leqq \left(\frac{3e}{j}\right)^j 3^j \left[\alpha^{-j} M_0 + \frac{\alpha^{3-j}}{3!} M_1 \right] \quad (j = 1,2).$$

Now we prove

THEOREM 6.6. *If* $f \in W_{3,0,0}$ *on* $0 \leqq x \leqq 1$, *then* f *coincides on* $[0, 1]$ *with a real entire function of exponential type not exceeding* $\lambda_{0,1}$ *and the* $L_{3,0,0}$ *series representation* (3.1) *holds.*

PROOF. Observing that $(d/dt)M_{3,0}(t) = -M_{3,2}(t)$ and using integra tion by parts, we have

$$\int_0^1 f(x)M_{3,2}(\lambda_{0,1}-\lambda_{0,1}x)\,dx$$
$$= \frac{f(1)}{\lambda_{0,1}} - \frac{1}{\lambda_{0,1}}\int_0^1 f'(x)M_{3,0}(\lambda_{0,1}-\lambda_{0,1}x)\,dx.$$

Repeating this process twice, we get

$$\int_0^1 f(x)M_{3,2}(\lambda_{0,1}-\lambda_{0,1}x)\,dx$$
$$(6.11)\qquad = \frac{f(1)}{\lambda_{0,1}} - \frac{f'(0)M_{3,1}(\lambda_{0,1})}{\lambda_{0,1}^2} - \frac{f''(0)M_{3,2}(\lambda_{0,1})}{\lambda_{0,1}^3}$$
$$-\frac{1}{\lambda_{0,1}^3}\int_0^1 f'''(x)M_{3,2}(\lambda_{0,1}-\lambda_{0,1}x)\,dx.$$

Now $M_{3,\nu}(x) \geqq 0$ for $0 \leqq x \leqq \lambda_{\nu,1}$ $(\nu = 1, 2)$, and $\lambda_{0,1} < \lambda_{1,1} < \lambda_{2,1}$. Therefore $M_{3,\nu}(\lambda_{0,1}) \geqq 0$ $(\nu = 1, 2)$. Further, since $f(x) \in W_{3,0,0}$, we have $f(1) \geqq 0$, $f'(0) \leqq 0$, and $f''(0) \leqq 0$. Hence from (6.11) we have

$$(6.12)\qquad \int_0^1 f(x)M_{3,2}(\lambda_{0,1}-\lambda_{0,1}x)\,dx \geqq \frac{1}{\lambda_{0,1}^3} f'''(x)M_{3,2}(\lambda_{0,1}-\lambda_{0,1}x)\,dx.$$

If $f(x) \in W_{3,0,0}$, then $-f'''(x)$ also belongs to $W_{3,0,0}$. Hence applying (6.12) successively k times we get

$$\frac{(-1)^k}{\lambda_{0,1}^{3k}}\int_0^1 f^{(3k)}(x)M_{3,2}(\lambda_{0,1}-\lambda_{0,1}x)\,dx$$
$$\leqq \int_0^1 f(x)M_{3,2}(\lambda_{0,1}-\lambda_{0,1}x)\,dx = A.$$

Let $0 < b < 1$. Then *a fortiori*

$$\frac{(-1)^k}{\lambda_{0,1}^{3k}}\int_0^b f^{(3k)}(x)M_{3,2}(\lambda_{0,1}-\lambda_{0,1}x)\,dx \leqq A.$$

Elementary considerations show that for $0 \leqq x \leqq b$ $(0 < b < 1)$,

$$\min_{0\leqq x\leqq b} M_{3,2}(\lambda_{0,1}-\lambda_{0,1}x) = D > 0,$$

so that

$$(-1)^k \int_0^b f^{(3k)}(x)\,dx < \frac{A\lambda_{0,1}^{3k}}{D}.$$

Therefore by Lemma 6.4

$$(-1)^k f^{(3k)}(x) < 2A\lambda_{0,1}^{3k}/\mathrm{b}.$$

By Lemma 6.5 we then have

$$(-1)^k f^{(3k+j)}(x) = O(\lambda_{0,1}^{3k+j}) \qquad \text{as } k \to \infty \text{ for } j = 0, 1, 2.$$

Thus we have

$$f^{(n)}(x) = O(\lambda_{0,1}^n) \quad \text{uniformly in } [0, b] \text{ as } n \to \infty.$$

This shows that $f(x)$ is entire of exponential type $\leqq \lambda_{0,1}$ and the theorem is proved.

THEOREM 6.7. *If the series*

$$\sum_{n=0}^{\infty} \{b_{3n}B_{3n}(x) + a_{3n+1}A_{3n+1}(x) + a_{3n+2}A_{3n+2}(x)\} \tag{6.13}$$

converges for a single value x_0 $(0 < x_0 < 1)$ *then it converges uniformly in* $0 \leqq x \leqq 1$ *to a function* $f(x)$. *Furthermore, the series*

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\lambda_{0,1}^{3n+1}} \left\{ b_{3n} - \frac{M_{3,1}(\lambda_{0,1})}{\lambda_{0,1}} a_{3n+1} - \frac{M_{3,2}(\lambda_{0,1})}{\lambda_{0,1}^2} a_{3n+2} \right\} \tag{6.14}$$

converges and we have for $0 < x < 1$
converges and we have for $0 \leqq x \leqq 1$

$$\begin{aligned} f^{(3k)}(x) = \sum_{n=0}^{\infty} \{b_{3(n+k)}B_{3n}(x) &+ a_{3(n+k)+1}A_{3n+1}(x) \\ &+ a_{3(n+k)+2}A_{3n+2}(x)\}. \end{aligned} \tag{6.15}$$

PROOF. As the proof is straightforward, we will give only its outline. By (5.1), (5.2) and Lemma 5.3, it is easy to see that the series

$$
\begin{aligned}
\sum_{n=0}^{\infty} \Bigg\{ & b_{3n}\left[B_{3n}(x_0) - \frac{(-1)^n}{\lambda_{0,1}^{3n+1}}\right] \\
(6.16) \qquad & + a_{3n+1}\left[A_{3n+1}(x_0) - \frac{(-1)^{n+1}M_{3,1}(\lambda_{0,1})}{\lambda_{0,1}^{3n+2}}\right] \\
& + a_{3n+2}\left[A_{3n+2}(x_0) - \frac{(-1)^{n+1}M_{3,2}(\lambda_{0,1})}{\lambda_{0,1}^{3n+3}}\right]\Bigg\}
\end{aligned}
$$

converges absolutely. Subtracting (6.13), convergent for $x = x_0$, from (6.16) we prove that (6.14) converges and further that (6.13) converges uniformly for $0 \leqq x \leqq 1$.

The uniform convergence of the series in (6.15) is proved by noting the uniform convergence in $0 \leqq x \leqq 1$ of both the series

$$
\begin{aligned}
\sum_{n=0}^{\infty} \Bigg\{ & b_{3(n+k)}\left[B_{3n}(x) - \frac{(-1)^n}{\lambda_{0,1}^{3n+1}}\right] \\
& + a_{3(n+k)+1}A_{3n+1}(x) - \left[\frac{(-1)^{n+1}M_{3,1}(\lambda_{0,1})}{\lambda_{0,1}^{3n+2}}\right] \\
& + a_{3(n+k)+2}\left[A_{3n+2}(x) - \frac{(-1)^{n+1}M_{3,2}(\lambda_{0,1})}{\lambda_{0,1}^{3n+3}}\right]\Bigg\}
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{n=0}^{\infty} \Bigg\{ & \frac{(-1)^n b_{3(n+k)}}{\lambda_{0,1}^{3n+1}} + \frac{(-1)^{n+1}a_{3(n+k)+1}M_{3,1}(\lambda_{0,1})}{\lambda_{0,1}^{3n+2}} \\
& + \frac{(-1)^{n+1}a_{3(n+k)+2}M_{3,2}(\lambda_{0,1})}{\lambda_{0,1}^{3n+3}}\Bigg\}.
\end{aligned}
$$

This completes the proof.

Lemma 6.8. *If $f(x) \in W_{3,0,0}$ in $0 \leqq x \leqq 1$, then there exist constants C_8, C_9 such that*

$$
(6.17) \qquad
\begin{aligned}
0 \leqq (-1)^k f^{(3k)}(x) &\leqq C_8 \left(\frac{\lambda_{0,1}}{x}\right)^{3k} \\
0 \leqq (-1)^k f^{(3k)}(x) &\leqq C_9 \left(\frac{\lambda_{0,1}}{1-x}\right)^{3k}
\end{aligned}
\qquad (k \to \infty).
$$

Proof. If $f(x) \in W_{3,0,0}$ in $a \leqq x \leqq b$ then $F(x) = f(a + bx - ax) \in W_{3,0,0}$ on $0 \leqq x \leqq 1$. Therefore by Theorem 6.6 we have

$$(6.18)\qquad \begin{aligned} F^{(3k)}(0) &= (b-a)^{3k} f^{(3k)}(a) = O(\lambda_{0,1}^{3k}) \\ F^{(3k)}(1) &= (b-a)^{3k} f^{(3k)}(b) = O(\lambda_{0,1}^{3k}) \end{aligned} \qquad (k \to \infty).$$

First putting $a = 0$, $b = x < 1$ and then $a = x > 0$, $b = 1$ in (6.18) we get (6.17).

THEOREM 6.9. *If $f(x) \in W_{3,0,0}$ in $a \leqq x \leqq b$ where $b - a > 1$ then $f(x)$ is an entire function of exponential type less than $\lambda_{0,1}$ and the $L_{3,0,0}$-series representation holds for any z in the complex plane.*

PROOF. By (6.18) we have

$$|f^{(3k)}(x)| \leqq C_{10}(\lambda_{0,1}/(b-x))^{3k} \qquad (a \leqq x \leqq b).$$

Choose c so that $b - c > 1$. Then we get

$$|f^{(3k)}(x)| \leqq C_{10}(\lambda_{0,1}/(b-c))^{3k} \qquad (a \leqq x \leqq c).$$

Setting $\lambda_{0,1}/(b-c) = q$ and applying Lemma 6.5 we get

$$f^{(n)}(x) = O(q^n) \qquad (n \to \infty),$$

uniformly in $a \leqq x \leqq c$. This shows that $f(x)$ is entire of exponential type $q < \lambda_{0,1}$, which proves the theorem.

7. **Minimal $W_{3,0,0}$ functions.** The sufficient condition of Theorem 6.9 for the representation of a function by a $L_{3,0,0}$ series is not necessary. For example $N_{3,0}(x)$ has the $L_{3,0,0}$ representation

$$N_{3,0}(x) = N_{3,0}(1) \sum_{k=0}^{\infty} B_{3k}(x)$$

yet it is not a $W_{3,0,0}$ function. Also $M_{3,0}(\lambda_{0,1}x) \in W_{3,0,0}$, but it has no $L_{3,0,0}$ representation. In order to obtain a necessary and sufficient condition, we introduce the class of minimal $W_{3,0,0}$ functions.

DEFINITION 7.1. A real valued function $f(x)$ defined on $0 \leqq x \leqq 1$ is said to be a minimal $W_{3,0,0}$ function ($f(x) \in$ minimal $W_{3,0,0}$) on $[0, 1]$ if $f(x) \in W_{3,0,0}$ on $[0, 1]$ and if $f(x) - \epsilon M_{3,0}(\lambda_{0,1}x) \notin W_{3,0,0}$ on $[0, 1]$ for some $\epsilon > 0$.

Thus $f(x) = 0$ and $f(x) = M_{3,0}(x)$ can be easily seen to be minimal $W_{3,0,0}$ functions.

THEOREM 7.2. *If the series*

$$\sum_{n=0}^{\infty} \left\{(-1)^n b_{3n} B_{3n}(x) - \frac{(-1)^n M_{3,1}(\lambda_{0,1})}{\lambda_{0,1}} a_{3n+1} A_{3n+1}(x) - \frac{(-1)^n M_{3,2}(\lambda_{0,1})}{\lambda_{0,1}^2} a_{3n+2} A_{3n+2}(x) \right\} \tag{7.1}$$

with $b_{3n} \geqq 0$, $a_{3n+1} \geqq 0$ *and* $a_{3n+2} \geqq 0$ $(n = 0, 1, \cdots)$ *converges to* $f(x)$, *then* $f(x) \in$ *minimal* $W_{3,0,0}$ *on* $[0, 1]$.

Proof. Using Theorem 6.7 and (4.7), simple calculations show that $f(x) \in W_{3,0,0}$ $0 \leqq x \leqq 1$, thus satisfying the first part of Definition 7.1.

Further

$$(-1)^k f^{(3k)}(x) = \sum_{n=k}^{\infty} \left\{(-1)^{n-k} b_{3n} B_{3(n-k)}(x) + \frac{(-1)^{n+1-k} M_{3,1}(\lambda_{0,1})}{\lambda_{0,1}} a_{3n+1} A_{3(n-k)+1}(x) + \frac{(-1)^{n+1-k} M_{3,2}(\lambda_{0,1})}{\lambda_{0,1}^2} a_{3n+2} A_{3(n-k)+2}(x) \right\}. \tag{7.2}$$

Using (5.7) and (5.8) in (7.2), we get $(-1)^k f^{(3k)}(x) \leqq C_9 \lambda_{0,1}^{3k} T_k$ where

$$T_k = \sum_{n=k}^{\infty} \lambda_{0,1}^{-3n} \left\{ b_{3n} + \frac{M_{3,1}(\lambda_{0,1}) a_{3n+1}}{\lambda_{0,1}} + \frac{M_{3,2}(\lambda_{0,1}) a_{3n+2}}{\lambda_{0,1}^2} \right\}.$$

From Theorem 6.7, the series (6.14) converges absolutely so that for a given $\epsilon > 0$ and x_0, $0 < x_0 < 1$, there exists an integer k_0 sufficiently large such that $C_9 T_{k_0} - \epsilon M_{3,0}(\lambda_{0,1} x_0) < 0$. In other words,

$$(-1)^k [f(x) - \epsilon M_{3,0}(\lambda_{0,1} x)]^{(3k)} < 0 \quad \text{at } x = x_0.$$

This completes the proof.

Lemma 7.3. *If* (i) $f'(0) \leqq 0$, $f''(0) \leqq 0$, (ii) $f(x) \geqq 0$, $f'''(x) \leqq 0$ *for* $0 \leqq x \leqq 1$ *and if* (iii) $f(x_0) > \lambda_{0,1}^3 \epsilon/2$ *for some* x_0, $0 \leqq x_0 \leqq 1$, *then*

$$f(x) \geqq \epsilon M_{3,0}(\lambda_{0,1} x) \qquad (0 < x < 1). \tag{7.3}$$

Proof. We first recall from (2.4) that

$$6^{1/3} < \lambda_{0,1} < (12)^{1/3}. \tag{7.4}$$

From (iii), (6.5) and (7.4) we have for $0 \leqq x \leqq x_0$, $f(x) \geqq f(x_0) >$

$\lambda_{0,1}^3\epsilon/2 > \epsilon$. Therefore (7.3) follows if we show that

$$1 \geqq M_{3,0}(\lambda_{0,1}x) = 1 - \frac{\lambda_{0,1}^3 x^3}{3!} + \frac{\lambda_{0,1}^6 x^6}{6!} - \cdots. \tag{7.5}$$

(7.5) is equivalent to $0 \geqq -\lambda_{0,1}^3 x^3 h_1(x)$, where

$$h_1(x) = \left(\frac{1}{3!} - \frac{\lambda_{0,1}^3 x^3}{6!}\right) + \left(\frac{\lambda_{0,1}^6 x^6}{9!} - \frac{\lambda_{0,1}^9 x^9}{12!}\right) + \cdots.$$

Using (7.4), it is easily seen that each bracket is a nonnegative quantity Thus (7.3) is proved for $0 \leqq x \leqq x_0$.

If $x_0 = 1$, there is nothing else to prove. Let therefore $x_0 \leqq x < 1$ From (iii), (6.6) and (7.4), we have $f(x) \geqq (1-x)\lambda_{0,1}^3\epsilon/2$. Hence, (7.3 follows if

$$\lambda_{0,1}^3(1-x)/2 \geqq M_{3,0}(\lambda_{0,1}x) \qquad (x_0 \leqq x < 1),$$

or equivalently if

$$\lambda_{0,1}^3 x/2 \geqq M_{3,0}(\lambda_{0,1} - \lambda_{0,1}x) \qquad (0 \leqq x \leqq 1 - x_0 < 1). \tag{7.6}$$

Both sides of (7.6) vanish at $x = 0$. Hence it is enough to show that

$$\lambda_{0,1}^3/2 \geqq \lambda_{0,1} \max_{0 \leqq x < 1} M_{3,2}(\lambda_{0,1} - \lambda_{0,1}x). \tag{7.7}$$

Simple considerations show that $M_{3,2}(\lambda_{0,1} - \lambda_{0,1}x)$ is decreasing i $0 \leqq x < 1$ and therefore has the maximum $M_{3,2}(\lambda_{0,1})$. Thus it is see from (7.7) that we have to show that

$$\frac{\lambda_{0,1}^3}{2} \geqq \lambda_{0,1}M_{3,2}(\lambda_{0,1}) = \lambda_{0,1}\left[\frac{\lambda_{0,1}^2}{2!} - \frac{\lambda_{0,1}^5}{5!} + \cdots\right]$$

or equivalently that

$$0 \geqq -\lambda_{0,1}^6 h_2(\lambda_{0,1}),$$

where

$$h_2(\lambda_{0,1}) = \left(\frac{1}{5!} - \frac{\lambda_{0,1}^3}{8!}\right) + \left(\frac{\lambda_{0,1}^6}{11!} - \frac{\lambda_{0,1}^9}{14!}\right) + \cdots.$$

From (7.4) we see that $h_2(\lambda_{0,1}) \geqq 0$. This completes the proof of th lemma.

THEOREM 7.4. *If $f(x) \in$ minimal $W_{3,0,0}$ on $0 \leqq x \leqq 1$ then it ca be expanded in a convergent $L_{3,0,0}$ series.*

PROOF. Let

$$S_n(x) = \sum_{k=0}^{n} \Big\{ f^{(3k)}(1)B_{3k}(x) + f^{(3k+1)}(0)A_{3k+1}(x) + f^{(3k+2)}(0)\,A_{3k+2}(x) \Big\}. \tag{7.8}$$

As $f(x) \in W_{3,0,0}$, we see from Lemma 4.1 that

$$S_n(x) \leqq f(x) \qquad (0 \leqq x \leqq 1;\ n = 0, 1, \cdots).$$

$S_n(x)$ is also seen to be a nondecreasing function of n for each x. Hence $S_n(x) \to g(x)$ (say) as $n \to \infty$. We claim that $g(x) = f(x)$. For if $g(x) \not\equiv f(x)$, then for some x_0 in $[0, 1]$

$$f(x_0) - \lim_{n\to\infty} S_n(x_0) = \Delta > 0,$$

so that

$$f(x_0) - S_n(x_0) = \int_0^1 K_n(x_0, t) f^{(3n)}(t)\, dt \geqq \Delta \qquad (n = 1, 2, \cdots). \tag{7.9}$$

Since $f(x) \in$ minimal $W_{3,0,0}$, $f(x) - \epsilon M_{3,0}(\lambda_{0,1}x) \notin W_{3,0,0}$ for every $\epsilon > 0$. But we have

$$\frac{d^{3n+j}}{dx^{3n+j}} \{(-1)^{n+1}[f(x) - \epsilon M_{3,0}(\lambda_{0,1}x)]_{x=0}$$
$$= (-1)^{n+1} f^{(3n+j)}(0) \geqq 0 \qquad (j = 1, 2).$$

Therefore there exists an integer n_1 and x_1 $(0 < x_1 < 1)$ such that

$$(-1)^{n_1} f^{(3n_1)}(x_1) - \epsilon\lambda_{0,1}^{3n_1} M_{3,0}(\lambda_{0,1}x_1) < 0. \tag{7.10}$$

By Lemma 7.3, (7.10) implies that $(-1)^{n_1} f^{(3n_1)}(x) < \epsilon\lambda_{0,1}^{3n_1+3}/2$. Hence by Lemma 5.4

$$\int_0^1 K_{n_1}(x_0, t) f^{(3n_1)}(t)\, dt \leqq \frac{\epsilon\lambda_{0,1}^{3n_1} C_3}{2\lambda_{0,1}^{3n_1}}. \tag{7.11}$$

Choosing $\epsilon < 2\,\Delta/C_3\lambda_{0,1}^{3}$, we get from (7.11) that

$$\int_0^1 K_{n_1}(x_0, t) f^{(3n_1)}(t)\, dt < \Delta,$$

which contradicts (7.9).

Hence $g(x) = f(x)$ and the theorem is proved.

This leads us to state

THEOREM 6.5. *A necessary and sufficient condition that $f(x)$ be represented by an absolutely convergent $L_{3,0,0}$ series is that it is the difference of two minimal $W_{3,0,0}$ functions in* $0 \leqq x \leqq 1$.

PROOF. The proof of this theorem is similar to that of Widder's theorem for completely convex functions and is therefore omitted.

ADDED IN PROOF: Professor G. Polya has kindly pointed out that an essential property of the zeros of $M_{p,j}(x)$ used here was given by him as far back as 1929 in *Jahresbericht d. deutschen Math. Vereinigung*, 38 (1929) and also in *Comptes Rendus* (Paris), 183, 467–468.

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UNIVERSITY OF MALAYA, KUALA LUMPUR, MALAYASIA

UNIVERSITY OF ALBERTA, EDMONTON 7, ALBERTA, CANADA

Classes of Functions Associated With Six Interpolation Problems

Matrix of interpolation problem (1)	Associated class of functions (2)	Properties satisfied by functions in column (2), in addition to (i) and (ii) given below(*) (3)	Extremal function (4)
$M_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$W_{3,0,0}$	(iii) $(-1)^{k+1} f^{(3k+1)}(0) \geq 0$ (iv) $(-1)^{k+1} f^{(3k+2)}(0) \geq 0$	$M_{3,0}(\lambda_{0,1}^{(3)}x)$
$M_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$W_{3,1,0}$	(iii) $(-1)^{k} f^{(3k+1)}(1) \geq 0$ (iv) $(-1)^{k+1} f^{(3k+2)}(0) \geq 0$	$M_{3,1}(\lambda_{0,1}^{(3)}x)$
$M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$W_{3,2,0}$	(iii) $(-1)^{k} f^{(3k+1)}(0) \geq 0$ (iv) $(-1)^{k} f^{(3k+2)}(1) \geq 0$	$M_{3,2}(\lambda_{0,1}^{(3)}x)$
$M_4 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$W_{3,1,1}$	(iii) $(-1)^{k+1} f^{(3k+2)}(0) \geq 0$	$M_{3,1}(\lambda_{1,1}^{(3)}x)$

(*) (i) $f \in C^{\infty}[0,1]$

(ii) $(-1)^k f^{(3k)}(x) \geq 0$; $(0 \leq x \leq 1;\ k = 0, 1, 2, \cdots)$.

Table I (continued)

Matrix of interpolation problem (1)	Associated class of functions (2)	Properties satisfied by functions in column (2), in addition to (i) and (ii) given below(*) (3)	Extremal function (4)
$M_5 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$W_{3,2,1}$	(iii) $(-1)^k f^{(3k+1)}(0) \geq 0$ (iv) $(-1)^k f^{(3k+1)}(1) \geq 0$	$M_{3,2}(\lambda_{1,1}^{(3)}x)$
$M_6 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$W_{3,2,2}$	(iii) $(-1)^k f^{(3k+1)}(0) \geq 0$	$M_{3,2}(\lambda_{2,1}^{(3)}x)$

(*) (i) $f \in C^\infty[0,1]$

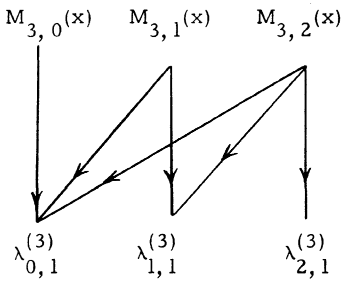
(ii) $(-1)^k f^{(3k)}(x) \geq 0$; $(0 \leq x \leq 1;\ k = 0, 1, 2, \cdots)$.

Table II

Classes of Functions Associated With the Interpolation Problems Treated Earlier by Bernstein, Widder, Whittaker, Schoenberg and Poritsky

Matrix of the interpolation problem (1)	Associated class of functions (2)	Properties satisfied by function in (2) in addition to property (i) given below(**) (3)	Extremal function (4)
(1)	completely monotonic	(ii) $(-1)^k f^{(k)}(x) \geq 0$ $0 \leq x < \infty$	0
$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	completely convex	(ii) $(-1)^k f^{(2k)}(x) \geq 0$ $0 \leq x \leq 1$	$\sin \pi x$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	c-c*	(ii) $(-1)^k f^{(2k)}(x) \geq 0$ $0 \leq x \leq 1$ (iii) $(-1)^k f^{(2k+1)}(0) \geq 0$	$\cos\left(\frac{\pi x}{2}\right)$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	c-c**	(ii) $(-1)^k f^{(2k)}(x) \geq 0$ $0 \leq x \leq 1$ (iii) $(-1)^k f^{(2k+1)}(1) \geq 0$	$\sin\left(\frac{\pi x}{2}\right)$

(**)(i) $f(x)$ possesses derivatives of all orders in the relevant interval.



Schematic Method for Obtaining the Six Extremal Functions

$M_{3,0}(\lambda_{01}^{(3)}x)$, $M_{3,1}(\lambda_{0,1}^{(3)}x)$, $M_{3,1}(\lambda_{1,1}^{(3)}x)$, $M_{3,2}(\lambda_{0,1}^{(3)}x)$, $M_{3,2}(\lambda_{1,1}^{(3)}x)$,

$M_{3,2}(\lambda_{2,1}^{(3)}x)$.

Table III

Generating Functions of Fundamental Polynomials and Kernels Associated with Six Interpolation Problems.

Matrix	Generating functions of fundamental polynomials	Fundamental polynomials for k = 0	Kernel
M_1	$\sum_{k=0}^{\infty} t^{3k} B_{3k}^{(1)}(x) = \frac{N_{3,0}(tx)}{N_{3,0}(t)}$	$B_0^{(1)}(x) = 1$	$2K_1^{(1)}(x,t) = \begin{cases} (x-t)^2 - (1-t)^2, & 0 \leq t < x \leq 1 \\ -(1-t)^2, & 0 \leq x \leq t \leq 1 \end{cases}$
	$\sum_{k=0}^{\infty} t^{3k+1} A_{3k+1}^{(1)}(x) = N_{3,1}(tx) - \frac{N_{3,1}(t)\, N_{3,0}(tx)}{N_{3,0}(t)}$	$A_1^{(1)}(x) = x - 1$	
	$\sum_{k=0}^{\infty} t^{3k+2} A_{3k+2}^{(1)}(x) = N_{3,2}(tx) - \frac{N_{3,2}(t)\, N_{3,0}(tx)}{N_{3,0}(t)}$	$A_2^{(1)}(x) = \frac{x^2-1}{2}$	
M_2	$\sum_{k=0}^{\infty} t^{3k+1} B_{3k+1}^{(2)}(x) = \frac{N_{3,1}(tx)}{N_{3,0}(t)}$	$B_1^{(2)}(x) = x$	$2K_1^{(2)}(x,t) = \begin{cases} (x-t)^2 - 2x(1-t), & 0 \leq t < x \leq 1 \\ -2x(1-t), & 0 \leq x \leq t \leq 1 \end{cases}$
	$\sum_{k=0}^{\infty} t^{3k} A_{3k}^{(2)}(x) = N_{3,0}(tx) - \frac{N_{3,2}(t)\, N_{3,1}(tx)}{N_{3,0}(t)}$	$A_0^{(2)}(x) = 1$	
	$\sum_{k=0}^{\infty} t^{3k+2} A_{3k+2}^{(2)}(x) = N_{3,2}(tx) - \frac{N_{3,1}(t)\, N_{3,1}(tx)}{N_{3,0}(t)}$	$A_2^{(2)}(x) = \frac{x(x-2)}{2}$	

Table III (continued)

Matrix	Generating functions of fundamental polynomials	Fundamental polynomials for k = 0	Kernel
	$\sum_{k=0}^{\infty} t^{3k+2} B_{3k+2}^{(3)}(x) = \frac{N_{3,2}(tx)}{N_{3,0}(t)}$	$B_2^{(3)}(x) = \frac{x^2}{2}$	
M_3	$\sum_{k=0}^{\infty} t^{3k} A_{3k}^{(3)}(x) = N_{3,0}(tx) - \frac{N_{3,1}(t)\, N_{3,2}(tx)}{N_{3,0}(t)}$	$A_0^{(3)}(x) = 0$	$2K_1^{(3)}(x,t) = \begin{cases} -2xt + t^2, & 0 \leq t < x \leq 1 \\ -x^2, & 0 \leq x \leq t \leq 1 \end{cases}$
	$\sum_{k=0}^{\infty} t^{3k+1} A_{3k+1}^{(3)}(x) = N_{3,1}(tx) - \frac{N_{3,2}(t)\, N_{3,2}(tx)}{N_{3,0}(t)}$	$A_1^{(3)}(x) = x$	
	$\sum_{k=0}^{\infty} t^{3k} B_{3k}^{(4)}(x) = \frac{N_{3,1}(tx)}{N_{3,1}(t)}$	$B_0^{(4)}(x) = x$	
M_4	$\sum_{k=0}^{\infty} t^{3k} A_{3k}^{(4)}(x) = N_{3,0}(tx) - \frac{N_{3,0}(t)\, N_{3,1}(tx)}{N_{3,1}(t)}$	$A_0^{(4)}(x) = 1 - x$	$2K_1^{(4)}(x,t) = \begin{cases} (x-t)^2 - x(1-t)^2, & 0 \leq t < x \leq 1 \\ -x(1-t)^2, & 0 \leq x \leq t \leq 1 \end{cases}$
	$\sum_{k=0}^{\infty} t^{3k+2} A_{3k+2}^{(4)}(x) = N_{3,2}(tx) - \frac{N_{3,2}(t)\, N_{3,1}(tx)}{N_{3,1}(t)}$	$A_0^{(4)}(x) = \frac{x(x-1)}{2}$	

Table III (continued)

Matrix	Generating functions of fundamental polynomials	Fundamental polynomials for k = 0	Kernel
M_5	$\sum_{k=0}^{\infty} t^{3k+1} B_{3k+1}^{(5)}(x) = \dfrac{N_{3,2}(tx)}{N_{3,1}(t)}$	$B_1^{(5)}(x) = \dfrac{x^2}{2}$	$2K_1^{(5)}(x,t) = \begin{cases} (x-t)^2 - x^2(1-t), & 0 \leq t < x \leq 1 \\ -x^2(1-t), & 0 \leq x \leq t \leq 1 \end{cases}$
	$\sum_{k=0}^{\infty} t^{3k} A_{3k}^{(5)}(x) = N_{3,0}(tx) - \dfrac{N_{3,2}(t)\, N_{3,2}(tx)}{N_{3,1}(t)}$	$A_0^{(5)}(x) = 1$	
	$\sum_{k=0}^{\infty} t^{3k+1} A_{3k+1}^{(5)}(x) = N_{3,1}(tx) - \dfrac{N_{3,0}(t)\, N_{3,2}(tx)}{N_{3,1}(t)}$	$A_1^{(5)}(x) = \dfrac{x(2-x)}{2}$	
M_6	$\sum_{k=0}^{\infty} t^{3k} B_{3k}^{(6)}(x) = \dfrac{N_{3,2}(tx)}{N_{3,2}(t)}$	$B_0^{(6)}(x) = x^2$	$2K_1^{(6)}(x,t) = \begin{cases} t^2(1-x^2) - 2tx(1-x), & 0 \leq t < x \leq 1 \\ -x^2(1-t)^2, & 0 \leq x \leq t \leq 1 \end{cases}$
	$\sum_{k=0}^{\infty} t^{3k} A_{3k}^{(6)}(x) = N_{3,0}(tx) - \dfrac{N_{3,0}(t)\, N_{3,2}(tx)}{N_{3,2}(t)}$	$A_0^{(6)}(x) = 1-x^2$	
	$\sum_{k=0}^{\infty} t^{3k+1} A_{3k+1}^{(6)}(x) = N_{3,1}(tx) - \dfrac{N_{3,1}(t)\, N_{3,2}(tx)}{N_{3,2}(t)}$	$A_1^{(6)}(x) = x(1-x)$	