ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 3, Number 1, Winter 1973

MATHEMATICAL QUESTIONS RELATING TO VISCOUS FLUID FLOW IN AN EXTERIOR DOMAIN

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The material presented here elaborates somewhat the content of six expository lectures given at the University of Arizona in April 1970. The presentation is necessarily sketchy, and is intended only as an outline — to some extent historical — and source of references for a body of distinct but interrelated contributions, all motivated by a common physical problem. Many details have been suppressed or insufficiently indicated; the reader who is new to the subject should not expect to be able to fill them in with ease, although my earlier expository article [5] should serve as a supplementary guide. I do hope to have made accessible a common thread that might be hard to perceive from the individual papers, and to have suggested, at least by inference, a range of open problems that invite new ideas and methods.

Received by the editors October 21, 1971.
AMS (MOS) subject classifications (1969). Primary 7635; Secondary 4531.

The topics did not divide evenly among the lectures, and I have therefore reorganized the material into sections of varying length, according to the topic. The reader may nevertheless perceive in the presentation traces of the original didactic style.

I wish to thank Clifton Suitt* and David Zachmann, who attended the lectures and prepared an invaluable first draft for the manuscript. I modeled the present text on their outline, and they must share responsibility for its virtues and its faults. It is a pleasure for me to thank also many department members, to whose personal warmth and hospitality I owe so much of my feeling that my visit in Arizona was far too brief.

1. **General considerations.** Our attention will center on the problem of determining a stationary flow, in $n = 2$ or 3 dimensions, of an incompressible viscous fluid past an isolated rigid body Ω that is in uniform motion in the fluid. It is assumed that the fluid extends to infinity in the domain $\mathcal{E}$ outisde Ω and that the motion is governed by the Navier-Stokes equations

$$\begin{aligned} \rho \partial w/\partial t - \mu \Delta w + \rho w \cdot \nabla w + \nabla p = 0, \\ \nabla \cdot w = 0. \end{aligned} \tag{1}$$

Here w is the velocity vector, p the (scalar) pressure, ρ the density and μ the viscosity coefficient. The quantities ρ and μ are assumed constant. The first (vector) equation (1) expresses the equilibrium of forces at each point in the fluid; the last equation (1) is the condition for conservation of mass. In what follows we shall assume $\mu = \rho = 1$; the general case can always be reduced to this one by a coordinate transformation.

Solutions of (1) will be sought for which an equilibrium configuration has been obtained, so that the velocity relative to points of Ω is time independent. In a Galilean reference frame attached to Ω, (1) then becomes

$$\begin{aligned} \Delta w - w \cdot \nabla w - \nabla p = 0, \\ \nabla \cdot w = 0 \end{aligned} \tag{2}$$

with a condition at infinity,

$$\lim_{x \to \infty} w(x) = w^{\infty} \tag{3}$$

for some fixed (usually nonzero) vector w^{∞}. On $\Sigma = \partial \Omega$, the (physical) adherence condition is

*I must relate with sadness the untimely death of this capable young person.

$$\lim_{x \to \Sigma} w(x) = 0. \tag{4}$$

It will be desirable to consider also the more general condition

$$\lim_{x \to \Sigma} w(x) = w^*(x) \tag{5}$$

for prescribed w^* on Σ.

It will be convenient in much of what follows to assume the origin of coordinates interior to Ω.

A direct study of (2) would be difficult because of the nonlinear term, and it is natural to start with the equations for an infinitesimally slow motion, obtained by linearizing about the particular solution $w \equiv 0$,

$$\begin{aligned} \Delta w - \nabla p &= 0, \\ \nabla \cdot w &= 0. \end{aligned} \tag{6}$$

These equations were first studied by G. G. Stokes, who found, in 1850, an explicit solution, with adherence on the boundary, for a flow past a sphere [1],

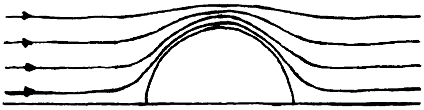
$$w(x) = w^\infty - \frac{3}{4a} \nabla \wedge r^2 \nabla \wedge \frac{w^\infty}{r} - \frac{a}{4} \nabla \wedge \nabla \wedge \frac{w^\infty}{r}, \quad r = |x|, \tag{7}$$

where a is the radius of the sphere. The solution is obtained by noting that (6) implies $\Delta p = 0$, so that p can be expanded in spherical functions, which simplify under the expected symmetry conditions.

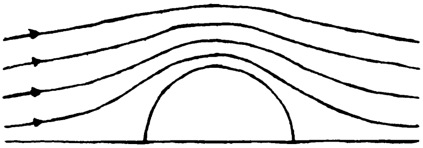
It should be noted that the solution (7) is physically unrealistic; it is symmetric with respect to the origin, and does not exhibit the anticipated "wake" property behind the body. For most commonly observed motions, predictions based on (7) would be seriously in error; nevertheless, (7) has been widely (and successfully) applied as an asymptotic formula for vanishingly small velocity field.[1]

Stokes sought also to solve the two-dimensional problem (flow past a circular cylinder) by the same method, but found it was not possible to satisfy all conditions on the coefficients in a formal expansion. He concluded (correctly) that no solution exists. This "Stokes paradox" of hydrodynamics holds in a much more general context, and will be discussed in §3.

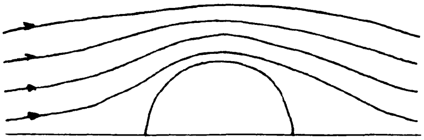
In Fig. 1, the solution of Stokes is illustrated and compared with flow patterns arising from other hypotheses. Potential flow corresponds to vanishing vorticity ($\nabla \wedge w = 0$) and w^* tangential on Σ. Figs. 1c, d arise from (21), equations linearized about w^∞. Figs. 1c, d, e, f are obtained by numerical procedures [33], [34].



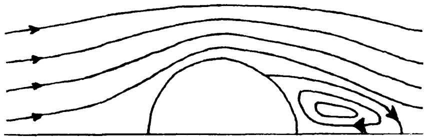
a Potential flow



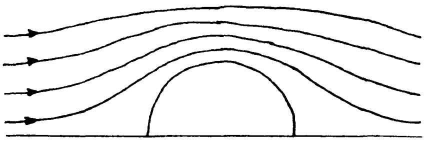
b Stokes flow



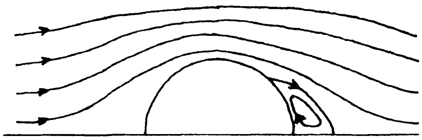
c Oseen flow; $|w^{\infty}| = 1$



d Oseen flow; $|w^{\infty}| = 20$



e Navier-Stokes flow; $|w^{\infty}| = 1$



f Navier-Stokes flow; $|w^{\infty}| = 20$

FIGURE 1. Flow patterns past a unit sphere; $\rho = \mu = 1$. See also [33], [34], from which c, d, e, f are taken.

2. **The boundary value problems; Stokes equations.** We outline here a systematic study of the system (6) due to Odqvist[2] [3], who exploited some striking analogies with potential theory. His starting point was the fundamental tensor pair $\boldsymbol{E} = (E_{ij})$, $\boldsymbol{e} = (e_j)$, which had already been given by Lorentz [**6**]:

(8)
$$E_{ij}(x) = \frac{1}{4\pi}\left(\delta_{ij}\log\frac{1}{|x|} + \frac{x_i x_j}{|x|^2}\right), \qquad n = 2,$$
$$= \frac{1}{8\pi}\left(\delta_{ij}\frac{1}{|x|} + \frac{x_i x_j}{|x|^3}\right), \qquad n = 3,$$
$$e_j(x) = \frac{1}{2\pi}\frac{\partial}{\partial x_j}\log|x|, \qquad n = 2,$$
$$= \frac{1}{4\pi}\frac{\partial}{\partial x_j}\frac{1}{|x|}, \qquad n = 3.$$

For fixed j, each column vector $E_{ij}(x)$ is a solution of (6) with corresponding "pressure" $e_j(x)$, and exhibits a fundamental singularity at $x = 0$. One is led to the representation for a solution $w(x)$ in the *interior* Ω of a smooth surface (curve) Σ,

$$w(x) = \oint_\Sigma [w(y) \cdot TE(x - y) - E(x - y) \cdot Tw(y)] \cdot d\boldsymbol{\sigma}_y \tag{9}$$

which is analogous to the Green's representation for a harmonic function. Here the "*stress tensor*" Tw, defined by

$$(Tw)_{ij} = -p\delta_{ij} + (\partial w_i/\partial x_j + \partial w_j/\partial x_i) ,$$

replaces the normal derivative of potential theory, and the tensor E replaces the (scalar) fundamental solution. Odqvist then introduces (vector) potentials of simple and double layers

$$U = \oint_\Sigma E \cdot \varphi \, d\sigma, \qquad P = \oint_\Sigma e \cdot \varphi \, d\sigma,$$
$$V = \oint_\Sigma TE \cdot \psi \cdot d\boldsymbol{\sigma}, \qquad Q = \oint_\Sigma Te \cdot \psi \cdot d\boldsymbol{\sigma},$$

and in analogy with potential theory establishes jump relations at Σ for V and for TU. One is led to consider two problems, analogous to the Dirichlet and Neumann problems of potential theory:

1. *"Dirichlet" problem*: $w\,|_\Sigma = w^*$, prescribed.
2. *"Neumann" problem*: $Tw \cdot \nu\,|_\Sigma = f^*$, prescribed.

Here ν is the unit exterior normal to Σ. As the notation indicates, the "Neumann" problem consists in prescribing a force distribution on Σ. For the interior and exterior problems, respectively, we write D_i, D_e, or N_i, N_e. For the exterior problems, solutions are sought that converge to zero[3] at suitable rates at infinity.

Let us seek a solution of D_i, in $n = 3$ dimensions, as a double layer potential. The jump relation leads [3] to a Fredholm equation of the second kind

$$w_i^* = \varphi_i + \int_\Sigma K_{ij}\varphi_j \, d\sigma \tag{10}$$

for the surface density $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$. The kernel $K_{ij}(\boldsymbol{x}, \boldsymbol{y})$, $\boldsymbol{x}$, $\boldsymbol{y} \in \Sigma$, has an integrable singularity at $\boldsymbol{x} = \boldsymbol{y}$.

A solution of (10) exists exactly for those data $\boldsymbol{w}^*$ that are orthogonal to all solutions of the homogeneous transposed system

$$0 = \psi_i + \int_\Sigma K_{ji}\psi_j \, d\sigma. \tag{11}$$

But (11) is the special case of zero data in the integral equation

$$f_i^* = \psi_i + \int_\Sigma K_{ij}\psi_j \, d\sigma \tag{12}$$

arising from a representation of a solution of N_e as a simple layer. This interpretation leads easily to the result that the only solution of (12) is $\boldsymbol{\psi} = c\boldsymbol{\nu}$. It follows that *a solution of D_i can always be found if the data w^* satisfy*

$$\int_\Sigma w^* \cdot d\sigma = 0. \tag{13}$$

The condition is also necessary, as is seen by integrating the last equation in (6) by parts.

Now let us seek a solution of N_i as a simple layer. The homogeneous adjoint admits in this case exactly six independent solutions, $\psi_j = \{\delta_{ij}\}$; $\psi_j = \{\delta_{ij}x_i\}$, corresponding to the independent motions of a rigid body. *It follows that N_i can be solved if the net force and moment on Σ due to the prescribed data both vanish.* Again a direct integration shows these conditions to be necessary.

Similarly, if D_e is to be solved by a double layer potential, six conditions must be satisfied by the data. In this case the conditions are not necessary, and Odqvist showed that *by adjoining simple layer potentials, D_e can be solved for any prescribed data w^*.*

The two-dimensional discussion parallels that for the case $n = 3$ except for this last point. In solving D_e, three conditions, corresponding to the rigid body motions, must be satisfied by the data. If one seeks to satisfy these conditions by adjoining simple layer potentials, the corresponding flow velocities can become infinite at infinity. Thus, in this case, data at infinity can no longer be imposed. This leads again to the Stokes paradox, and provides a clue towards its clarification.

3. **The Stokes paradox.** Let us consider in some detail the behavior of solutions $\boldsymbol{w}(\boldsymbol{x})$ of D_e in $n = 2$ or 3 dimensions. We follow here the

method of [17]. Suppose $|w(x)| = o(r)$ as $r = |x| \to \infty$. We write, for an annular domain $\mathcal{E}_R$ bounded by Σ and by a sphere Σ_R of (large) radius R,

$$(14)\qquad \begin{aligned} w(x) &= \oint_{\Sigma} (w \cdot TE - E \cdot Tw) \cdot d\boldsymbol{\sigma} \\ &\quad + \oint_{\Sigma_R} (w \cdot TE - E \cdot Tw) \cdot d\boldsymbol{\sigma}. \end{aligned}$$

The last term on the right is thus expressed by the other two terms and hence, for fixed $x \in \mathcal{E}_R$, is independent of R; denote it by $Q(x)$.

Let $x \in \mathcal{E}$, let S be a unit sphere, centered at x. One verifies easily that $\oint_S E(x - y) \cdot d\boldsymbol{\sigma} = 0$; thus the above existence theorem for D_i yields the existence of a Green's tensor $G(x, y)$, that is, a fundamental tensor all of whose components vanish when $y \in S$. Thus, the representation (9) yields[4]

$$w(x) = \oint_S w \cdot TG \cdot d\boldsymbol{\sigma}.$$

Keeping S fixed, this relation can be differentiated under the sign, and yields the preliminary estimates $|\nabla w| = o(r)$, $|\Delta w| = o(r)$. Hence, from (6), $|p| = o(r^2)$. Placing these results in the expression for $Q(x)$ and using the explicit knowledge of E, we find $\lim_{R\to\infty} D^2Q(x) = 0$, for any second derivative $D^2Q(x)$. But $Q(x)$ is independent of R; hence $D^2Q(x) \equiv 0$, from which $Q(x) \equiv A \cdot x + w^\infty$, for some constant matrix A and constant vector w^∞. By hypothesis, $|w(x)| = o(r)$; hence, using (8) and (14), we find $A = 0$. There follows

$$(15)\qquad w(x) = w^\infty + \oint_{\Sigma} (w \cdot TE - E \cdot Tw) \cdot d\boldsymbol{\sigma}.$$

Now observe $\mathcal{F} = -\oint_{\Sigma} Tw \cdot d\boldsymbol{\sigma}$ is the net force exerted on Σ in the flow. Using the mean value theorem on $E(x - y)$ for $y \in \Sigma$ and x large, *we obtain the general asymptotic estimate*

$$(16)\qquad w(x) = w^\infty + E \cdot \mathcal{F} + O(r^{1-n}), \qquad n = 2, 3,$$

valid for all solutions in $\mathcal{E}$ satisfying $|w(x)| = o(r)$. This result can be differentiated formally in any direction.

Note that w^∞ is uniquely determined even though the second term in (16) will, in general (for $n = 2$), become logarithmically infinite at infinity.[5]

We introduce now the deformation tensor **def** w, defined by

$$(\text{def}\, w)_{ij} = \tfrac{1}{2}\,(\partial w_i/\partial x_j + \partial w_j/\partial x_i).$$

A formal integration by parts yields, for the solution $w(x)$,

$$(17)\qquad \oint_{\Sigma + \Sigma_R} (w - w^\infty) \cdot Tw \cdot d\mathfrak{d} = 2 \int_{\mathcal{E}_R} (\text{def}\, w)^2\, dx.$$

If $n = 3$, (16) implies the integral over Σ_R vanishes in the limit. We find, in particular,

THEOREM 3A. *Let $n = 3$ and let $w(x)$ be a solution of* (6) *in $\mathcal{E}$, with $|w(x)| = o(r)$ and $w(x) = 0$ on Σ. Then there exists a finite $w^\infty =$* $\lim_{x\to\infty} w(x)$, and

$$(18)\qquad \mathfrak{F} \cdot w^\infty = 2 \int_{\mathcal{E}} (\text{def}\, w)^2\, dx.$$

Thus, in any physical flow there is always a nonzero resistance force on Σ in the direction w^∞. Physically, (18) asserts that the work done per unit time in sustaining the motion equals the rate of dissipation of kinetic energy into heat in the motion.

Now let $n = 2$. Since $4\pi E_{ij} = \delta_{ij} \log (1/r) + x_i x_j / r^2$, a condition $|w(x)| = o(\log r)$ implies, by (16), $\mathfrak{F} = 0$. But this result implies, in turn, that the outer boundary integral in (17) vanishes in the limit, so that (18) again holds, with $\mathfrak{F} = 0$. Hence $\text{def}\, w \equiv 0$ in $\mathcal{E}$, so that $w(x)$ represents a rigid motion. Since $w(x) = 0$ on Σ, we find

THEOREM 3B. *If $n = 2$ there is no solution $w(x)$ of* (6) *in $\mathcal{E}$ for which $w = 0$ on Σ and $|w(x)| = o(\log r)$ at infinity.*

This is the precise expression of the Stokes paradox. The paradox will be clarified further in §6, where it will be shown to arise from a nonuniformity in the perturbation to the vanishing velocity field, owing to the infinite size of the domain. For three-dimensional flows the paradox is less striking but it persists nevertheless, and is evidenced by the physically unrealistic behavior of the flow field and the poor correspondence of functionals of the flow (e.g., the force of resistance) with experimental measurements.

It is the Stokes paradox that leads to the essential difficulties in attempts to solve the exterior problem for the Navier-Stokes equations, even in three dimensions, as the solution apparently cannot be obtained directly from a perturbation procedure starting from solutions of (6). The remainder of these lectures will be devoted to ways of overcoming this difficulty.

4. **Another approach; the nonlinear problem.** The method of Odqvist (§2) permitted him to obtain general estimates on the singu-

larity of the Green's tensor for (6) in an interior domain [3]. Odqvist used this tensor to formulate an integral equation satisfied by any solution of D_i for (2). By a perturbation procedure, he was able to solve the equation to obtain a solution of D_i, provided the data are sufficiently small.[6] This general approach will be adopted for the study of the exterior problem in §5; in the meantime, let us consider what can be accomplished for D_i by a more abstract method. We follow Ladyzhenskaia; to fix the ideas, we solve again D_i for (6), using the method of projection in Hilbert space [4, p. 38]. This method is variational, and deals with generalized function classes. The essential step is to obtain an a priori bound on the Dirichlet integral

$$D[w] = \int_\Omega |\nabla w|^2 \, dx$$

for any "solution" of (6) in Ω, such that $w = w^*$ on $\partial\Omega = \Sigma$. We assume that Σ, w^* are suitably smooth and that (13) holds. We construct a vector field $\boldsymbol{\zeta}(x)$ such that $\nabla \cdot \boldsymbol{\zeta} = 0$ in Ω, $\boldsymbol{\zeta} = w^*$ on Σ, and show that this can be done in such a way that $D[\zeta] < M(\Sigma, w^*) < \infty$. Introducing a norm $\| \ \| = D^{1/2}$ in the class J of vector fields $v(x)$ with compact support in Ω, for which $\nabla \cdot v = 0$ and $\|v\| < \infty$, we seek a generalized solution $w(x)$ of (6) for which $w - \boldsymbol{\zeta}$ lies in the closure H (with respect to $\| \ \|$) of J. The system (6) implies, for $w(x)$,

$$(19) \qquad [w - \boldsymbol{\zeta}, \boldsymbol{\varphi}] = -[\boldsymbol{\zeta}, \boldsymbol{\varphi}], \quad \text{any } \boldsymbol{\varphi} \in H,$$

where $[\ ]$ denotes the scalar product corresponding to $\| \ \|$. The right side of (19) is a linear functional on $\boldsymbol{\varphi} \in H$; by the Riesz theorem, there exists $\mathfrak{n} \in H$ such that $[\boldsymbol{\zeta}, \boldsymbol{\varphi}] = [\mathfrak{n}, \boldsymbol{\varphi}]$. *The vector field* $w = \boldsymbol{\zeta} - \mathfrak{n}$ *is the unique generalized solution of* (6), (5), *and is easily shown to be identical to the smooth solution discussed in* §2. A direct proof that the solution is smooth at interior points can be obtained using the fundamental tensor (8); to prove continuity at the boundary requires, as of this writing, use of the Green's tensor, whose existence and requisite smoothness have been shown only by the methods of §2.

Let us now return to the original nonlinear system (2) in Ω. It is a remarkable fact that *again a solution exists for any data* w^* *with* $\oint w^* \cdot d\boldsymbol{\sigma} = 0$. The underlying reason for this is the a priori estimate, discovered first by Leray [8] in 1933, that

$$(20) \qquad D[w] < L(\Sigma; w^*) < \infty$$

(see also [2, p. 210]) where L does not depend on the particular solution considered. From this result one may use fixed point methods to prove the existence of a solution of D_i for (2), for *any* data satisfying (13). One may proceed either directly in the class of smooth functions, using the Odqvist integral equation [8, 2, p. 226], or else one may solve the abstract equation in Hilbert space that is obtained by applying the Riesz theorem, as above, to the nonlinear equations [4, p. 113 ff.]. This latter procedure leads in a fairly straightforward way to the existence of a solution in a generalized function class; the smoothness of the solution, as for the linearized equations, must be proved a posteriori.[7] Experimentally, stationary solutions are not observed for large data, and it is essentially certain that in this situation Leray's solutions are not unique and are unstable. But what is crucial for what follows is that the solutions exist and satisfy (2), (5).

Consider the exterior problem for (2) with data (3), (5). Leray [8] started with an annular domain $\mathcal{E}_R$ bounded by Σ and by a sphere (circle) Σ_R of large radius R. Leray's method shows that *if the data on* Σ_R *have the constant value* w^∞, *then* (20) *holds with* $L = L(\Sigma, w^*, w^\infty)$ *independent of* R. From this result, Leray showed *the existence of a sequence of solutions, as* $R \to \infty$, *that converge in* $\mathcal{E}$, *uniformly in any* $\mathcal{E}_R$, *to a (smooth) solution in* $\mathcal{E}$. *The data* w^* *on* Σ *are achieved strictly,* while if $n = 3$ he showed that *the data at infinity are achieved in the generalized sense*

$$\int_{\mathcal{E}} \frac{|w(x) - w^\infty|^2}{|x - y|^2}\,dx < C < \infty$$

uniformly for all $y \in \mathcal{E}$. It was later shown [9], [7], [4], [5] that *if* $n = 3$, *then* $w(x) \to w^\infty$ *continuously and that all deriviatives of* $w(x)$ *tend to zero.* However, no further information on asymptotic behavior of these solutions has been obtained. Conditions ensuring uniqueness have not been found, nor is it known whether the physically expected wake region appears. In two dimensions, even less is known, and the possibility that, e.g., for the physical data $w^* = 0$ the solution vanishes identically, has not been excluded. The method has led no further.

5. **The Oseen equations; physically reasonable (PR) solutions.** A somewhat different approach to the exterior problem was taken by Oseen [**10**] in 1927. Oseen considered the equations, obtained by linearizing (2) about the solution $w(x) \equiv w^\infty$,

$$\begin{aligned} \Delta w - w^\infty \cdot \nabla w - \nabla p &= 0, \\ \nabla \cdot w &= 0, \end{aligned} \tag{21}$$

and he studied them by methods analogous to those used by Odqvist for the Stokes equations. A fundamental tensor is in this case not so easy to construct, but Oseen found one by using a combination of invariance considerations and judicious guesswork. He first showed that for the Stokes equations (6) one can write

$$E_{ij} = (\delta_{ij}\,\Delta - \partial^2/\partial x_i \partial x_j)\Phi, \qquad e_i = -\,\partial \Delta \Phi / \partial x_i \tag{22}$$

with

$$\Phi = \frac{1}{4\pi}(r^2 \log r - r^2), \qquad n = 2,$$

$$= \frac{1}{8\pi} r, \qquad n = 3.$$

For the system (21), he found a similar representation, but with Φ replaced by

$$\mathcal{O} = -\frac{1}{8\pi\sigma}\int_0^{\sigma s} \frac{1 - e^{-\alpha}}{\alpha}\, d\alpha, \qquad n = 3, \tag{23}$$

with $\sigma = |w^\infty|/2$ and $s = r + (x \cdot w^\infty)/|w^\infty|$. Then, corresponding to (21),

$$\hat{E}_{ij} = (\delta_{ij}\,\Delta - \partial^2/\partial x_i \partial x_j)\mathcal{O}, \qquad \hat{e}_i = -\partial(\Delta - 2w^\infty \cdot \nabla)\mathcal{O}/\partial x_i. \tag{24}$$

For $n = 2$, $\mathcal{O}(\sigma s)$ becomes a rather complicated relation involving Bessel functions; cf. [**11**].

The tensor $\hat{E}$, e has local properties similar to those of E, e, but its asymptotic behavior for large $|x|$ is strikingly different. It exhibits a wake region in the direction w^∞, and it vanishes at infinity both for $n = 2$ and $n = 3$. One may expect, then, a more realistic behavior at infinity for flows computed using (21); in particular, there is no Stokes paradox. Nevertheless, the correspondence with real flows is not good; cf. the discussion in [**12**], [**13**].

We note the important homogeneity relation

$$\hat{E}(x; w^\infty) = \hat{E}(|w^\infty| x; w^\infty/|w^\infty|), \qquad \hat{e}(x; w^\infty) = |w^\infty| \hat{e}(|w^\infty| x; w^\infty/|w^\infty|) \tag{25}$$

and the asymptotic properties, for any fixed vector α,

$$\hat{E}(x;\alpha) = \hat{E}(x;0) + \text{const} + o(1), \qquad T\hat{E}(x;\alpha) = T\hat{E}(|\alpha| x; 0) + o(1), \quad \text{as } |\alpha| x \to 0. \tag{26}$$

Existence theorems for (21), with particular reference to an exterior domain with $n = 3$, are established in [14]. From these results and (24) one is led to the existence of a Green's tensor $\hat{\boldsymbol{G}}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{w}^\infty)$ for (21) in $\mathcal{E}$. If we assume for the moment the analogous asymptotic behavior of integrals over Σ_R as was established for the Stokes equations in §3, and if we set $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{w}(\boldsymbol{x}) - \boldsymbol{w}^\infty$, we obtain the representation for a (supposed) solution $\boldsymbol{w}(\boldsymbol{x})$ of (2), (3), (5) in $\mathcal{E}$,

$$\boldsymbol{u}(\boldsymbol{x}) = \hat{\boldsymbol{u}}(\boldsymbol{x}) - \int_{\mathcal{E}} \boldsymbol{u} \cdot \boldsymbol{u} \cdot \nabla \hat{\boldsymbol{G}}\, dy \equiv \boldsymbol{T}\boldsymbol{u} \tag{27}$$

where $\hat{\boldsymbol{u}}(\boldsymbol{x})$ is a solution of (21), (5) that vanishes at infinity. In deriving (27), we used the identity $\int_{\mathcal{E}} \hat{\boldsymbol{G}} \cdot \boldsymbol{u} \cdot \nabla \boldsymbol{u}\, dy = - \int_{\mathcal{E}} \boldsymbol{u} \cdot \boldsymbol{u} \cdot \nabla \hat{\boldsymbol{G}}\, dy$, which holds because $\nabla \cdot \boldsymbol{u} = 0$.

Conversely, any solution of (27) that vanishes at infinity defines a solution $\boldsymbol{w}(\boldsymbol{x})$ of the boundary problem (2), (3), (5) for the Navier-Stokes equations.

We proceed to solve (27) for small data. The crucial step in the proof is

LEMMA. *If $n = 3$, there exists $H < \infty$, depending only on the geometry, such that*

$$|\boldsymbol{x}| \int_{\mathcal{E}} |\boldsymbol{y}|^{-2} |\nabla_y \hat{\boldsymbol{G}}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{w}^\infty)|\, dy < H \tag{28}$$

uniformly in $\boldsymbol{x}$ and in $\boldsymbol{w}^\infty$, as $\boldsymbol{w}^\infty \to 0$.

The lemma is established in [14]. From it, one concludes that if a norm $\|\boldsymbol{u}\| = \sup_{\mathcal{E}} |\boldsymbol{x}|\, |\boldsymbol{u}(\boldsymbol{x})|$ is introduced, then if $\|\hat{\boldsymbol{u}}\| < 1/4H$ the operator $\boldsymbol{T}\boldsymbol{u}$ will be a contraction of the sphere S_H, $\|\boldsymbol{u}\| < 1/2H$, and will carry $S_H \to S_H$. Thus, $\boldsymbol{T}\boldsymbol{u}$ admits a fixed point in S_H. Using this result and properties established by Odqvist [3] for the Green's tensor for (6), we obtain the result

THEOREM 5A. *For $n = 3$, if Σ is sufficiently smooth and $\boldsymbol{w}^* - \boldsymbol{w}^\infty$ is sufficiently small,*[8] *then there is a solution $\boldsymbol{w}(\boldsymbol{x})$ of (2) in $\mathcal{E}$, such that $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{w}^*$ on Σ, $\boldsymbol{w}(\boldsymbol{x}) \to \boldsymbol{w}^\infty$ at infinity. The solution is locally smooth, and there holds $|\boldsymbol{w}(\boldsymbol{x}) - \boldsymbol{w}^\infty| < C|\boldsymbol{x}|^{-1}$ as $\boldsymbol{x} \to \infty$.*

A careful study of these solutions indicates that they have the physically expected asymptotic properties. More generally, we may introduce [15], [14] a class of "physically reasonable" (PR) solutions, defined by the requirement that, for some $\epsilon > 0$, $|\boldsymbol{x}|^{1/2+\epsilon}|\boldsymbol{w}(\boldsymbol{x}) - \boldsymbol{w}^\infty| < C < \infty$ in $\mathcal{E}$. We then have, for $n = 3$, THEOREM 5B. *If $w^\infty \neq 0$, every solution of class PR exhibits a paraboloidal wake region, opening about an axis Z directed along the vector w^∞. If φ is the polar angle between Z and a ray from a fixed point on Z to a variable point $x \in \mathcal{E}$, then*

$$(29)\quad \begin{aligned} |w(x) - w^\infty| &< C|x|^{-1}, &\quad &\text{if } |\varphi| < \pi|x|^{-1/2}, \\ &< C|x|^{-(1+\sigma)}, &\quad &\text{if } |\varphi| < \pi|x|^{-(1-\sigma)/2}, \quad 0 \leqq \sigma \leqq 1, \end{aligned}$$

(see Fig. 1).

This result is qualitatively best possible. In addition the energy relation

$$(30)\qquad \mathcal{F} \cdot w^\infty = 2 \int_{\mathcal{E}} (\operatorname{def} w)^2\, dx$$

holds whenever $w^* \equiv 0$. Thus, the force of resistance cannot vanish in a nontrivial flow.

The properties of contraction mappings show that the solution of Theorem 5A is unique in S_H. However, a much stronger result holds [14].

THEOREM 5C. *Suppose $|x|\ |w(x) - w^\infty| \leqq \frac{1}{2}$ in $\mathcal{E}$. Then $w(x)$ is unique among all solutions in $\mathcal{E}$ of class PR that achieve the same data on Σ and at infinity.*

PROOF. Let $v(x)$ be another such solution. The function $\mathbf{n}(x) = w(x) - v(x)$ achieves zero data and satisfies an equation

$$\Delta \mathbf{n} - \mathbf{n} \cdot \nabla \mathbf{n} - \nabla q = (\mathbf{n} \cdot \nabla w - w \cdot \nabla \mathbf{n})$$

for some scalar $q(x)$. We multiply by $\mathbf{n}$ and integrate over $\mathcal{E}_R$. After some formal integrations by parts, and noting that the surface integrals over Σ_R vanish in the limit (by Theorem 5B), we find

$$-\int_{\mathcal{E}} |\nabla \mathbf{n}|^2\, dx = \int_{\mathcal{E}} \mathbf{n} \cdot \mathbf{n} \cdot \nabla w\, dx.$$

Now

$$\begin{aligned} \int_{\mathcal{E}_R} \mathbf{n} \cdot \mathbf{n} \cdot \nabla w\, dx &= \int_{\mathcal{E}_R} \mathbf{n} \cdot \mathbf{n} \cdot \nabla (w - w^\infty)\, dx \\ &= -\int_{\mathcal{E}_R} (w - w^\infty) \cdot \mathbf{n} \cdot \nabla \mathbf{n}\, dx \\ &\quad + \oint_{\Sigma_R} \mathbf{n} \cdot (w - w^\infty)(\mathbf{n} \cdot d\boldsymbol{\sigma}) \end{aligned}$$

and again the boundary term vanishes in the limit. Thus,

$$\int_{\mathcal{E}} |\nabla \mathfrak{n}|^2 \, dx = \int_{\mathcal{E}} (w - w^{\infty}) \cdot \mathfrak{n} \cdot \nabla \mathfrak{n} \, dx \leqq \frac{1}{2} \int_{\mathcal{E}} |x|^{-1} |\mathfrak{n} \cdot \nabla \mathfrak{n}| \, dx$$
$$\leqq \frac{1}{2} \left\{ \int_{\mathcal{E}} |x|^{-2} \mathfrak{n}^2 \, dx \int_{\mathcal{E}} |\nabla \mathfrak{n}|^2 \, dx \right\}^{1/2}.$$

A form of the Poincaré inequality that holds for exterior domains (Lemma 3.4 in [2]) shows that if $\mathfrak{n} = 0$ on Σ and $\mathfrak{n} \to 0$ at infinity, then

$$\int_{\mathcal{E}} |x|^{-2} \mathfrak{n}^2 \, dx \leqq 4 \int_{\mathcal{E}} |\nabla \mathfrak{n}|^2 \, dx \tag{31}$$

equality holding only if $\mathfrak{n} \equiv 0$ in $\mathcal{E}$. Thus,

$$\int_{\mathcal{E}} |\nabla \mathfrak{n}|^2 \, dx < \int_{\mathcal{E}} |\nabla \mathfrak{n}|^2 \, dx$$

unless $\mathfrak{n} \equiv 0$ in $\mathcal{E}$. Q.E.D.

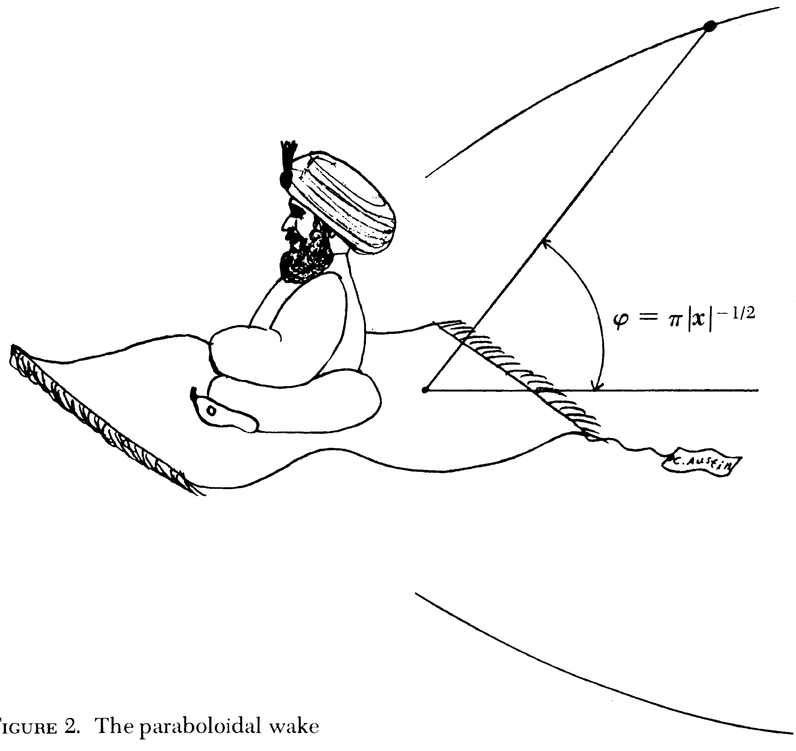


FIGURE 2. The paraboloidal wake

6. **The two-dimensional problem; linearized equations.** We study this case again with the aid of the Oseen equations (21). However, the preceding discussion does not carry over directly, as the solutions of (21), in view of the Stokes paradox, will necessarily become singular in some way with vanishing physical data. For $n = 2$, an analogue of the preceding basic lemma (28)

$$|x|^{(1-\epsilon)/2} \int_{\mathcal{E}} |y|^{-1} |\nabla \hat{G}(x, y; w^{\infty})| \, dy < H(w^{\infty}; \epsilon) < \infty, \tag{32}$$

any $\epsilon > 0$, still holds for any fixed $w^{\infty} \neq 0$, but it is unlikely that it holds uniformly as $w^{\infty} \to 0$. Thus, the preceding method yields the existence of a solution of (2), (3), (5) for data w^* in some functional neighborhood $\mathcal{N}$ of w^{∞}, but $\mathcal{N}$ shrinks as $w^{\infty} \to 0$. Without further information the existence of a solution for the physical data $w^* = 0$ cannot be inferred from (32), even for small $|w^{\infty}|$. We proceed therefore to study in some detail the asymptotic behavior of solutions of (21) in $\mathcal{E}$ as $w^{\infty} \to 0$. We shall obtain as a consequence not only an existence theorem for the nonlinear problem (2), (3), (5) in $\mathcal{E}$, but also a new clarification of the Stokes paradox from the point of view of singular perturbation theory.

It is convenient to write[9] $w^{\infty} = \lambda \boldsymbol{\alpha}$ where $\boldsymbol{\alpha}$ is a fixed unit vector and $0 < \lambda \leqq 1$. Then (21) becomes

$$\begin{aligned} \Delta w - \lambda \boldsymbol{\alpha} \cdot \nabla w - \nabla p &= 0, \\ \nabla \cdot w &= 0, \end{aligned} \tag{33}$$

the linearity of which permits us to write the prescribed data in the form

$$\begin{aligned} w(x) &\to w^* \quad \text{on } \Sigma, \\ w(x) &\to 0 \quad \text{at infinity}. \end{aligned} \tag{34}$$

The following results summarize the qualitative behavior of the solutions as $\lambda \to 0$, when $n = 2$.

1. *Given $\lambda \neq 0$, u^*, there is a unique solution $u(x; \lambda)$ of* (21), (34) *in $\mathcal{E}$.*

2. *As $\lambda \to 0$ the solutions $u(x; \lambda)$ remain uniformly bounded in Dirichlet norm; that is, there is a constant A such that*

$$\int_{\mathcal{E}} |\nabla u(x; \lambda)|^2 \, dx < A < \infty$$

uniformly in λ in $0 < \lambda \leqq 1$.

3. *There exists* $\boldsymbol{u}^0(\boldsymbol{x}) = \lim_{\lambda\to 0} \boldsymbol{u}(\boldsymbol{x};\lambda)$ *uniformly in any set* $\mathscr{E}_R$, $R < \infty$; $\boldsymbol{u}^0(\boldsymbol{x})$ *satisfies the Stokes equations* (6) *and* $\boldsymbol{u}^0(\boldsymbol{x}) \to \boldsymbol{u}^*$ *on* Σ.

4. *The function* $\boldsymbol{u}^0(\boldsymbol{x})$ *is characterized*[10] *as the unique solution of* (6), *such that* $\boldsymbol{u}^0(\boldsymbol{x}) \to \boldsymbol{u}^*$ *on* Σ *and* $\int_{\mathscr{E}} |\nabla \boldsymbol{u}|^2\, dx < \infty$.

5. *Let* $\mathcal{G}(\lambda)$ *be the force on* Σ *due to the flow* $\boldsymbol{u}(\boldsymbol{x};\lambda)$; *let* $\mathcal{G}^0$ *be the force arising from the flow* $\boldsymbol{u}^0(\boldsymbol{x})$. *There holds* $\mathcal{G}^0 = \lim_{\lambda\to 0} \mathcal{G}(\lambda)$.

6. $\mathcal{G}^0 = 0$.

7. *There exists* $\boldsymbol{u}_0{}^\infty = \lim_{x\to\infty} \boldsymbol{u}^0(\boldsymbol{x})$, *and* $|\boldsymbol{u}_0{}^\infty| \neq \infty$. *In general,* $\boldsymbol{u}_0{}^\infty \neq 0$.

8. *There holds* $\boldsymbol{u}_0{}^\infty = (1/4\pi)\lim_{\lambda\to 0} \mathcal{G}(\lambda) \log(1/\lambda)$.

We note from 7 that the limiting condition at infinity is in general lost in the perturbation. Property 8 has a consequence that is important for the nonlinear theory.

There holds $|\mathcal{G}(\lambda)| < C/(\log 1/\lambda)$ *as* $\lambda \to 0$. $\mathcal{G}(\lambda)$ *has the asymptotic direction* $\boldsymbol{u}_0{}^\infty$.

From 4 and 8 we find

Suppose the physical data $\boldsymbol{u}^* = \boldsymbol{w}^\infty$ *are imposed. Then* $\mathcal{G}(\lambda)$ *is asymptotically independent of the shape or size of the obstacle* Σ.

For by 4 there holds in this case $\boldsymbol{u}^0(\boldsymbol{x}) \equiv \boldsymbol{u}^* = \boldsymbol{w}^\infty$, so that $\boldsymbol{u}_0{}^\infty = \boldsymbol{w}^\infty$, regardless of the choice of Σ.

Detailed proofs of 1 through 8 appear in [**16**]. We outline here some of the salient features:

1. The proof of existence is analogous to that for the case $n = 3$, but in some technical respects more difficult.[11] See [**16**].

2. Let $\boldsymbol{\zeta}(\boldsymbol{x})$ be a vector field with compact support in $\mathscr{E}$, with $\boldsymbol{\zeta} = \boldsymbol{u}^*$ on Σ and $\nabla \cdot \boldsymbol{\zeta} = 0$. Then $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{u} - \boldsymbol{\zeta}$ satisfies

$$\Delta \boldsymbol{v} - \lambda \boldsymbol{\alpha} \cdot \nabla \boldsymbol{v} - \nabla p = -\Delta \boldsymbol{\zeta} + \lambda \boldsymbol{\alpha} \cdot \nabla \boldsymbol{\zeta}$$

and $\boldsymbol{v} = 0$ on Σ and at infinity. Multiplying by $\boldsymbol{v}$ and integrating over $\mathscr{E}_R$, and noting that all boundary integrals vanish in the limit $R \to \infty$, we find

$$-\int_{\mathscr{E}} |\nabla \boldsymbol{v}|^2\, dx = \int_{\mathscr{A}} \nabla \boldsymbol{\zeta} \cdot \nabla \boldsymbol{v}\, dx - \lambda \int_{\mathscr{A}} \boldsymbol{\zeta} \cdot \boldsymbol{\alpha} \cdot \nabla \boldsymbol{v}\, dx$$

where $\mathscr{A}$ is the support of $\boldsymbol{\zeta}$. Hence

$$\int_{\mathscr{E}} |\nabla \boldsymbol{v}|^2\, dx \leqq C(1+\lambda) \left[\int_{\mathscr{E}} |\nabla \boldsymbol{v}|^2\, dx \right]^{1/2},$$

where C depends only on $\boldsymbol{u}^*$ and on Σ. This result implies in turn the stated bound for $\boldsymbol{u}(\boldsymbol{x})$.

3. Given $x \in \mathcal{E}$, fix R so that x lies in $\mathcal{E}_{R/2}$. Let $G(x, y)$ be the Green's tensor for the Stokes equations in $\mathcal{E}_R$. Then in $\mathcal{E}_R$ there holds

$$
\begin{aligned}
u(x;\lambda) &= \oint_{\Sigma} u^* \cdot TG \cdot d\mathfrak{d} + \oint_{\Sigma_R} u \cdot TG \cdot d\mathfrak{d} + \lambda \int_{\mathcal{E}_R} G \cdot \alpha \cdot \nabla u \, dy \\
&= u^{(1)} \qquad\qquad\quad + u^{(2)} \qquad\qquad\quad + u^{(3)}.
\end{aligned} \tag{35}
$$

Here $u^{(1)}$ is a solution of (6) with $u^{(1)} = u^*$ on Σ , $u^{(1)} = 0$ on Σ_R. It is independent of λ, and can be shown [3] to be smooth in $\mathcal{E}_R$, depending only on Σ, u^*, and R. Since x is bounded from Σ_R, there holds, for fixed R, independent of λ,

$$|u^{(2)}(x)|^2 \leqq C \oint_{\Sigma_R} u^2 \, d\mathfrak{d} \leqq C_1 \int_{\mathcal{E}_R} |\nabla u|^2 \, dx + C_2$$

where C_1 and C_2 depend only on Σ, u^*, R. The integral on the right is again bounded, by 2. Finally,

$$|u^{(3)}(x)|^2 \leqq C\lambda^2 \int |\nabla u|^2 \, dx$$

which is bounded, for $0 < \lambda \leqq 1$, by 2. Thus $u(x;\lambda)$ is bounded in any set $\mathcal{E}_R$, independent of λ. Placing this result back into (35) permits, successively, the proof of the (Hölder) equicontinuity of $u(x;\lambda)$ and then of its derivatives, up to second order, as $\lambda \to 0$. Thus, a sequence $\lambda_j \to 0$ can be chosen for which the $u(x;\lambda_j)$ converge in $\mathcal{E}$ to a solution $u^0(x)$ of the limiting equations (6).

4. The representation (16) can be shown to hold for any solution with finite Dirichlet integral [17, Theorem 1]. Thus, for the difference $W(x)$ of two solutions, there holds (as in (16))

$$W(x) = W^\infty + E \cdot \vartheta + \text{ terms with finite Dirichlet integral.}$$

But

$$\int_{\mathcal{E}} |\nabla E_{11}|^2 \, dx = \int_{\mathcal{E}} |\nabla E_{22}|^2 \, dx = \infty\,;$$

hence $\vartheta = 0$. Since $W = 0$ on Σ, we conclude, using (16), that (18) holds; hence $\operatorname{def} W \equiv 0$ in $\mathcal{E}$; hence W represents a rigid motion. Since $W = 0$ on Σ, there follows $W \equiv 0$ in $\mathcal{E}$.

5 follows from the convergence of the derivatives of $u(x;\lambda)$ as $\lambda \to 0$.

6 and 7 follow from the same reasoning used to prove 4.

8. Since $\lim_{x\to\infty} u(x;\lambda) = 0$, we can establish, as was done for (15), the representation

$$u(x;\lambda) = -\oint_\Sigma \hat{E}\cdot Tu\cdot d\mathfrak{d} + \oint_\Sigma u^*\cdot T\hat{E}\cdot d\mathfrak{d} + \lambda\oint_\Sigma (\hat{E}\cdot u^*)\alpha\cdot d\mathfrak{d}\,. \tag{36}$$

The relations (26) imply

$$\begin{aligned}\hat{E}_{ij}(x;\lambda) &= \hat{E}_{ij}(\lambda x;1)\\ &= \frac{1}{4\pi}\,\delta_{ij}\,\log\frac{1}{\lambda} + \hat{E}_{ij}(x) + \text{const} + o(1)\end{aligned}$$

as $\lambda x \to 0$. Hence, for fixed $x \in \mathcal{E}$,

$$\begin{aligned}u(x;\lambda) = \frac{1}{4\pi}\,\mathfrak{F}(\lambda)\left(\log\frac{1}{\lambda} + \text{const}\right)\\ - \oint_\Sigma \hat{E}\cdot Tu\cdot d\mathfrak{d} + \oint_\Sigma u^*\cdot T\hat{E}\cdot d\mathfrak{d} + o(1)\end{aligned} \tag{37}$$

as $\lambda \to 0$. Since all other terms tend to limits as $\lambda \to 0$, there exists also

$$\lim_{\lambda\to 0} \mathfrak{F}(\lambda)\log\frac{1}{\lambda} = 4\pi u_0^{\infty}. \tag{38}$$

Passing to the limit in (37) as $\lambda \to 0$ for fixed x yields

$$u^0(x) = u_0^{\infty} - \oint_\Sigma E\cdot Tu^0\cdot d\mathfrak{d} + \oint_\Sigma u^*\cdot TE\cdot d\mathfrak{d}. \tag{39}$$

But from (38) follows $\oint_\Sigma Tu^0\cdot d\mathfrak{d} = \mathfrak{F}^0 = 0$. Hence, applying the mean value theorem to $E(x-y)$ in (39), we see that both integrals vanish at $x = \infty$, so that $\lim_{x\to\infty} u^0(x) = u_0^{\infty}$.

A closer study of the convergence process yields the following result: *Let* $0 < \epsilon < \frac{1}{2}$. *Set*

$$\begin{aligned}h_i(\zeta) &= \log 2/|\zeta| && \text{if } 0 < |\zeta| \leqq 1,\\ &= \begin{cases}|\zeta|^{-1/2} & \text{if } |\zeta| > 1, \quad i = 1,\\ |\zeta|^{-(1-\epsilon)/2} & \text{if } |\zeta| > 1, \quad i = 2.\end{cases}\end{aligned}$$

Assume the flow so oriented that $w^{\infty} = (b, 0)$. *Then*

$$|u_i(x;\lambda)| < C h_i(\lambda x)\,\frac{1}{\log 1/\lambda}, \qquad i = 1, 2, \tag{40}$$

uniformly in $\mathcal{E}$, *as* $\lambda \to 0$. In particular, the solutions $u(x;\lambda)$ are equibounded in magnitude in the perturbation.

7. The two-dimensional problem; Navier-Stokes equations. We now apply the above results to the nonlinear stationary equations (2), with $n = 2$. If we set $\boldsymbol{u}(\boldsymbol{x}; \lambda) = (\boldsymbol{w}(\boldsymbol{x}; \lambda) - \lambda\boldsymbol{\alpha})/\lambda$, then (2) becomes

$$\begin{aligned} &\Delta \boldsymbol{u} - \lambda\boldsymbol{\alpha} \cdot \nabla \boldsymbol{u} - \nabla p = \lambda \boldsymbol{u} \cdot \nabla \boldsymbol{u}, \\ &\nabla \cdot \boldsymbol{u} = 0, \end{aligned} \tag{41}$$

with boundary conditions

$$\begin{aligned} &\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}^* = (\boldsymbol{w}^* - \lambda\boldsymbol{\alpha})/\lambda \quad \text{on } \Sigma, \\ &\lim_{x \to \infty} \boldsymbol{u}(\boldsymbol{x}) = 0. \end{aligned} \tag{42}$$

The above existence theorem for the Oseen equations yields the existence of a Green's tensor $\hat{G}(\boldsymbol{x}, \boldsymbol{y}; \lambda)$ for (41), and leads, as before, to an integral equation

$$\boldsymbol{u}(\boldsymbol{x}; \lambda) = \hat{\boldsymbol{u}}(\boldsymbol{x}; \lambda) - \lambda \int_{\mathcal{E}} \boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{y}) \cdot \nabla \hat{G}(\boldsymbol{x}, \boldsymbol{y}; \lambda)\, dy \equiv T\boldsymbol{u} \tag{43}$$

where $\hat{\boldsymbol{u}}$ is the solution of (33) satisfying (42). The crucial step now becomes

LEMMA. *If $n = 2$, there exists $H < \infty$, depending only on the geometry, such that*

$$\lambda \int_{\mathcal{E}} h_j(\lambda y) h_k(\lambda y) \left| \frac{\partial \hat{G}_{ij}(\boldsymbol{x}, \boldsymbol{y}; \lambda)}{\partial y_k} \right| dy < H h_i(\lambda x) \tag{44}$$

uniformly in x and in λ, as $\lambda \to 0$.

The lemma is proved in [18]. It implies that $T\boldsymbol{u}$ contracts the function sphere S_H, $|u_i(x; \lambda)| < (1/2H) h_i(\lambda x)$, and that it carries $S_H \to S_H$ if $|\hat{u}(x; \lambda)| < (1/4H) h_i(x; \lambda)$. As indicated above, $|\hat{u}(x; \lambda)| < C h_i(x; \lambda) \cdot (1/\log 1/\lambda)$. Here C depends on the geometry and on u^*, which in turn depends on λ. But if $\boldsymbol{u}^*$ remains bounded and smooth as $\lambda \to 0$, then C will remain bounded. We are led to the result ($n = 2$)

THEOREM 7A. *If Σ is of class $\mathcal{C}^{(3)}$ and if data $\boldsymbol{w}^*(x; \lambda)$ are prescribed on Σ, so that $\boldsymbol{u}^*$, together with its tangential derivatives to third order, remains uniformly bounded as $\lambda \to 0$, then for all sufficiently small λ a solution $\boldsymbol{w}(x; \lambda)$ of (2) exists in $\mathcal{E}$, such that $\boldsymbol{w}(x; \lambda) = \boldsymbol{w}^*$ on Σ and* $\lim_{x \to \infty} \boldsymbol{w}(\boldsymbol{x}; \lambda) = \lambda\boldsymbol{\alpha}$.

COROLLARY. *If $n = 2$, there is a solution $\boldsymbol{w}(\boldsymbol{x})$ of (2) in $\mathcal{E}$, satisfying the physical conditions (3), (4), whenever $|\boldsymbol{w}^\infty|$ is sufficiently small.*

The solutions constructed above are unique in S_H. But in contrast to the three-dimensional case, it is not known whether they are unique in the large and any reasonable sense. They do, however, exhibit a parabolic wake region [11] in the direction $\boldsymbol{w}^\infty$. We have also the following limiting property, which seems to have no three-dimensional counterpart:

THEOREM 7B. *Suppose* $\boldsymbol{u}^*(\boldsymbol{x};\lambda) \to \boldsymbol{u}_0^*(\boldsymbol{x})$ *as* $\lambda \to 0$, *uniformly on* Σ *together with derivatives to third order. Then the solutions* $\boldsymbol{u}(\boldsymbol{x};\lambda)$ *of* (41) *converge in* $\mathcal{E}$, *uniformly in any* $\mathcal{E}_R$, *to a solution* $\boldsymbol{u}^0(\boldsymbol{x})$ *of* (6). *There exists* $\boldsymbol{u}_0{}^\infty = \lim_{x\to\infty} \boldsymbol{u}^0(\boldsymbol{x})$, *and if* $\mathcal{F}(\lambda)$ *is the force on* Σ *due to the flow* $\boldsymbol{w}(\boldsymbol{x};\lambda) = \lambda[\boldsymbol{u}(\boldsymbol{x};\lambda) + \boldsymbol{\alpha}]$, *there holds* $4\pi \boldsymbol{u}_0{}^\infty = \lim_{\lambda\to 0}(1/\lambda)\,\mathcal{F}(\lambda)\log 1/\lambda$.

As with the linearized equations, we have

COROLLARY. *For the physical problem with boundary conditions* (3), (4), $\mathcal{F}(\lambda)$ *is asymptotically a pure resistance and is asymptotically independent of* Σ.

In analogy with the case $n = 3$, we may introduce a class PR of (two-dimensional) solutions of (2) in $\mathcal{E}$ satisfying $|\boldsymbol{x}|^{1/4+\epsilon}|\boldsymbol{w}(\boldsymbol{x}) - \boldsymbol{w}^\infty| < C < \infty$. D. R. Smith has shown [11] that all such solutions exhibit a parabolic wake region opening in the direction $\boldsymbol{w}^\infty$, and satisfy estimates analogous to (29).

8. **The vorticity at infinity.** The asymptotic properties of PR solutions for $n = 2$ or 3 have been investigated in some detail by D. Clark [19]. We cite, in particular, the result: *Let* $n = 3$ *and let* $\boldsymbol{w}(\boldsymbol{x})$ *be a solution of class PR in* $\mathcal{E}$, *with* $\boldsymbol{w}^\infty \neq 0$. *Then along any ray extending to infinity in a direction not coinciding with that of* $\boldsymbol{w}^\infty$, *there holds* $|\mathrm{rot}\, \boldsymbol{w}(\boldsymbol{x})| < C_1 e^{-C_2|\boldsymbol{x}|}$ *for certain positive constants* C_1, C_2. The result shows that any PR flow becomes asymptotically (and exponentially) potential in directions leading out of the wake. In the direction $\boldsymbol{w}^\infty$, Clark's estimate no longer holds.

Clark also obtained an analogous result in the case $n = 2$.

9. **Relations between Leray's solutions and PR solutions.** We note that Leray's solution of the exterior problem (§4) was based essentially on an a priori estimate for the Dirichlet integral $\int_{\mathcal{E}} |\nabla \boldsymbol{w}|^2\, dx \equiv D[\boldsymbol{w}]$ that holds for all solutions of the type he considered. In contrast, the construction of a PR solution (§§5–7) has made—at least as far as the nonlinear equations are concerned—no use at all of the Dirichlet integral. It would be desirable to determine conditions under which the two solutions could be identified, especially in view of Theorem

5C; in this respect, a recent result of Heywood (§14) may be an important first step.

In analogy with the class PR, and motivated by Leray's existence proof, we may consider the class D of all solutions $w(x)$ in $\mathcal{E}$ such that $D[w] < \infty$. A strengthened form of Theorem 5B (Theorems 5.3, 5.4 in [14]), when used in conjunction with Leray's procedure, yields the result

THEOREM 9A. *Let $w(x) \in PR$ in $\mathcal{E}$, $w(x) = w^*$ on Σ. Then $w(x) \in D$, and furthermore $D[w] < M < \infty$, depending only on Σ, on w^* and on w^∞.*

That is, if (say, for $n = 3$) $w(x) = w^*$ on Σ, $|x|^{1/2+\epsilon}|w(x) - w^\infty| < C < \infty$ in $\mathcal{E}$, then $w(x)$ is a priori bounded in the class D, depending only on Σ and on the data w^*, w^∞, and *not* on C.

Theorem 9A is given in [2, p. 213], for the case $n = 3$. If $n = 2$, it follows in the same way from the (later) results of Smith [**11**].

The result suggests that solutions in PR could be constructed by Leray's procedure. However, even this limited step toward identification of the classes remains open.

Solutions in class D are in general not known to be in class PR. If $n = 2$, the example

$$w = u - iv = i(1 - \alpha)r^{-\alpha}e^{-i\theta} + (1 + \alpha)e^{i\theta}/r$$

defines for any real α a solution $w(x)$ of (2) for all $r = |x| > 0$; if $\alpha > 0$, then $w \in D$, but if $\alpha \leqq \frac{1}{4}$, then $w \notin$ PR.

For $n = 3$, the situation is less clear. We do have the result [9], [7], [4], [5]

THEOREM 9B. *Every solution of class D tends to a finite limit w^∞ at infinity.*

The hypothesis $w(x) \in D$ has yielded no further information on the asymptotic structure of the solutions. The following easily proved result [2, p. 229] is, however, suggestive.

Let $w(x)$ be any vector valued function having finite Dirichlet integral in a three-dimensional neighborhood of infinity, and such that $\lim_{x\to\infty} w(x) = w^\infty$. Then on almost every ray from the origin there holds $|x|^{1/2}|w(x) - w^\infty| < C < \infty$ for some constant C.

One might expect that if $w(x)$ were known to be a solution in $\mathcal{E}$, this additional knowledge would suffice to put $w(x)$ in PR. As yet, it has not.

In the case $n = 2$, the knowledge that $w(x) \in D$ in $\mathcal{E}$ has in most respects led to less information about asymptotic behavior than in the three-dimensional case. Nevertheless, Gilbarg and Weinberger have shown in an unpublished work that *if* $w(x) \in D$ *in the entire plane, then* $w(x)$ *represents a uniform flow,* $w(x) \equiv$ const. The corresponding result for $n = 3$ has been shown [15], [14] only under the presumably stronger hypothesis $w(x) \in$ PR. (The proof for $n = 2$ relies on the maximum principle for the vorticity, for which no three-dimensional analogue is available.)

10. **Other solutions in** $\mathcal{E}$. Theorem 9B suggests again a question, whether every solution of (2) continuous at infinity is in class D. Again, no answer is as yet available. However, it can be shown [15] that *if* $w(x) \to w^{\infty}$ *at infinity, then the derivatives of* $w(x)$ *of all orders vanish at infinity.*

This result raises in turn the question, whether every such solution admits an asymptotic expansion in terms of prescribed functions, analogous to the expansion of a harmonic function in spherical harmonics. If $n = 2$, the answer is negative, as one sees by considering the example of §9 for small positive α. We may note also that the choice $\alpha = 0$ yields a solution bounded and discontinuous at infinity, a situation that could not occur in potential flow.

11. **The representation formula.** An integration by parts, analogous to the derivation of Green's formula of potential theory, yields the representation (cf. (9))

$$\begin{aligned} w(x) &= \int_{\mathfrak{D}} E(x-y)\cdot w(y)\cdot \nabla w(y)\,dy \\ &\quad + \oint_{\Sigma} (w\cdot TE - E\cdot Tw)\cdot d\mathbf{s}_y \\ &= -\int_{\mathfrak{D}} w\cdot w\cdot \nabla E\,dy \\ &\quad + \oint_{\Sigma} [w\cdot TE - E\cdot Tw + (E\cdot w)w]\cdot d\mathbf{s} \end{aligned}$$

for any solution $w(x)$ of (2) in a (smoothly) bounded domain $\mathfrak{D}$. In an exterior domain $\mathcal{E}$, we obtain, using the Oseen tensor (24) corresponding to the assumed limiting velocity w^{∞}, the formula

$$\begin{aligned} u(x) &= -\int_{\mathcal{E}_R} u\cdot u\cdot \nabla \hat{E}\,dy \\ &\quad + \oint_{\Sigma+\Sigma_R} [u\cdot T\hat{E} - \hat{E}\cdot Tu + (\hat{E}\cdot u)w]\cdot d\mathbf{s} \end{aligned}$$

for the difference $u(x) = w(x) - w^{\infty}$, in the region $\mathcal{E}_R$ bounded by Σ and by a sphere (circle) of large radius Σ_R. In discussing the integral over Σ_R as $R \to \infty$, the following cases have a special interest:

(i) $w(x) \in D, n = 3$,
(ii) $w(x) \in \mathrm{PR}$,
(iii) $\int^{\infty} (m^2(\rho)/\rho)\, d\rho < \infty$, where $m(\rho) = \max_{|x| \geqq \rho} |u(x)|$.

None of these hypotheses permit a direct estimate of the integral based on orders of magnitude. Nevertheless, we have [**15**]

THEOREM 11A. *Under any of the hypotheses* (i), (ii), *or* (iii), *there holds*

$$\lim_{R\to\infty} \oint_{\Sigma_R} [u \cdot T\hat{E} - \hat{E} \cdot Tu + (\hat{E} \cdot u)w] \cdot d\boldsymbol{\sigma} = 0,$$

and

$$w(x) = w^{\infty} - \int_{\mathcal{E}} u \cdot u \cdot \nabla \hat{E}\, dy + \oint_{\Sigma} [u \cdot T\hat{E} - \hat{E} \cdot Tu + (\hat{E} \cdot u)w] \cdot d\boldsymbol{\sigma}$$

throughout $\mathcal{E}$.

This representation has been useful in a number of contexts, and it is the basis for the derivation of asymptotic properties of PR solutions; cf. [**15**], [**14**], [**11**]. Some of these properties are indicated in Theorem 5B and in §7. Further applications appear in [**15**, pp. 413–417] and in [**14**, p. 393], for $n = 3$, and in [**11**, pp. 366–372] for $n = 2$. We note here a particular result (cf. Berker [**31**]).

THEOREM 11B. *Let* $w(x)$ *be a solution of* (2) *in* $\mathcal{E}$, *and suppose* $w(x) = 0$ *on* Σ. *If there is a limiting velocity* w^{∞} *such that*

$$|w(x) - w^{\infty}| = o(|x|^{-1}), \qquad n = 3,$$
$$= o(|x|^{-1/2}), \qquad n = 2,$$

then $w(x) \equiv 0$ *in* $\mathcal{E}$.

OUTLINE OF PROOF. Let

$$|x| = r, \qquad \sigma(r) = r^{-1}, \qquad n = 3,$$
$$= r^{-1/2} \qquad n = 2.$$

We note the hypotheses imply, in particular, $w(x) \in \mathrm{PR}$. The estimates of [**14**] or of [**11**] imply $|\int_{\mathcal{E}} u \cdot u \cdot \nabla \hat{E}\, dy| = o(\sigma(r))$. Also, from (24) or its analogue for $n = 2$ we find $|T\hat{E}(x, y)| = o(\sigma(r))$ for $y \in \Sigma$. Thus by hypothesis and the above representation,

$|\oint_\Sigma \hat{E} \cdot Tu \cdot d\mathfrak{d}| = o(\sigma(r))$. Writing $\hat{E}(x; y) = \hat{E}(x; 0) + \Lambda(x; y)$, we find by the mean value theorem that $|\Lambda(x; y)| = o(\sigma(r))$ uniformly for $y \in \Sigma$. Hence $|w(x) - w^\infty| = |\hat{E}(x; 0) \cdot \oint_\Sigma Tu \cdot d\mathfrak{d}| + o(\sigma(r))$. But $(1/\sigma(r))|\hat{E}(x; 0)|$ does not vanish at infinity, hence $-\oint_\Sigma Tu \cdot d\mathfrak{d} = -\oint_\Sigma Tw \cdot d\mathfrak{d} = 0$, that is, there can be no net force on Σ. Applying (30) (which holds also for $n = 2$) we obtain **def** $w \equiv 0$; that is, $w(x)$ represents a rigid motion in $\mathcal{E}$. Since $w(x) = 0$ on Σ there follows $w(x) \equiv 0$ in $\mathcal{E}$, which was to be shown.

In general, the expression $w^\infty + \hat{E}(x; 0) \cdot \oint_\Sigma Tw \cdot d\mathfrak{d}$ yields the first terms of an asymptotic expansion of a PR solution at infinity [**15**], [**14**]; thus, as in potential flow, the force on Σ is completely determined by the first order terms of the expansion.

Some of the theory of PR solutions for $n = 2$ has been rederived from another point of view by K. I. Babenko [**32**], who obtained a formula for the "lift" (force orthogonal to w^∞) on Σ in terms of the asymptotic circulation of the flow.

We note the hypothesis (i) requires $n = 3$. For $n = 2$ the condition is not yet known to imply the representation, even for solutions that are continuous at infinity (cf. the remarks at the end of §9).

In the paper of Smith [**11**, p. 349], the result is stated for $n = 2$ under a weakened form of (ii). The proof is however not complete.

12. **Stationary perturbations; flows at low Reynolds' number.** In view of the Stokes paradox, it is important to determine the sense in which the solution $U(x)$ of the linearized problem (6), (3), (4) can be interpreted as an approximation to a solution of (2), (3), (4) when the data are small. In two dimensions, this question is improperly posed, as (6), (3), (4) in general admits no solution. However, if $n = 3$, then despite the expected singularity in the perturbation (cf. the remarks in §1), all solutions $w(x)$ of (2) in $\mathcal{E}$ with $w(x) = 0$ on Σ and $w(x) \to \lambda w^\infty$ at infinity $(0 < \lambda \leqq 1)$ satisfy $|(1/\lambda)w(x) - U(x)| < C\lambda^{1/2}$ uniformly in $\mathcal{E}$, with a fixed constant C independent of the particular solution [**2**]. This result holds not only for solutions in PR (for which uniqueness holds for small λ) but for arbitrary perturbations of solutions in D; it justifies the use of (7) as an approximate equation to facilitate the calculation of slow flows (flows at low Reynolds' number).

More generally, one may study various classes of perturbations of a uniform flow to a solution $w(x)$ of the general boundary problem (2), (3), (5). To do so, it is convenient to introduce the field $u(x; \lambda) = (1/\lambda)[w(x) - w^\infty]$, which satisfies

$$\Delta u - w^\infty \cdot \nabla u - \nabla p = \lambda u \cdot \nabla u,$$

$$\nabla \cdot u = 0.$$

We consider a family of solutions $u(x; \lambda)$ for which the boundary data $u^* = (1/\lambda)(w^* - w^\infty)$ remain fixed, as $\lambda \to 0$. If the data at infinity are also varied (as in the above example), we prescribe $\lim_{x\to\infty} u(x; \lambda) = u^\infty \neq 0$. Otherwise (as in the procedure used for the existence proofs in §§5–7) we set $u^\infty = 0$. These situations are studied in detail in [14], for perturbations either in PR or in D. The results are summarized in Table 1. Here $U(x)$ denotes the solution of the limiting equation ($\lambda = 0$) with the same data. We note that Case 1 is the only one for which an asymptotic expansion appears. This is the expansion that arises in the successive approximation procedure used in the usual demonstration of the fixed point principle on which the existence theorems of §§5–7 were based.

The estimates of Table I were developed in [14] only for the case $n = 3$. However, it follows from the material of §§6–7 that Case 1 applies also if $n = 2$, with the modification that the global estimate must be changed to $|u(x; \lambda) - U(x)| < Cr^{-1/2+\epsilon}$, any $\epsilon > 0$.

TABLE 1

Case	Flow Class	Limiting Perturbation	Uniform Estimate in $\mathcal{E}$
1	PR	zero	$u(x; \lambda) = U(x) + \sum_1^\infty u_j(x)\lambda^j$ $\lvert u(x; \lambda) - U(x)\rvert < C\lambda r^{-1}$
2	PR	$u^\infty \neq 0$	$\lvert u(x; \lambda) - U(x)\rvert$ $< C \min \{\lambda \log \lambda, r^{-1} \log r\}$
3	D	zero	$\lvert u(x; \lambda) - U(x)\rvert < C\lambda$ $\lvert\nabla u(x; \lambda) - \nabla U(x)\rvert < C(\lambda^2 + \lambda r^{-1/2})$
4	D	$u^\infty \neq 0$	$\lvert u(x; \lambda) - U(x)\rvert < C(\lambda + \lambda^{1/2} r^{-1})$ $\lvert\nabla u(x; \lambda) - \nabla U(x)\rvert < C(\lambda^2 + \lambda r^{-1/2})$

13. **Stability questions.** What happens when a solution $\overline{w}(x)$ of (2) of class PR is subjected to a small disturbance $u_0(x)$? The question has been studied by J. G. Heywood [**20**], who constructed generalized solutions of the time dependent equations (1) with initial data $\overline{w}(x) + u_0(x)$ and with boundary data $\overline{w}^*(x)$, $\overline{w}^\infty$. Heywood found that these solutions are unique, and that they exist for all time and return (in a generalized sense) to $\overline{w}(x)$, provided (a) $w(x)$ satisfies the conditions of the uniqueness theorem (§5) and (b) $u_0(x)$ is appropriately small. The procedure combines the method used by Ladyzhenskaia [**21**], [**4**] to solve the initial-boundary problem in a bounded domain,

with the technique used for the stationary uniqueness result (§5).

To emphasize the physical significance of Heywood's result, the coefficients will not be normalized, and we state the results in terms of the "kinematic viscosity coefficient" $\nu = \rho/\mu$. For a (supposed) time dependent solution $\boldsymbol{w}(\boldsymbol{x}; t)$, the function $\boldsymbol{u}(\boldsymbol{x}; t) = \boldsymbol{w}(\boldsymbol{x}; t) - \overline{\boldsymbol{w}}(\boldsymbol{x})$ satisfies

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \overline{\boldsymbol{w}} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \overline{\boldsymbol{w}}, \qquad \nabla \cdot \boldsymbol{u} = 0. \tag{45}$$

We introduce the class $D(\mathcal{E})$ of vector fields $\boldsymbol{\varphi}(\boldsymbol{x}) \in \mathcal{C}_0^{\infty}(\mathcal{E})$ with $\nabla \cdot \boldsymbol{\varphi} = 0$, and the closures $J(\Omega)$ and $J_1(\Omega)$ in the respective norms $\{\int_{\mathcal{E}} |\boldsymbol{\varphi}|^2 \, d\boldsymbol{x}\}^{1/2} = \|\boldsymbol{\varphi}\|_{\mathcal{E}}$ and $(\|\boldsymbol{\varphi}\|_{\mathcal{E}}^2 + \|\nabla \boldsymbol{\varphi}\|_{\mathcal{E}}^2)^{1/2}$. Similarly, we consider in the cylinder $Q_T = \mathcal{E} \times [0, T]$ the class $D(Q_T)$ of vector fields $\boldsymbol{\varphi}(\boldsymbol{x}; t) \in \mathcal{C}^{\infty}(Q_T)$ with $\boldsymbol{\varphi}(\boldsymbol{x}; t) \in D(\mathcal{E})$ for each $t \in [0, T]$, and the corresponding closures $J(Q_T)$, $J_1(Q_T)$. If $\boldsymbol{\varphi} \in D(Q_T)$ and $\boldsymbol{u}(\boldsymbol{x}; t)$ is a smooth solution of (45) in Q_T, we multiply (45) by $\boldsymbol{\varphi}(\boldsymbol{x}; t)$ and integrate by parts to obtain

$$\int_{Q_T} \{[\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \overline{\boldsymbol{w}} - \overline{\boldsymbol{w}} \cdot \nabla \boldsymbol{u}]\boldsymbol{\varphi} - \nu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\varphi}\} \, dx \, dt = 0. \tag{46}$$

A function $\boldsymbol{u}(\boldsymbol{x}; t) \in J_1(Q_T)$ having a time derivative $\boldsymbol{u}_t(\boldsymbol{x}; t) \in J_1(Q_T)$ will be called a generalized solution of (45) in Q_T provided (46) holds for any $\varphi \in J_1(Q_T)$. Clearly any smooth solution is a generalized solution; a generalized solution, if it is smooth, is a solution (cf. [4, p. 144]). We seek to construct a generalized solution that assumes the initial data $\boldsymbol{u}_0(\boldsymbol{x})$ in a reasonable sense. We follow a method due originally to Galerkin, and first applied to the Navier-Stokes equations by E. Hopf [35].

Let $\{\boldsymbol{a}^l(\boldsymbol{x})\}$ be a complete set of functions in $J_1(\mathcal{E})$; for convenience, we choose the $\{\boldsymbol{a}^l\}$ to be orthonormal in $J(\mathcal{E})$ and $\boldsymbol{a}^l \in \mathcal{C}_0^{\infty}(\mathcal{E})$. We fix $\boldsymbol{a}^1(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x})/\|\boldsymbol{u}_0\|$ and set

$$\boldsymbol{u}^k(\boldsymbol{x}, t) = \sum_{l=1}^{k} c_{kl}(t)\boldsymbol{a}^l(\boldsymbol{x}) \tag{47}$$

with coefficients $c_{kl}(t)$ determined by the condition that (46) should hold with φ replaced by any of the functions $\boldsymbol{a}^1, \cdots, \boldsymbol{a}^l$ and for arbitrary T. Letting (,) denote scalar product in J, we obtain, from (46) and (47),

$$(\boldsymbol{u}_t^k, \boldsymbol{a}^l) + \nu(\nabla \boldsymbol{u}^k, \nabla \boldsymbol{a}^l) = -(\boldsymbol{u}^k \cdot \nabla \boldsymbol{u}^k, \boldsymbol{a}^l) - (\bar{\boldsymbol{w}} \cdot \nabla \boldsymbol{u}^k, \boldsymbol{a}^l) - (\boldsymbol{u}^k \cdot \nabla \bar{\boldsymbol{w}}, \boldsymbol{a}^l),$$

or

$$\frac{d}{dt} c_{kl}(t) = \sum_{m=1}^{k} c_{km}(t)(\nu\, \Delta \boldsymbol{a}^m - \bar{\boldsymbol{w}} \cdot \nabla \boldsymbol{a}^m - \boldsymbol{a}^m \cdot \nabla \boldsymbol{w}, \boldsymbol{a}^l) + \sum_{m,n=1}^{k} c_{km}(t) c_{kn}(t)(-\boldsymbol{a}^m \cdot \nabla \boldsymbol{a}^n, \boldsymbol{a}^l). \tag{48}$$

The general existence theorem for systems of ordinary equations shows that for any fixed k there is a unique solution set $\{c_{kl}(t)\}$ on some interval $[0, T)$. We shall obtain conditions to ensure that $\sum_{m=1}^{k} c_{km}^2(t) < A < \infty$ in any interval of existence; this result implies that the solution can be continued for all $t > 0$. Since $\sum_{m=1}^{k} c_{km}^2(t) = \|\boldsymbol{u}^k(t)\|^2$, we see that what is needed is an energy estimate on the approximating solutions $\boldsymbol{u}_k(t)$. A corresponding bound on $\|\boldsymbol{u}_t^k(t)\|$ will then ensure convergence, as $k \to \infty$, to a generalized solution of (45).

Multiply each equation (48) by $c_{kl}(t)$ and sum, noting that $(\boldsymbol{u}^k \cdot \nabla \boldsymbol{u}^k, \boldsymbol{u}^k) = (\bar{\boldsymbol{w}} \cdot \nabla \boldsymbol{u}^k, \boldsymbol{u}^k) = 0$. We get

$$\frac{1}{2}\frac{d}{dt} \|\boldsymbol{u}^k\|^2 + \nu \|\nabla \boldsymbol{u}^k\|^2 = -(\boldsymbol{u}^k \cdot \nabla \bar{\boldsymbol{w}}, \boldsymbol{u}^k). \tag{49}$$

Similarly, differentiating (48) in t and multiplying by $(d/dt)c_{kl}(t)$ we find

$$\frac{1}{2}\frac{d}{dt} \|\boldsymbol{u}_t^k\|^2 + \nu \|\nabla \boldsymbol{u}_t^k\|^2 = -(\boldsymbol{u}_t^k \cdot \nabla \boldsymbol{u}^k, \boldsymbol{u}_t^k) - (\boldsymbol{u}_t^k \cdot \nabla \bar{\boldsymbol{w}}, \boldsymbol{u}_t^k). \tag{50}$$

In what follows k will be fixed, and for notational simplicity we delete it from the formulas. We estimate the right side of (49)

$$|(\boldsymbol{u} \cdot \nabla \bar{\boldsymbol{w}}, \boldsymbol{u})| = |\boldsymbol{u} \cdot (\bar{\boldsymbol{w}} - \boldsymbol{w}^\infty), \nabla \boldsymbol{u}| \leqq \|\nabla \boldsymbol{u}\| \, \|\boldsymbol{u} \cdot (\bar{\boldsymbol{w}} - \boldsymbol{w}^\infty)\| \leqq C \|\nabla \boldsymbol{u}\| \left\{ \int_{\mathcal{E}} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2} dx \right\}^{1/2} \leqq 2C \|\nabla \boldsymbol{u}\|$$

by (31), since $|\boldsymbol{x}| \, |\bar{\boldsymbol{w}}(\boldsymbol{x}) - \boldsymbol{w}^\infty| < C$ by (29). Thus, from (49),

$$\frac{1}{2}\frac{d}{dt} \|\boldsymbol{u}\|^2 + (\nu - 2C)\|\nabla \boldsymbol{u}\|^2 \leqq 0 \tag{51}$$

yielding the energy estimate

$$\tfrac{1}{2}\|u(t)\|^2 + (\nu - 2C)\int_0^t \|\nabla u\|^2\,d\tau \leqq \tfrac{1}{2}\|u(0)\|^2 \tag{52}$$

and providing an a priori bound for $\|u(t)\|$, depending only on the initial data, whenever $C \leqq \nu/2$. In particular, if $C \leqq \nu/2$ the solution of (48) can be continued for all positive t. We may also write

$$\begin{aligned}(\nu - 2C)\|\nabla u\|^2 &\leqq -\frac{1}{2}\frac{d}{dt}\int_{\mathscr{E}} u^2\,dx = -\int_{\mathscr{E}} uu_t\,dx \\ &\leqq \|u(t)\|\;\|u_t(t)\|\end{aligned}$$

so that, from (52),

$$(\nu - 2C)\|\nabla u\|^2 \leqq \|u(0)\|\;\|u_t(t)\|. \tag{53}$$

From (50) we may estimate $\|u_t(t)\|$. Exactly as above, we find

$$(u_t \cdot \nabla \bar{w}, u_t) \leqq 2C\|u_t(t)\|.$$

Also, one verifies easily $(u_t \cdot \nabla u,\ u_t) \leqq \|\nabla u\|\;\|u_t{}^2\|$ from which (cf. [22, p. 77])

$$\begin{aligned}(u_t \cdot \nabla u, u_t) &\leqq 3^{-3/4}\|\nabla u\|\;\|u_t\|^{1/2}\|\nabla u_t\|^{3/2} \\ &\leqq \tfrac{1}{4}3^{-3/4}\|\nabla u\|\{\lambda^{-3}\|u_t\|^2 + 3\lambda\|\nabla u_t\|^2\}\end{aligned}$$

for any $\lambda > 0$, by the Young's inequality $ab \leqq a^p/p + b^q/q$ with $p = 4$, $q = 4/3$. Setting $\lambda = 4(\nu - 2C)/3^{1/4}\|\nabla u\|$, we obtain from (50), whenever $\nu - 2C \geqq 0$,

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 \leqq \frac{1}{2^8}\frac{\|\nabla u\|^4}{(\nu - 2C)^3}\|u_t\|^2. \tag{54}$$

From (53), $(\nu - 2C)\|\nabla u\|^2 \leqq \|u(0)\|\;\|u_t\|$ so that (54) yields

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 \leqq \frac{\|u(0)\|\;\|\nabla u\|^2\|u_t\|^3}{2^8(\nu - 2C)^4} \tag{55}$$

from which,[12] applying (52) and setting $Q(t) = \|u_t\|$,

$$\frac{1}{Q^2}Q'(t) \leqq \frac{\|u(0)\|}{2^8(\nu - 2C)^4}\;\|\nabla u\|^2. \tag{56}$$

Integrating (56) and again applying (52),

$$1/Q(0) - 1/Q(t) \leqq \|u(0)\|^3/2^9(\nu - 2C)^5 \tag{57}$$

which provides an upper bound for $Q(t)$ whenever

$$Q(0) = \|u_t(0)\| < 2^9(\nu - 2C)^5/\|u_0\|^3.$$

Using the differentiated form of (48) one obtains the estimate

$$\|u_t(0)\| \leqq \|P\{\nu\Delta(u_0 + \bar{w}) - (u_0 + \bar{w}) \cdot \nabla(u_0 + \bar{w})\}\| \tag{58}$$

where P is the projection operator from $L^2(\Omega)$ into $J(\Omega)$. From (52), (57) and (53) we thus obtain bounds on $\|u\|$, $\|\nabla u\|$ and $\|u_t\|$ that are uniform in time, provided $2C < \nu$ and the initial disturbance $u_0 = u(x; 0)$ is sufficiently small.

We note also a further estimate that follows from these bounds. If the above choice for λ is replaced by $\lambda/2$, the inequality (54) becomes

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \frac{(\nu - 2C)}{2}\|\nabla u_t\|^2 \leqq \frac{1}{2^5}\frac{\|\nabla u\|^4}{(\nu - 2C)^3}\|u_t\|^2 \tag{59}$$

which, when the above bounds hold, can be integrated in time, yielding

$$\int_0^\infty \|\nabla u_t\|^2\, dt < H \tag{60}$$

where H depends only on $\nu - 2C$, on $\bar{w}$, and on the initial data $u(x; 0)$.

We recall that the above notation is somewhat misleading, in that the estimates have been obtained only for the kth approximating function $u^k(x; t)$ of the Galerkin procedure. All estimates are, however, independent of k, and this permits the passage to the limit, in L_2 norm for u^k, $u_t{}^k$, *and* ∇u^k, to obtain a (unique) generalized solution $u(x; t)$ of the initial value problem. For details we refer the reader to [**20**].

Finally it remains to show that the time dependent solution converges to $\bar{w}(x)$ as $t \to \infty$. Following Heywood [**20**], we observe that

$$\int_0^\infty \left|\frac{d}{dt}\|\nabla u\|^2\right| dt \leqq 2\left\{\int_0^\infty \|\nabla u\|^2\, dt\right\}^{1/2}\left\{\int_0^\infty \|\nabla u_t\|^2\, dt\right\}^{1/2}$$

$$\leqq H\|u(0)\|^2/(\nu - 2C)$$

by (52) and (60). Thus, $\int_0^\infty |d/dt\|\nabla u\|^2|\, dt$ and $\int_0{}^\infty \|\nabla u\|^2\, dt$ are both finite, from which we conclude $\|\nabla u\| = \|\nabla(w(x; t) - \bar{w}(x))\|$ tends to zero as $t \to \infty$. By (31), $\int_{\mathcal{E}_R} u^2(x; t)\, dx$ also tends to zero, for any fixed R.

We summarize the result.

THEOREM 13A. *Suppose the initial data* $u_0(x) = w(x; 0) - \bar{w}(x) = w_0(x) - \bar{w}(x)$ *satisfy*

(61) $$\|P(\nu\Delta \boldsymbol{w}_0 - \boldsymbol{w}_0 \cdot \nabla \boldsymbol{w}_0)\| \cdot \|\boldsymbol{u}_0\|^3 < 2^9(\nu - 2C)^5.$$

Then there is a unique generalized solution $\boldsymbol{u}(\boldsymbol{x}; t)$ *of* (45) *in* $\mathcal{E}$ *(and correspondingly a generalized solution* $\boldsymbol{w}(\boldsymbol{x}; t)$ *of* (1) *in* $\mathcal{E}$*) that achieves the initial data in* L^2 *norm; for this solution there holds*

$$\lim_{t\to\infty} \int_{\mathcal{E}} |\nabla(\boldsymbol{w} - \overline{\boldsymbol{w}})|^2 \, dx = 0,$$

and

$$\lim_{t\to\infty} \int_{\mathcal{E}_R} |\boldsymbol{w} - \overline{\boldsymbol{w}}|^2 \, dx = 0$$

for any fixed R.

The solution $\boldsymbol{w}(\boldsymbol{x}; t)$ is known to possess considerably more local smoothness than is indicated above. It has not yet been shown, however, that $\boldsymbol{w}(\boldsymbol{x}; t)$ tends to the (smooth) $\overline{\boldsymbol{w}}(\boldsymbol{x})$ in a uniform pointwise sense. See, however, the remarks below on the paper of Cannon and Knightly (§15).

14. **Time dependent boundary data; physical realizability of stationary solutions.** We note that the above discussion has been restricted to disturbances $u_0(x)$ that are square integrable over $\mathcal{E}$. This property is retained by the generalized solution for all time. The base flow $\overline{\boldsymbol{w}}(x) - \boldsymbol{w}^\infty$ is, however, in general not square integrable. It will be so exactly in those cases for which the force on Σ and the momentum flux across Σ sum to zero (cf. [**14**, p. 393], see also [**23**]). These solutions have been studied by Heywood in a paper that is now in preparation, in which he shows that *if, in addition, the solution is uniformly small in* $\mathcal{E}$, *then it can be attained as a generalized limit, as* $t \to \infty$, *of a time dependent solution* $\boldsymbol{w}(\boldsymbol{x}; t)$, *with* $\boldsymbol{w}(\boldsymbol{x}; 0) \equiv 0$. Further, *given any solution* $\overline{\boldsymbol{w}}(\boldsymbol{x})$ *satisfying* $\|\overline{\boldsymbol{w}}(\boldsymbol{x}) - \boldsymbol{w}^\infty\| < \infty$ *and* $|\overline{\boldsymbol{w}}(\boldsymbol{x}) - \boldsymbol{w}^\infty|\ |x| < \nu/2$ *in* $\mathcal{E}$, *then no other solution* $\overline{\boldsymbol{v}}(\boldsymbol{x})$ *with the same data on* Σ *and at infinity can be attained by a solution* $\boldsymbol{w}(\boldsymbol{x}; t)$ *starting from rest, in the sense* $\|\boldsymbol{w}(\boldsymbol{x}; t) - \overline{\boldsymbol{v}}\|_{\mathcal{E}_R} \to 0$ for each $R < \infty$. Thus, *if, in the case considered, a Leray solution* (§4) *exists that differs from a given PR solution, it cannot be attained as a limiting configuration of a physically acceptable time dependent motion that is initially at rest.*

The idea of the proof is as follows: Suppose there is a stationary solution $\overline{\boldsymbol{v}}(\boldsymbol{x})$ and a corresponding time dependent solution $\boldsymbol{w}(\boldsymbol{x}; t)$ with $\boldsymbol{w}(\boldsymbol{x}; 0) = 0$, such that $\|\boldsymbol{w}(\boldsymbol{x}; t) - \overline{\boldsymbol{v}}(\boldsymbol{x})\|_{\mathcal{E}_R} \to 0$ as $t \to \infty$. For the particular stationary solution $\overline{\boldsymbol{w}}(\boldsymbol{x})$ indicated above, Heywood derives the energy inequality

$$\frac{1}{2}\|u(t)\|^2 + C\int_{T_0}^{t} \|\nabla u(\tau)\|^2\, d\tau \leqq \frac{1}{2}\|u(T_0)\|^2 \tag{62}$$

for the "disturbance flow" $u(x;t) = w(x;t) - \bar{w}(x)$, for all sufficiently large T_0. Since $\|w - \bar{v}\|_{\mathcal{E}_R} = \|u - (\bar{v} - \bar{w})\|_{\mathcal{E}_R} \to 0$, there must hold $\|(\bar{v} - \bar{w})\|_{\mathcal{E}_R} = 0$, for otherwise $\lim_{t\to\infty} \int_{T_0}^{t} \|\nabla u\|^2\, d\tau = \infty$, contradicting (62).

15. **Continuous dependence theorems.** We call attention finally to an important paper by Cannon and Knightly [24], in which, under remarkably mild hypotheses on asymptotic behavior (in x), the (pointwise) continuous dependence on boundary and initial data of solutions of (1) is proved. The result is obtained through a nice combination of potential theoretic estimates, energy estimates, and embedding theorems. The method of proof yields as a corollary the result that if the perturbed solutions studied by Heywood (§13) satisfy the general hypotheses, then his solutions converge pointwise to the initial (stationary) solution. It seems essentially certain that Heywood's solutions have this property, but it has not yet been proved.

The hypotheses of Cannon and Knightly are those introduced by D. Graffi for a general uniqueness theorem [25], and Graffi's method is integral to their procedure. Graffi's result applies in particular to the situations considered in this report, and has of course an independent interest.

Notes added in proof. 1. The conjecture of §15 that Heywood's solutions converge pointwise to the stationary solution has been proved by Knightly (informal communication).

2. I have just seen a preprint of a new work of Babenko [*On the stationary solutions of the problem of flow of a viscous incompressible fluid*, Lenin Inst. for Applied Mathematics, Academy of Sciences of the USSR, Preprint No. 40, Moscow 1972] in which is proved that if $n = 3$, every solution in class D is also in class PR. This result settles the question raised in §9 above. I am not yet in a position to report on this basic contribution in detail.

Footnotes

[1]p. 109: A notable example is the Millikan oil drop experiment for determining the charge on the electron. A proof of the asymptotic correctness of (7) appears in [2]. See also §12.

[2]p. 110: Outlines of Odqvist's results appear in [4], [5]. For earlier work on this problem, see Crudeli [37] and Lichtenstein [36].

[3]p. 111: The linearity of (6) permits the case of arbitrary data w^∞ to be reduced immediately to this one.

[4]p. 113: The existence theorem can be avoided at this point by use of the "truncated tensor" first introduced by Fujita [7, p. 77]; cf. the discussion in [**5**, p. 139].

[5]p. 113: This fact was exploited in [**30**] to obtain a correction term to the force predicted by (7).

[6]p. 115: The same result had been published previously by Lichtenstein [**36**] using other methods. The approach of Odqvist formed the basis for the later developments described in §§5—12 of this report. It was used also by McCready [**26**] in an important study of the analogue of N_i for the Navier-Stokes equations (2). See also the work of Knightly [**27**] and of Cannon and Knightly [**28**].

[7]p. 116: The above discussion was intentionally cursory, as the techniques needed for this existence proof—although basic for the general theory of (2)—are peripheral to the main thrust of these lectures. See, however, the material directly following and in §13. Other modifications of Leray's original proof have been given by Fujita [7] and by Shinbrot [**29**].

[8]p. 118: It suffices that Σ admit local parameters of class $\mathcal{C}^{(3)}$ and that $w^* - w^\infty$, together with its derivatives up to third order in these parameters, be sufficiently small in magnitude.

[9]p. 121: Here and in what follows, λ is equivalent to the Reynolds' number of the flows considered. We study flow at low Reynolds' number by choosing a model in which all parameters except w^∞ are fixed, and then letting $w^\infty \to 0$ with fixed direction.

[10]p. 122: Such a characterization does not suffice for the solutions of (21), nor does it for those of (6) in $n > 2$ dimensions.

[11]p. 122: In these respects our discussion has been simplified by D. Gilbarg (oral communication).

[12]p. 134: It suffices, of course, to restrict attention to intervals in which an inequality $Q(t) > Q_0 > 0$ holds. Thus, division by $Q(t)$ is permissible.

The research described in this report was supported in part by AFOSR contract F44620-69-C-0106 and in part by NSF grant GP-16115 with Stanford University. Some of the work was completed during my tenure in Paris as a Guggenheim Fellow.

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