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INVARIANT MEANS ON ALMOST PERIODIC FUNCTIONS AND FIXED POINT PROPERTIES

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1. **Introduction.** Consider on a topological semigroup S the following fixed point properties:

(F) For any separately continuous, equicontinuous and affine action of S on a compact convex subset K of a separated locally convex space, K has a common fixed point for S.

(G) For any separately continuous and nonexpansive action of S on a compact subset K of a separated locally convex space, K has a common fixed point for S.

Recently, Holmes and the author have proved in [**10**, Corollary 1] that if S is *left reversible* (i.e., any two nonempty closed right ideals of S have nonvoid intersection; see [**1**, p. 34]), then S has property (G). For discrete left reversible semigroups, this latter result is due to T. Mitchell [**14**]; the implication was first proved by De Marr in [**6**, p. 1139] for commuting semigroups and then by W. Takahashi [**16**, p. 384] for discrete left amenable semigroups (i.e., the space of bounded real valued functions on the semigroup has a left invariant mean; see Day [**2**]).

A well-known theorem of Kakutani [**7**, p. 457] shows that if S is a group, then S has property (F). This result has also been generalised recently by Šneperman [**19**] and [**20**] to the class of left reversible discrete semigroups.

Note that, as known, any commuting semigroup is left amenable (see Day [**2**, p. 516]) and any left amenable discrete semigroup is left reversible (see Granirer [**8**, p. 371]).

The main purpose of this paper is to show that, for any topological semigroup S, the existence of a left invariant mean on $AP(S)$, the space of strongly almost periodic functions on S, is equivalent to *each* of the two fixed point properties (F) and (G).

Since if S is left reversible then $AP(S)$ has a left invariant mean (note that the converse is false; see [**10**, §4]), it follows that our result generalises Šneperman's fixed point theorem in [**7**, p. 457] and a fixed

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point theorem of Holmes and the author in [**10**, Corollary 1].

We also show that $AP(S)$ has a *multiplicative* left invariant mean if and only if whenever S is a separately continuous and equicontinuous action on a compact Hausdorff space X, X has a common fixed point for S.

2. **Preliminaries and notations.** For any set A, 1_A will denote the characteristic function on A and $|A|$ will denote the cardinality of A.

If A is a subset of a topological space E, then $\bar{A}$ will denote the closure of A in E. If in addition E is a linear topological space, then [$\overline{\text{co}}A$] co$A$ will denote the [closed] convex hull of A in E.

Throughout this paper, S will denote a *topological semigroup*; that is, S is a semigroup with a Hausdorff topology such that, for each $a \in S$, the two mappings from S into S defined by $s \to as$ and $s \to sa$ for all $s \in S$ are continuous.

An *action* of S on a topological space X is a mapping ψ from $S \times X$ into X, denoted by $\psi(s, x) = s \cdot x$, $s \in S$ and $x \in X$, such that $(s_1 s_2) \cdot x = s_1 \cdot (s_2 \cdot x)$ for all s_1, $s_2 \in S$ and $x \in X$. The action is *separately co itinuous* if the mapping ψ is continuous in each of the variables when the other is kept fixed.

When X is a convex subset of a linear topological space, then an action of S on X is *affine* if for each $s \in S$, the mapping from X into X, defined by $x \to s \cdot x$ for all $x \in X$, is affine.

For any topological space X, let $C(X)$ be the space of bounded continuous real valued functions on X. Let A be a sup norm closed subspace of $C(X)$ containing constants, then an element $\phi \in A^*$, the conjugate space of A, is a *mean* if $\|\phi\| = \phi(1_X) = 1$. If in addition A is an algebra, then $\phi \in A^*$ is *multiplicative* if $\phi(fg) = \phi(f)\phi(g)$ for all $f, g \in A$.

For each $a \in S$, define the two mappings r_a, l_a from $C(S)$ into $C(S)$ by $r_a f(s) = f(sa)$ and $l_a f(s) = f(as)$ for all $s \in S$ and $f \in C(S)$. Let A be a translation invariant (i.e., $r_a(A) \subseteq A$ and $l_a(A) \subseteq A$ for all $a \in S$) sup norm closed subspace of $C(S)$ containing constants. Then a mean ϕ on A is a *left invariant mean* (denoted by LIM) if $\phi(l_a f) = \phi(f)$ for all $f \in A$ and $a \in S$.

A function $f \in C(S)$ is *strongly almost periodic* if $\mathcal{L}O(f) = \{l_a f; a \in S\}$ is relatively compact in the sup norm topology of $C(S)$. Then, as known [**5**, p. 80], $AP(S)$, the space of strongly almost periodic functions on S, is a sup norm closed translation invariant subalgebra of $C(S)$ containing constants. Furthermore, $f \in AP(S)$ if and only if $\mathcal{R}O(f) = \{r_a f; a \in S\}$ is relatively compact in the sup norm topology of $C(S)$.

3. **Equicontinuous actions.** An action of S on a compact Hausdorff space X is *equicontinuous* if, for each $y \in X$ and $U \in \mathcal{U}$, where $\mathcal{U}$ is the *unique* uniformity which determines the topology of X (see [11, p. 197]), there is a V in $\mathcal{U}$ such that $(sx, sy) \in U$ for all $s \in S$ whenever $(x, y) \in V$.

The following lemma is crucial to the rest of our work:

Lemma 3.1. *If the action of S on a compact Hausdorff space Y is separately continuous and equicontinuous and $y \in Y$, then $T_y(C(Y)) \subseteq AP(S)$, where $T_yf(s) = f(s \cdot y)$ for all $s \in S$ and $f \in C(Y)$.*

Proof. Let $f \in C(Y)$ be fixed. We first show that the mapping T from Y into $C(S)$ defined by $z \to T_zf$ for all $z \in Y$ is continuous when $C(S)$ has the sup norm topology. Let $z \in Y$ and $\epsilon > 0$. By compactness of Y, we may choose $U \in \mathcal{U}$, where $\mathcal{U}$ is the unique uniformity which determines the topology on Y, such that if $(x, y) \in U$ then $|f(x) - f(y)| < \epsilon/2$ (see [11, p. 198]). By equicontinuity of S on Y, there exists $V \in \mathcal{U}$ such that whenever $(x, z) \in V$, then $(sx, sz) \in U$ for all $s \in S$. Consequently if $y \in \{y \in Y; (y, z) \in V\}$, which is a neighbourhood of z, then

$$\sup\{|T_yf(s) - T_zf(s)|; s \in S\} < \epsilon.$$

To complete the proof, we let $y \in Y$ be fixed and $O(y) = \{sy; s \in S\}$. Then, for each $f \in C(Y)$, $\overline{O(T_yf)} = \{T_zf; z \in \overline{O(y)}\}$ is compact since $\overline{O(y)}$ is compact and T is continuous.

Theorem 3.2. *$AP(S)$ has a LIM if and only if*

(F) *for any separately continuous, equicontinuous and affine action of S on a compact convex subset K of a separated locally convex space, K has a common fixed point for S.*

Proof. Let ψ be a LIM on $AP(S)$ and let $A(K)$ be the closed subspace of $C(K)$ consisting of all real valued continuous affine functions on K. As known [2, p. 513], there is a net of finite means $\psi_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha p_{s_i^\alpha}$, $\lambda_i^\alpha > 0$, and $\sum_{i=1}^{n_\alpha} \lambda_i^\alpha = 1$, such that $\lim_\alpha \psi_\alpha(f) = \psi(f)$ for all $f \in AP(S)$, where $p_s(f) = f(s)$ for all $s \in S$. Let $y \in K$ be fixed and z be a cluster point of the net $\{\sum_{i=1}^{n_\alpha} \lambda_i^\alpha s_i^\alpha y\}$ in K. Then, for each $h \in A(K)$, we have $T_yh \in AP(S)$ (Lemma 3.1) and hence

$$h(s \cdot z) = \psi(l_s(T_yh)) = \psi(T_y(h)) = h(z)$$

for all $s \in S$, where the first equality follows by virtue of the affineness of S and h. Since $A(K)$ separates points (see [15, p. 31]), it follows that z is a fixed point for S.

Conversely, observe that the semigroup S acts linearly on $AP(S)^*$, by $s \to l_s^*$, and it leaves the set (K, weak*) of the means on $AP(S)$ invariant. To see that the action of S on K is separately continuous, it is sufficient to prove that $a \to l_a^*(m)(f) = m(l_a f)$ is continuous for each $m \in K$ and $f \in AP(S)$. Since $\mathcal{L}O(f)$ is norm relatively compact, the norm topology in $\mathcal{L}O(f)$ is the same as the topology of pointwise convergence. Since $a \to (l_a f)(t) = f(at)$ is continuous for each $t \in S$, the map $a \to l_a f$ is a continuous map $S \to (\mathcal{L}O(f)$, norm). Hence $a \to l_a^*(m)$ is continuous on S into (K, weak*). For each $f \in AP(S)$, let p_f be a pseudonorm on $AP(S)^*$ defined by $p_f(\phi) = \sup\{|\phi(l_a f)|,\ |\phi(f)|;\ a \in S\}$ for each $\phi \in AP(S)^*$, and let $Q = \{p_f;\ f \in AP(S)\}$. Then clearly the action of S on $AP(S)^*$ (and therefore on K) is equicontinuous with respect to the topology determined by Q. Since, on K, weak* topology agrees with the topology of uniform convergence on totally bounded subsets of $AP(S)$, Q determines the weak* topology on K. Hence the action of S on (K, weak*) is both affine and equicontinuous. Consequently any fixed point in K under this action is a left invariant mean on $AP(S)$.

Theorem 3.2 yields the following generalisation of Šneperman's fixed point theorem [**19**] and [**20**]:

COROLLARY 3.3. *If S is left reversible, then S has property* (F).

PROOF. Let $\bar{S}^a$ denote the strongly almost periodic compactification of S (see [**5**, p. 90]). Then S is left reversible implies that $\bar{S}^a$ is also left reversible. Consequently it follows from [**5**, Lemma 2.8] that $AP(S)$ has a LIM.

REMARK 3.4. The converse of Corollary 3.3 is certainly false, since there exist topological semigroups S such that $AP(S)$ (or even $C(S)$) has a LIM and yet S is not left reversible (see [**10**, §4]).

Our next result shows that when $AP(S)$ has a multiplicative LIM, then S has fixed property much stronger than (F) (compare with Mitchell [**13**, Theorems 1 and 3] and the author [**12**, Theorem 2.2]).

THEOREM 3.5. *Let n be a positive integer. Then $AP(S)$ has a LIM of the form $(1/n)\sum_{i=1}^{n}\phi_i$, where each ϕ_i is a multiplicative mean on $AP(S)$, if and only if*

Q(n) whenever S is a separately continuous and equicontinuous action on a compact Hausdorff space X, there exists a nonempty finite subset $F \subseteq X$, $|F| \leqq n$, $|F|$ divides n, such that $s \cdot F = F$ for all $s \in S$.

PROOF. Let $y \in X$. By Lemma 3.1, we may define, for each $i = 1, \cdots, n$, a multiplicative mean on $C(X)$ by $\psi_i(f) = \phi_i(T_y f)$ for all $f \in C(X)$. By compactness of X, there exists $x_i \in X$ such that

$\psi_i(f) = f(x_i)$ for all $f \in C(X)$, $i = 1, \cdots, n$ (see [7, p. 278]). Let Y be the set of distinct elements in $\{x_1, \cdots, x_n\}$; it follows easily from the invariance of $(1/n) \sum_{i=1}^n \phi_i$ that $s \cdot Y = Y$ for all $s \in S$.

Let H be the set of distinct elements in $\{\phi_1, \cdots, \phi_n\}$. Since finite subsets in the set of multiplicative means of $AP(S)$ are linearly independent, it follows that $L_s H = H$ for all $s \in S$, where $L_s\phi(f) = \phi(l_s f)$ for all $\phi \in H$, $f \in AP(S)$. Let G be the factor semigroup of S defined by the equivalence relation (E): a (E) b if and only if $L_a\phi = L_b\phi$ for all $\phi \in H$. Then G may be regarded as a finite group of transformations from Y onto Y defined by $\bar{s}y = sy$ for all $s \in S$, where $\bar{s}$ is the homomorphic image of s in G. Let $F = \{sx_1; s \in S\}$. Then $sF = F$ for all $s \in S$, and $|F|$ divides $|G|$. Consequently, if we can show that

$$(*) \qquad |G| \text{ divides } n$$

then F is the required invariant subset of X.

To prove $(*)$, we let $S_1, \cdots, S_m$ be the distinct cosets of S by (E). Then, as easily seen, each S_i is an open and closed subset of S and $O(1_{S_i}) = \{1_{S_j}; j = 1, \cdots, m\}$. Hence $1_{S_i} \in AP(S)$ for each $i = 1, \cdots, m$. For each $f \in C(G)$, define $\pi f(s) = f(\bar{s})$ for all $s \in S$. Then $\psi_i = \pi^*\phi_i$, $i = 1, \cdots, m$, are multiplicative means on $C(G)$ and $\psi = (1/n) \sum_{E_1}^{n} \psi_i$ is even a LIM. Hence, there exists $g_i \in G$, such that $\psi(f) = (1/n) \sum_{i=1}^n f(g_i)$ for all $f \in C(G)$. On the other hand, we have, by the uniqueness of LIM on $C(G)$, that $\psi(f) = (1/|G|) \sum \{f(g); g \in G\}$ for all $f \in C(G)$. Consequently, $|G|$ divides n.

To prove the converse, we consider the equicontinuous action of S on (X, weak*) the set of multiplicative means of $AP(S)$ defined by $(s, m) \to l_s^*(m)$ (see the proof of Theorem 3.2). Let F be a nonempty finite subset of X such that $l_s^*(F) = F$ for all $s \in S$ and $|F|$ divides n. Then $(k/n) \sum \{\phi; \phi \in F\}$, where $k \cdot |F| = n$, is a LIM on $AP(S)$.

For $n = 1$, we have

THEOREM 3.6. *$AP(S)$ has multiplicative LIM if and only if whenever S is a separately continuous and equicontinuous action on a compact Hausdorff space, X has a common fixed point for* S.

EXAMPLE. Let T be a regular Hausdorff topological space such that $C(T)$ consists of only constant functions (see [9]). Define on T the multiplication $a \cdot b = a$ for any $a, b \in T$. If G is a finite group of n elements and S is the product topological semigroup $T \times G$, with product topology, then it is easy to see that $AP(S) = C(S)$ has a LIM of the form $(1/n) \sum_1^n \phi_i$, where $\{\phi_1, \phi_2, \cdots, \phi_n\}$ are *distinct* multiplicative means on $AP(S)$ (see [18, Proposition 6.4]).

4. **Nonexpansive actions.** In this section we shall be concerned with a special kind of equicontinuous (but *not* necessarily affine) action on a compact convex subset of a locally convex space E, namely, the nonexpansive actions.

Let Q denote a (fixed) family of continuous seminorms on E which determine the topology of E. Then an action of S on X is Q-nonexpansive if $p(s \cdot x - s \cdot y) \leqq p(x - y)$ for all $s \in S$, $x, y \in X$ and $p \in Q$.

THEOREM 4.1. *$AP(S)$ has LIM if and only if*

(G) *whenever S is a separately continuous and Q-nonexpansive action on a compact convex subset K of a separated locally convex space E, K has a common fixed point for S.*

PROOF. Assume that $AP(S)$ has a LIM ψ. By Zorn's lemma, there exists a nonempty compact convex subset X of K which is minimal with respect to being closed, convex and invariant under each element of S. A second application of Zorn's lemma shows that there exists a nonempty subset F of X which is minimal with respect to being closed and invariant under each element of S. Let $y \in F$. Using Lemma 3.1, we may define a mean ϕ on $C(F)$ by $\phi(f) = \psi(T_y f)$ for all $f \in C(F)$. Then, as readily checked, $\phi({}_sf) = \phi(f)$ for all $s \in S$ and $f \in C(F)$, where ${}_sf(x) = f(s \cdot x)$ for all $x \in F$. Furthermore, using Riesz Representation Theorem, the functional ϕ defines a (regular) probability measure μ on F such that $\mu(A) = \mu(a^{-1}A)$ for all $a \in S$, and for each Borel subset A of F. Let $\mathcal{G}$ be the family of all closed subsets A of F such that $\mu(A) = 1$, and let $F_0 = \bigcap \mathcal{G}$ which is nonempty. If $A \in \mathcal{G}$ and $s \in S$, then $s^{-1}A \in \mathcal{G}$. Hence $s^{-1}F_0 \supset F_0$ or $F_0 \supset sF_0$. By minimality of F, $F = F_0$. Since $\mu(aF) = \mu(a^{-1}(aF)) = \mu(F) = 1$, $aF \in \mathcal{G}$ for each $a \in S$. Therefore $F \supset aF \supset F_0 = F$; hence $aF = F$.

We now follow an idea similar to that in [**6**, Lemma 2]. If F consists of only one point, we are done. Otherwise, there exists a continuous seminorm p in Q such that $r = \sup\{p(x - y); x, y \in F\} > 0$. Then, as known (see De Marr [**6**, Lemma 1] replacing the norm by p), there exists $\mu \in \overline{\text{co}}\, F$ such that

$$r_0 = \{\sup p(\mu - x); x \in F\} < r.$$

Let $X_0 = X \cap (\bigcap \{B_p[x, r_0]; x \in F\})$, where

$$B_p[x, r_0] = \{y \in F; p(y - x) \leqq r_0\}.$$

Then $\mu \in X_0$ and X_0 is a nonempty closed convex *proper* subset of X. Furthermore, if $x \in X_0$, then $x \in X$ and $F \subseteq B_p[x, r_0]$. Hence, for any $a \in S$, $F = aF \subseteq B_p[s \cdot x, r_0]$ by nonexpansiveness of S on X. It follows that $sX_0 \subseteq X_0$ for all $s \in S$, contradicting the minimality of X. Consequently, F must consist of a single point.

The proof of the converse is identical to that of Theorem 3.2 noting that the linear action of S on (K, weak*) the set of means on $AP(S)$ is even Q-nonexpansive.

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