ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 3, Number 1, Winter 1973

ON THE SIGN OF THE GREEN'S FUNCTION BEYOND THE INTERVAL OF DISCONJUGACY[1]

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1. **Introduction.** We are concerned with the nth order quasi differential equation

$$(\mathrm{E}_n) \qquad L_n[y] \equiv (D_{n-1}y)' + \sum_{i=1}^{n-1} f_{ni} D_{i-1} y = 0$$

where the quasi derivatives are given by

$$D_0 y = y, \qquad D_1 y = f_{12}^{-1} y',$$

$$D_i y = \frac{1}{f_{i,i+1}} \left[(D_{i-1}y)' + \sum_{j=1}^{i-1} f_{ij} D_{j-1} y \right],$$

$i = 2, \cdots, n-1$. We assume that the functions $f_{ij}(x)$, $i, j = 1, \cdots, n$, are continuous on $(-\infty, \infty)$, $f_{ij}(x) \equiv 0$, if $i + j$ is even or $j > i + 1$, $f_{i,i+1}(x) > 0$ on $(-\infty, \infty)$ for $i = 1, \cdots, n-1$. This nth order canonical form (Zettl [1]) was introduced by J. H. Barrett [2] for $n = 3$ and $n = 4$. If $[\alpha, \beta]$ is an interval of disconjugacy, the sign of the Green's function for the multi-point boundary value problems of de la Vallée Poussin is well known [3]. For the classical third order linear differential equation, the Green's function for either of the two point boundary value problems of de la Vallée Poussin conserves its sign if and only if $[\alpha, \beta]$ is an interval of disconjugacy [4]. The main result of Aliev [5, Theorem 4] is to show that, for the classical fourth order linear differential equation, it is not necessary that $[\alpha, \beta]$ be an interval of disconjugacy in order for the Green's function for either the (3, 1)- or (1, 3)-problem to conserve its sign. In particular he shows that if

$$\alpha < \beta < \min[r_{31}(\alpha), r_{22}(\alpha)] \quad \{\alpha < \beta < \min[r_{13}(\alpha), r_{22}(\alpha)]\}$$

($r_{ij}(\alpha)$ is defined in §2), then the Green's function for the (3, 1)-problem {(1, 3)-problem} is negative in the open square $(\alpha, \beta) \times (\alpha, \beta)$. His

Received by the editors March 17, 1971 and, in revised form, August 19, 1971.
AMS (MOS) subject classifications (1970). Primary 34A40, 34C10; Secondary 34B05.
[1]This work was supported in part by NSF grant GP-17321.

proof does not extend to higher order equations. The main result of this paper is to generalize this result of Aliev's to the nth order case for the equation (E_n).

2. **Preliminaries and main results.** An adjoint differential equation [1] to (E_n) is

$$(E_n{}^+) \qquad L_n{}^+[y] = (D_{n-1}^+ y)' + \sum_{i=1}^{n-1} f_{n+1-i,1} D_{i-1}^+ y$$

where

$$D_0{}^+ y = y, \qquad D_1{}^+ y = f_{n-1,n}^{-1} y',$$

$$D_i{}^+ y = \frac{1}{f_{n-i,n+1-i}} \left[(D_{i-1}^+ y)' + \sum_{j=1}^{i-1} f_{n+1-j,n+1-i} D_{j-1}^+ y \right],$$

$i = 2, \cdots, n-1$.

We say that $u(x)$ has a zero of multiplicity at least p at α provided $D_i u(\alpha) = 0$, $i = 0, \cdots, p-1$. A solution y of (E_n) is said to have an (i, j)-pair of zeros, $1 \leqq i, j \leqq n$, on $[t, b]$ provided there are numbers α, β such that $t \leqq \alpha < \beta \leqq b$ and y has a zero of order at least i at α and at least j at β. For each t, the extended real number $r_{ij}(t)$, $1 \leqq i$, $j \leqq n$, $i + j = n$, is the infimum of the set of $b \in (t, \infty)$ such that there is a nontrivial solution of (E_n) having an (i, j)-pair of zeros on $[t, b]$. For convenience we use the notation $r_{n0}(t) = \infty$ and $r_{0n}(t) = \infty$. The functions $r_{ij}^+(t)$ are defined similarly for the adjoint equation $(E_n{}^+)$. It is easy to see that if $t \leqq \alpha < \beta < r_{ij}(t) \leqq \infty$, then there is a unique solution of (E_n) satisfying

$$D_p y(\alpha) = A_p, \qquad D_q y(\beta) = B_q,$$

$p = 0, \cdots, i-1$, $q = 0, \cdots, j-1$, where A_p and B_q are constants. We define a fundamental set of solutions $\{u_j(x, t)\}$, $j = 0, \cdots, n-1$, of (E_n) by the initial conditions at $x = t$,

$$D_i u_j(t, t) = \delta_{ij}, \qquad i, j = 0, \cdots, n-1.$$

A fundamental set of solutions $\{u_j{}^+(x, t)\}$, $j = 0, \cdots, n-1$, of the adjoint equation $(E_n{}^+)$ is similarly defined. Dolan [6] proved that

$$(1) \quad D_\alpha u_\beta(s, t) = (-1)^{\alpha+\beta} D_{n-\beta-1}^+ u_{n-\alpha-1}^+(t, s), \qquad \alpha, \beta = 0, \cdots, n-1,$$

for $n = 3$. His proof is valid for the nth order case. The author [7], [8] used this result extensively for $n = 4$.

We define the following "Wronskians"

$$W[u_{n-k}(s,t), \cdots, u_{n-1}(s,t)] = \det(D_q u_p(s,t))$$

and

$$W^+[u^+_{n-k}(t,s), \cdots, u^+_{n-1}(t,s)] = \det(D_q{}^+ u_p{}^+(t,s)),$$

$q = 0, \cdots, k-1$, $p = n-k, \cdots, n-1$. One can show by direct use of (1) that, for $1 \leqq k \leqq n-1$,

$$(2) \qquad \begin{aligned} &W[u_{n-k}(s,t), \cdots, u_{n-1}(s,t)] \\ &\qquad = (-1)^{(n-k)k} W^+[u^+_{n-k}(t,s), \cdots, u^+_{n-1}(t,s)]. \end{aligned}$$

The author has noticed that Hinton (Theorem 4.2 of [**9**]) verifies (2) in a different manner. It follows immediately from (2) that

$$(3) \qquad r_{\alpha\beta}(t) = r^+_{\beta\alpha}(t)$$

for $\alpha, \beta = 1, \cdots, n-1$, $\alpha + \beta = n$. Another interesting identity (for $n = 4$ see (3.26) of [**2**]) we will need is

$$(4) \qquad u^+_{n-1}(s,t) = W[u_1, \cdots, u_{n-1}](s,t).$$

We are now ready to state and prove our main results. In the following theorem $G_{n-1,1}(x,s)$ is the Green's function for the problem

$$L_n[y] = 0, \qquad D_p y(\alpha) = 0 = y(\beta),$$

$p = 0, \cdots, n-2$.

THEOREM 1. *If* $\alpha < \beta < \min[r_{n-1,1}(\alpha), r_{n-2,2}(\alpha)]$, *for* $n \geqq 3$, *then* $G_{n-1,1}(x,s) < 0$ *for* $x, s \in (\alpha, \beta)$.

PROOF. Let $G(x,s) \equiv G_{n-1,1}(x,s)$, then (see [**10**, (2.6), p. 191])

$$G(x,s) = K(x,s) + \sum_{j=0}^{n-1} c_j(s) u_j(x,\alpha)$$

where (see [**10**, (2.4), p. 190])

$$K(x,s) = \begin{cases} u_{n-1}(x,s), & \alpha \leqq s \leqq x \leqq \beta, \\ 0, & \alpha \leqq x \leqq s \leqq \beta. \end{cases}$$

Since G as a function of x satisfies the boundary conditions, we obtain $c_j(s) = 0$, $j = 0, \cdots, n-2$, and $c_{n-1}(s) = -u_{n-1}(\beta,s)/u_{n-1}(\beta,\alpha)$. Hence one can easily show that

$$G(x,s) = \frac{u_{n-1}(\beta,s)u_{n-1}(x,\alpha)}{u_{n-1}(\beta,\alpha)}, \qquad \alpha \leqq x \leqq s \leqq \beta,$$

(5)

$$= \frac{1}{u_{n-1}(\beta,\alpha)} \begin{vmatrix} u_{n-1}(\beta,\alpha) & u_{n-1}(\beta,s) \\ u_{n-1}(x,\alpha) & u_{n-1}(x,s) \end{vmatrix}, \qquad \alpha \leqq s \leqq x \leqq \beta.$$

Since $r_{n-1,1}(\alpha) > \beta$, $G(x,s) < 0$ for $\alpha < x \leqq s < \beta$. It is easy to see that to complete the proof of this theorem it suffices to show that

$$f(x) = \begin{vmatrix} u_{n-1}(\beta,\alpha) & u_{n-1}(\beta,s) \\ u_{n-1}(x,\alpha) & u_{n-1}(x,s) \end{vmatrix}$$

is different from zero for $s < x < \beta$. Assume the contrary, i.e., there is a point τ, $s < \tau < \beta$ such that $f(\tau) = 0$. Let

$$u(x) = u_{n-1}(\beta,\alpha)u_{n-1}(x,s)$$

and $v(x) = u_{n-1}(\beta,s)u_{n-1}(x,\alpha)$. Then $((u-v)/u)(\tau) = ((u-v)/u)(\beta) = 0$ and so by Rolle's theorem there is $\xi \in (\tau,\beta)$ with $((u-v)/u)'(\xi) = ((u'v-uv')/u^2)(\xi) = 0$. It follows that

$$\begin{vmatrix} u_{n-1}(\xi,s) & u_{n-1}(\xi,\alpha) \\ D_1u_{n-1}(\xi,s) & D_1u_{n-1}(\xi,\alpha) \end{vmatrix} = 0. \tag{6}$$

It follows from (1) that

$$\begin{vmatrix} u^+_{n-1}(s,\xi) & u^+_{n-1}(\alpha,\xi) \\ u^+_{n-2}(s,\xi) & u^+_{n-2}(\alpha,\xi) \end{vmatrix} = 0.$$

This implies there is a solution $w(x)$ of $(\mathrm{E}_n{}^+)$ which has a zero of order $n-2$ at ξ, a zero at s, and a zero at α. Since, by (3), $r^+_{1,n-1}(\alpha) = r_{n-1,1}(\alpha) > \xi$, $u^+_{n-1}(x,\xi) \neq 0$ for $\alpha \leqq x < \xi$. By Lemma 1.2 of [11] there is a nontrivial combination of $u^+_{n-1}(x,\xi)$ and $w(x)$ with a double zero in (α,s) and a zero at ξ of order $n-2$. This contradicts $r^+_{2,n-2}(\alpha) = r_{n-2,2}(\alpha) > \beta$.

Remark. The above theorem is true if the hypothesis is replaced by $\beta \leqq r_{n-2,2}(\alpha) < r_{n-1,1}(\alpha)$. There is an interesting analogy between Theorem 1 and the famous problem of Chaplygin. If $r_{n-1,1}(\alpha) > \beta$, $u^{(k)}(\alpha) = v^{(k)}(\alpha)$, $k = 0, 1, \cdots, n-1$, $L[u] \geqq L[v]$, where L is the classical nth order operator on $[\alpha,\beta]$, then $u(t) \geqq v(t)$ in $[\alpha,\beta]$ (a corresponding statement holds for (E_n)). This follows immediately from the statement $r_{n-1,1}(\alpha) > \beta$ implies $G_{n0}(x,s) \geqq 0$ for $\alpha \leqq s \leqq x < \beta$ where $G_{n0}(x,s)$ is the Green's function (Cauchy function) for the initial value problem at α. It follows from Theorem 1 that, if β satisfies the hypothesis of Theorem 1, $L_n[u] \geqq L_n[v]$ on $[\alpha,\beta]$, and

$$D_i u(\alpha) = D_i v(\alpha), \qquad u(\beta) = v(\beta), \qquad i = 0, \cdots, n-2,$$

then $u(x) \leqq v(x)$ on $[\alpha, \beta]$.

Let $G_{1,n-1}(x, s)$ be the Green's function for the problem

$$L_n[y] = 0, \qquad y(\alpha) = 0 = D_p y(\beta),$$

$p = 0, \cdots, n-2$.

THEOREM 2. *If* $\alpha < \beta < \min[r_{2,n-2}(\alpha), r_{1,n-1}(\alpha)]$, *then* $\operatorname{sgn} G_{1,n-1}(x, s) = (-1)^{n+1}$ *for* $x, s \in (\alpha, \beta)$.

PROOF. Let $G(x, s) \equiv G_{1,n-1}(x, s)$, then (see [**10**, (2.6), p. 191]) $G(x, s) = K(x, s) + \sum_{j=0}^{n-1} c_j(s) u_j(x, \alpha)$ where (see [**10**, (2.4), p. 190])

$$K(x, s) = \begin{cases} u_{n-1}(x, s), & \alpha \leqq s \leqq x \leqq \beta, \\ 0, & \alpha \leqq x \leqq s \leqq \beta. \end{cases}$$

Since $G(\alpha, s) = 0$, it follows that $c_0(s) = 0$. Also since G as a function of x has a zero of order $n-1$ at β, we must have

$$\begin{gathered} c_1(s) u_1(\beta, \alpha) + \cdots + c_{n-1}(s) u_{n-1}(\beta, \alpha) = -u_{n-1}(\beta, s), \\ c_1(s) D_1 u_1(\beta, \alpha) + \cdots + c_{n-1}(s) D_1 u_{n-1}(\beta, \alpha) = -D_1 u_{n-1}(\beta, s), \\ \cdots \qquad \cdots \qquad \cdots \\ c_1(s) D_{n-2} u_{n-2}(\beta, \alpha) + \cdots + c_{n-1}(s) D_{n-2} u_{n-1}(\beta, \alpha) = -D_{n-2} u_{n-1}(\beta, s). \end{gathered}$$

If we solve for $c_j(s)$, $j = 1, \cdots, n-1$, and set

$$\Gamma(x, s) = W[u_1, \cdots, u_{n-1}](\beta, \alpha) G(x, s),$$

then

$$\Gamma(x, s) = \begin{vmatrix} u_{n-1}(x, s) & u_1(x, \alpha) & \cdots & u_{n-1}(x, \alpha) \\ u_{n-1}(\beta, s) & u_1(\beta, \alpha) & \cdots & u_{n-1}(\beta, \alpha) \\ D_1 u_{n-1}(\beta, s) & D_1 u_1(\beta, \alpha) & \cdots & D_1 u_{n-1}(\beta, \alpha) \\ \cdots & \cdots & & \cdots \\ D_{n-2} u_{n-1}(\beta, s) & D_{n-2} u_1(\beta, \alpha) & \cdots & D_{n-2} u_{n-1}(\beta, \alpha) \end{vmatrix}$$

for $\alpha \leqq s \leqq x \leqq \beta$ and for $\alpha \leqq x \leqq s \leqq \beta$, $\Gamma(x, s)$ is the above determinant with its upper left-hand element replaced by zero. Since $W[u_1, \cdots, u_{n-1}](\beta, \alpha) = u_{n-1}^+(\beta, \alpha)$ (see (4)) and $u_{n-1}^+(\beta, \alpha) > 0$ (see (3)), it suffices to show that $\operatorname{sgn} \Gamma(x, s) = (-1)^{n+1}$ for $x, s \in (\alpha, \beta)$.

Let s be fixed and consider the case $s \leqq x \leqq \beta$. For these values of x we have, since G as a function of x has a zero of order $n-1$ at β (see [**10**, (IV), p. 192]), that $\Gamma(x, s) = Au_{n-1}(x, \beta)$ where A is a nonzero constant. Since $r_{1,n-1}(\alpha) > \beta$, $\Gamma(x, s)$ is of one sign for $s \leqq x < \beta$. Note that

$$D_{n-1}\Gamma(\beta, s) = (-1)^{n+1}W[u_{n-1}(\beta, s), u_1(\beta, \alpha), \cdots, u_{n-1}(\beta, \alpha)].$$

It follows from (1) that

$$W[u_{n-1}(\beta, s), u_1(\beta, \alpha), \cdots, u_{n-1}(\beta, \alpha)]$$

$$= (-1)^{n-1}\begin{vmatrix} u^+_{n-1}(s, \beta) & D^+_{n-2}u^+_{n-1}(\alpha, \beta) & \cdots & u^+_{n-1}(\alpha, \beta) \\ u^+_{n-2}(s, \beta) & D^+_{n-2}u^+_{n-2}(\alpha, \beta) & \cdots & u^+_{n-2}(\alpha, \beta) \\ \cdots & \cdots & & \cdots \\ u_0{}^+(s, \beta) & D^+_{n-2}u_0{}^+(\alpha, \beta) & \cdots & u_0{}^+(\alpha, \beta) \end{vmatrix}$$

Since $r^+_{n-1,1}(\alpha) = r_{1,n-1}(\alpha) > s$, $D_{n-1}\Gamma(\tau, s) \neq 0$ for all τ. From the continuous dependence of solutions on the initial point, $f(\tau) = D_{n-1}\Gamma(\tau, s)$ is of one sign. Hence

$$\operatorname{sgn} D_{n-1}\Gamma(\beta, s) = (-1)^{n+1}(-1)^{n-1} \operatorname{sgn} u^+_{n-1}(s, \alpha) = +1.$$

Hence $\operatorname{sgn} G(x, s) = (-1)^{n+1}$ for $\alpha < s \leqq x < \beta$. To complete the proof of this theorem it suffices to show that $G(x, s) \neq 0$ for $\alpha < x < s$. Note that, for $\alpha \leqq x \leqq s \leqq \beta$,

$$\Gamma(x, s) = \begin{vmatrix} 0 & u_1(x, \alpha) & \cdots & u_{n-1}(x, \alpha) \\ u_{n-1}(\beta, s) & u_1(\beta, \alpha) & \cdots & u_{n-1}(\beta, \alpha) \\ \cdots & \cdots & & \cdots \\ D_{n-2}u_{n-1}(\beta, s) & D_{n-2}u_1(\beta, \alpha) & \cdots & D_{n-2}u_{n-1}(\beta, \alpha) \end{vmatrix}$$

$$= \begin{vmatrix} u_{n-1}(x, s) & u_1(x, \alpha) & \cdots & u_{n-1}(x, \alpha) \\ u_{n-1}(\beta, s) & u_1(\beta, \alpha) & \cdots & u_{n-1}(\beta, \alpha) \\ \cdots & \cdots & & \cdots \\ D_{n-2}u_{n-1}(\beta, s) & D_{n-2}u_1(\beta, \alpha) & \cdots & D_{n-2}u_{n-1}(\beta, \alpha) \end{vmatrix}$$

$$- W[u_1, \cdots, u_{n-1}](\beta, \alpha)u_{n-1}(x, s).$$

Hence, for $\alpha \leqq x \leqq s \leqq \beta$, s fixed,

$$G(x, s) = Bu_{n-1}(x, \beta) - u_{n-1}(x, s)$$

where B is a constant. Hence if $G(x_0, s) = 0$ for $\alpha < x_0 < s$ then one can show in a manner similar to the proof of Theorem 1 that (see (6))

$$\begin{vmatrix} u_{n-1}(\xi, s) & u_{n-1}(\xi, \beta) \\ D_1 u_{n-1}(\xi, s) & D_1 u_{n-1}(\xi, \beta) \end{vmatrix} = 0$$

where $\alpha < \xi < s$. It then follows from (1) that

$$\begin{vmatrix} u^+_{n-2}(s, \xi) & u^+_{n-1}(s, \xi) \\ u^+_{n-2}(\beta, \xi) & u^+_{n-1}(\beta, \xi) \end{vmatrix} = 0.$$

Hence there is a nontrivial solution of $(E_n{}^+)$ with a zero of order $n-2$ at ξ, a zero at s, and a zero at β. We then proceed as in the proof of Theorem 1 that eventually leads to the contradiction of $r^+_{n-2,2}(\alpha) = r_{2,n-2}(\alpha) > \beta$.

REMARK. Theorem 2 is valid if the hypothesis is replaced by $\beta \leqq r_{2,n-2}(\alpha) < r_{1,n-1}(\alpha)$.

REMARK. It follows from Theorem 2, that if

$$\alpha < \beta < \min[r_{2,n-2}(\alpha), r_{1,n-1}(\alpha)],$$

$$L_n[u] \geqq L_n[v] \quad \text{on } [\alpha, \beta]$$

and

$$u(\alpha) = v(\alpha), \qquad D_i u(\beta) = D_i v(\beta), \qquad i = 0, \cdots, n-2,$$

then $(-1)^n u(x) \leqq (-1)^n v(x)$ on $[\alpha, \beta]$.

The assumption in Theorem 1, that $r_{n-1,1}(t) > \beta$, ensures the existence of $G_{n-1,1}(x, s)$. One wonders, however, if the assumption $r_{n-2,2}(t) > \beta$ is needed. Theorem 3 shows that you cannot remove this hypothesis in general. Azbelev, Hohryakov and Caljuk [4] proved that, for the classical third order linear differential equation, if $r_{12}(\alpha) < \beta < r_{21}(\alpha)$ $\{r_{21}(\alpha) < \beta < r_{12}(\alpha)\}$, then the Green's function $G_{21}(x, s)$ $\{G_{12}(x, s)\}$ changes sign on $(\alpha, \beta) \times (\alpha, \beta)$. Their result can easily be shown to be valid for (E_3). We now prove a corresponding result for (E_n).

THEOREM 3. *If* $r_{n-2,2}(\alpha) < \rho(\alpha) \equiv \min[r_{n-1,1}(\alpha), r_{n-3,3}(\alpha)]$, $n \geqq 3$, *then there is a* $\beta \in (r_{n-2,2}(\alpha), \rho(\alpha))$ *such that the Green's function* $G_{n-1,1}(x, s)$ *changes sign in* $(\alpha, \beta) \times (\alpha, \beta)$.

PROOF. From (5) we have, for $\alpha < \beta < r_{n-1,1}(\alpha)$, that

$$\frac{1}{f_{12}(\beta)} \frac{\partial G_{n-1,1}(x, s)}{\partial x}\Bigg|_{x=\beta}$$

$$= \frac{1}{u_{n-1}(\beta, \alpha)} \begin{vmatrix} u_{n-1}(\beta, \alpha) & u_{n-1}(\beta, s) \\ D_1 u_{n-1}(\beta, \alpha) & D_1 u_{n-1}(\beta, s) \end{vmatrix}, \qquad \alpha \leqq s \leqq \beta.$$

If we let $u(s)$ be this above expression, then, by using (1), we obtain

$$u(s) = (-1)^n \left[u^+_{n-2}(s, \beta) + \frac{D_1 u_{n-1}(\beta, \alpha)}{u_{n-1}(\beta, \alpha)} u^+_{n-1}(s, \beta) \right].$$

Hence $u(s)$ is a solution to the adjoint equation $(E_n{}^+)$ and

$$u(\alpha) = 0, \qquad D_i{}^+ u(\beta) = 0, \qquad i = 0, \cdots, n-3.$$

Now assume $\beta \in (r_{n-2,2}(\alpha), \rho(\alpha))$, then (see [12]), for those $t \in [\alpha, \rho(\alpha))$ for which $r_{n-2,2}(t) < \rho(\alpha)$, $r_{n-2,2}(t)$ is a continuously differentiable strictly increasing function and its range contains $[r_{n-2,2}(\alpha), \rho(\alpha))$. Since $r_{n-2,2}(t) = r^+_{2,n-2}(t)$, there is a $\tau \in (\alpha, \beta)$ such that $r^+_{2,n-2}(\tau) = \beta$. Hence there is a nontrivial solution $w(s)$ of $(E_n{}^+)$ with a zero of order at least 2 at τ and at least $n - 2$ at β. Since $r_{n-1,1}(t) \equiv r^+_{1,n-1}(t)$ and $r_{n-3,3}(t) \equiv r^+_{3,n-3}(t)$ and $\beta < \rho(\alpha)$, there are no nontrivial solutions of $(E_n{}^+)$ with a $(1, n-1)$- or $(3, n-3)$-pair of zeros on $[\alpha, \beta]$. Hence the zeros of $w(s)$ at τ and β are of exactly order 2 and $n - 2$, respectively. Let $\{\beta_n\}$ be a sequence of points in $(r_{n-2,2}(\alpha), \rho(\alpha))$ with limit $r_{n-2,2}(\alpha)$. Let $w_n(s)$ and τ_n correspond to $w(s)$ and τ above with β replaced by β_n. Now if all the solutions $w_n(s)$ have a zero in $[\alpha, \tau_n)$, since $\lim_{n\to\infty} \tau_n = \alpha$, one could use a standard compactness argument to show that there is a nontrivial solution of $(E_n{}^+)$ with at least a triple zero at α and a zero of order $n - 2$ at $r_{n-2,2}(\alpha)$ which is a contradiction. Hence we can assume that $\beta \in (r_{n-2,2}(\alpha), \rho(\alpha))$ is picked (sufficiently close to $r_{n-2,2}(\alpha)$) so that $w(s) \neq 0$ for $\alpha \leqq s < \tau$. If $w(s)$ has a zero in (τ, β), then there would be a linear combination of $w(s)$ and $u^+_{n-1}(s, \beta)$ with a double zero in (τ, β) and a zero of order $n - 2$ at β. This contradicts $r^+_{2,n-2}(\tau) = \beta$ and $r^+_{2,n-2}(t)$ strictly increasing. Hence $w(s)$ is of the same sign on $[\alpha, \tau) \cup (\tau, \beta)$. But since $u(s)$ is a nontrivial solution of $(E_n{}^+)$ which has a zero of order $n - 2$ at β and since $w(s)$ and $u^+_{n-1}(s, \beta)$ are linearly independent solutions of $(E_n{}^+)$ with zeros of order $n - 2$ and $n - 1$ at $s = \beta$ respectively, $u(s)$ is a nontrivial linear combination of $w(s)$ and $u^+_{n-1}(s, \beta)$. Since $u(\alpha) = 0$, $u^+_{n-1}(s, \beta)$ is of one sign on $[\alpha, \beta)$ and has a higher order zero at β than $w(s)$ which is of one sign on $[\alpha, \tau) \cup (\tau, \beta)$, $u(s)$ has a zero at some point $\gamma \in (\tau, \beta)$. The zero of $u(s)$ at γ is not a multiple zero as $r^+_{2,n-2}(\tau) = \beta$ and $r^+_{2,n-2}(t)$ is strictly increasing. If $u(s)$ has two or more distinct zeros in $[\tau, \beta)$, then there is a nontrivial linear combination of

$u^+_{n-1}(s,\beta)$ and $u(s)$ with a double zero in (τ,β) and a zero of order $n-2$ at β which is impossible. Since $D^+_{n-2}u(\beta) = (-1)^n$, $u(s) > 0$ on (γ,β) and $u(s) < 0$ on (τ,γ). Hence for $s_1 \in (\gamma,\beta)$,

$$\frac{\partial G_{n-1,1}(x,s)}{\partial x}\Bigg|_{(\beta,s_1)} > 0$$

which means, of course, that

$$\frac{\partial G_{n-1,1}(x,s_1)}{\partial x}\Bigg|_{x=\beta} > 0.$$

Therefore, since $G_{n-1,1}(x,s_1)|_{x=\beta} = 0$, we have that $G(x,s_1) < 0$ for $x < \beta$ sufficiently close to β. Similarly, if $s_2 \in (\tau,\gamma)$, we get that

$$\frac{\partial G_{n-1,1}(x,s_2)}{\partial x}\Bigg|_{x=\beta} < 0.$$

Since $G_{n-1,1}(\beta,s_2) = 0$, we have that $G_{n-1,1}(x,s_2) > 0$ for $x < \beta$ sufficiently close to β.

It is well known that for the differential equation

$$y^{(n)} - p(x)y = 0 \tag{7}$$

with $p(x) > 0$ and continuous on the real line that $r_{n-1,1}(t) = r_{n-3,3}(t) = \infty$. Hence for equations of the form (7) for which $r_{n-2,2}(t)$ exists, for some t, we have examples where the Green's function $G_{n-1,1}(x,s)$ for certain pairs (α,β) is not of constant sign. When $n = 4$, any conjugate differential equation of the form (7) satisfies the hypothesis of Theorem 3 and Theorem 4. For examples for $y^{(4)} + q(x)y = 0$ see [13]. J. H. Barrett (see the paragraph preceding Theorem 1.1 of [14]) points out that $r_{31}(t) = r_{13}(t) = \infty$ and states a very nice result [14, Theorem 3.5] concerning the existence of $r_{22}(t)$ for the fourth order equation

$$[(r(x)y'')' + q(x)y']' - p(x)y = 0$$

where $r(x) > 0$, $p(x) \geqq 0$. We mention two simple fourth order examples which satisfy the hypothesis of Theorems 3 and 4. For $y^{(4)} + y'' = 0$, we have $r_{22}(t) = t + 2\pi < r_{31}(t) = r_{13}(t) = \infty$. For $y^{(4)} - y = 0$, $r_{22}(t) \approx t + 3\pi/2 < r_{31}(t) = r_{13}(t) = \infty$.

The dual theorem of Theorem 3 we state without proof.

THEOREM 4. *If* $\alpha < r_{2,n-2}(\alpha) < \rho(\alpha) \equiv \min[r_{1,n-1}(\alpha),\ r_{3,n-3}(\alpha)]$, $n \geqq 3$, *then there is a* β *in* $(r_{2,n-2}(\alpha), \rho(\alpha))$ *such that the Green's function* $G_{1,n-1}(x,s)$ *changes sign in* $(\alpha,\beta) \times (\alpha,\beta)$.

We conclude this paper with a theorem which gives a large class of examples of fourth order differential equations where the Green's functions $G_{31}(x, s)$ and $G_{13}(x, s)$ are not of constant sign beyond the interval of disconjugacy. By (E_4) is selfadjoint [2] we mean $f_{12}(x) \equiv f_{34}(x)$ and $f_{21}(x) \equiv f_{43}(x)$; and by $\eta_1(\alpha)$ we mean the first conjugate point of $t = \alpha$ for (E_4).

THEOREM 5. *If* (E_4) *is conjugate and selfadjoint with* $f_{21}(x)$, $f_{32}(x)$, $f_{41}(x) \leqq 0$ *on* $(-\infty, \infty)$, *then, for* $\beta > \eta_1(\alpha)$, *the Green's functions* $G_{31}(x, s)$ *and* $G_{13}(x, s)$ *change sign on* $(\alpha, \beta) \times (\alpha, \beta)$.

PROOF. We merely sketch the proof of the first part of this theorem. The author has shown in his doctoral dissertation, that if $f_{21}(x)$, $f_{32}(x)$, $f_{41}(x)$, $f_{43}(x) \leqq 0$ on $(-\infty, \infty)$, then $r_{13}(t) = r_{31}(t) = r_{121}(t) = \infty$ for all t ($r_{121}(t) = \infty$ means there is no nontrivial solution of (E_4) with a (1, 2, 1) distribution of zeros on $[t, \infty)$). The proof that $G_{31}(x, s)$ changes sign on $(\alpha, \beta) \times (\alpha, \beta)$ for $\beta > \eta_1(\alpha) = r_{22}(\alpha)$ follows from the proof of Theorem 3. Since (E_4) is selfadjoint and $r_{121}(\alpha) = \infty$, for any $\beta > \eta_1(\alpha)$, the solution $w(s)$ in the proof of Theorem 3 is different from zero on $[\alpha, \tau)$. The remainder of the proof is similar to the last part of the proof of Theorem 3.

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