

## INDEPENDENT CLASSES OF SEMIPRIMAL ALGEBRAS

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**1. Introduction.** Recent investigations in the area of universal algebra have focused attention on the conditions under which an algebra subdirectly decomposes within a given class of algebras. See, for example, Astromoff [1], Foster and Pixley [2], [3], [5], Gould and Grätzer [6], Hu [8], and Jónsson [9]. In particular, it has been shown that each cluster  $K$  of primal or, more generally, semiprimal algebras determines a certain unique subdirect factorization for each algebra which satisfies the identities common to some finite subset of  $K$ . Primal clusters (see [3], [10]–[12], [14], and [15]) are known to exist in great profusion.

In this paper we show the existence of clusters of semiprimal algebras of rather diverse nature, thereby enhancing the scope of applicability of the more general semiprimal theory. Our main result is

**THEOREM 1.** *A family  $K$  of semiprimal algebras is a cluster if (a) each member of  $K$  is strongly surjective, (b) the members of  $K$  have pairwise nonisomorphic structures, and (c) for each  $A \in K$ , the intersection of all subalgebras of  $A$  is nonempty.*

We prove this result with an eye on the techniques of O'Keefe [10]–[12]. It is shown in §3 that  $K$  is a cluster if its members are pairwise independent. This pairwise independence for members of  $K$  is then established in §4.

**2. Basic definitions.** Let  $S$  be a fixed finitary species (or type) of universal algebra. All algebras to be considered are assumed to be of species  $S$ .

(1°) A *term* (or *expression*) is any indeterminate symbol  $x, y, x_1, y_1, \dots$  or any finite composition of indeterminate symbols via the primitive operations of  $S$ . All terms are written as italic caps,  $F, G, H$ , etc. We write  $F = G(A)$  when two terms  $F$  and  $G$  agree as functions within an algebra  $A$ .

(2°) Any finitary mapping  $f: A^n \rightarrow A$  for arbitrary  $n$  is called an *A-function*. Such a function is said to be *conservative* if for each subalgebra  $A'$  of  $A$ ,  $f(a_1, \dots, a_n) \in A'$  whenever  $a_1, \dots, a_n \in A'$ . An algebra  $A$  is *primal* (respectively, *semiprimal*) if it is finite, contains

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at least two elements, and each  $A$ -function (respectively, each conservative  $A$ -function) is equal, within  $A$ , to some term.

(3°) In an algebra  $A$ ,  $[a]$  will denote the subalgebra of  $A$  generated by  $a \in A$ . Let  $f(x_1, \dots, x_n)$  be a conservative  $A$ -function;  $f$  is said to be *surjective* if for each  $a \in A$  there correspond  $a_1, \dots, a_n \in [a]$  for which  $f(a_1, \dots, a_n) = a$ ;  $f$  is *strongly surjective* if it is surjective and if for each  $a \in A$  not contained in a minimal subalgebra of  $A$  the corresponding  $a_1, \dots, a_n$  above can further be chosen to satisfy the "normalized" conditions  $[a_i] = [a]$  for at most one  $i = 1, \dots, n$ . Subalgebras of  $A$  are *minimal* or *maximal* if they are minimal or maximal with respect to inclusion. An algebra  $A$  is said to be surjective (respectively, strongly surjective) if each primitive operation of  $A$  is surjective (respectively, strongly surjective).

(4°) Two algebras  $A$  and  $B$  have *nonisomorphic structures* if  $A'$  and  $B'$  are nonisomorphic whenever  $A'$  and  $B'$  are non-one-element subalgebras of  $A$  and  $B$ , respectively.

(5°) A finite set of algebras  $A_1, \dots, A_n$  is *independent* if there exists a single term  $F(x_1, \dots, x_n)$  satisfying  $F(x_1, \dots, x_n) = x_i(A_i)$ ,  $i = 1, \dots, n$ . A *cluster* is any family of algebras, each finite subset of which is independent.

**3. The factorization property.** An algebra  $A$  is said to satisfy the *factorization property* if

(\*) for each primitive operation  $f(x_1, \dots, x_r)$  of  $A$  and each term  $F(x)$  there exist terms  $F_1(x), \dots, F_r(x)$  such that

$$f(F_1(x), \dots, F_r(x)) = F(x)(A).$$

It was shown in [10] that property (\*) is equivalent to

(\*\*\*) for each term  $G(x_1, \dots, x_m)$  in which no indeterminate occurs twice and each term  $F(x_1, \dots, x_n)$  there exist terms  $F_i(x_1, \dots, x_n)$ ,  $i = 1, \dots, m$ , such that

$$G(F_1(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_n)) = F(x_1, \dots, x_n)(A).$$

**LEMMA 1.** *The factorization property holds in any surjective semiprimal algebra.*

**PROOF.** Let  $A = \{a_1, \dots, a_n\}$  be a semiprimal algebra,  $F(x)$  an arbitrary term, and  $f(x_1, \dots, x_r)$  a primitive operation of  $A$ . If  $A$  is surjective, then, for each  $i = 1, \dots, n$ , there exist elements  $a_{i1}, \dots, a_{ir} \in [F(a_i)]$  satisfying  $f(a_{i1}, \dots, a_{ir}) = F(a_i)$ . Since, additionally,  $a_{ij} \in [a_i]$ , the functions  $f_j: A \rightarrow A$ ,  $j = 1, \dots, r$ , defined by  $f_j(a_i) = a_{ij}$ , are conservative. Being so, there correspond terms  $F_j(x)$  for which  $F_j(x) = f_j(x)(A)$ . An element by element verification then establishes that  $f(F_1(x), \dots, F_r(x)) = F(x)(A)$ .

The factorization property is extremely useful in clarifying the

relationship between pairwise and general independence.

LEMMA 2 (O'KEEFE [10]). *Suppose that  $K = \{\cdots, A_i, \cdots\}$  is a set of pairwise independent algebras. If the factorization property holds in each  $A_i$ , then  $K$  is a cluster.*

4. **Pairwise independence.** With an eye on the preceding lemma, we now turn our attention completely to the case of pairwise independence, the chief result being Theorem 2 below. We establish Theorem 2 via a sequence of several lemmata, the first two being of a rather general nature.

LEMMA 3. *Let  $A$  and  $B$  be semiprimal algebras with nonisomorphic structures. Then, for each sequence  $F_1(x), \cdots, F_n(x)$  of terms, there exists a corresponding sequence of terms  $G_1(x), \cdots, G_n(x)$  such that*

$$\begin{aligned} G_1(x) &= \cdots = G_n(x)(B), \\ G_i(x) &= F_i(x)(A), \quad i = 1, \cdots, n. \end{aligned}$$

PROOF. This lemma was established in [10] for the special case in which both  $A$  and  $B$  are primal. The same proof presented there readily adapts to the more general situation in which  $A$  is primal and  $B$  is an arbitrary semiprimal. Of course, the lemma is trivially true if  $A$  contains but one element. Consequently, we can and do assume the lemma to be true for  $B$  and any minimal subalgebra of  $A$ . This represents the first step of a proof by induction. Assume, then, that the lemma is true for each maximal subalgebra of  $A$ . We now show that the lemma is true for  $A$  also. To this end, let  $A_1, \cdots, A_m$  denote the distinct maximal subalgebras of  $A$  and assume the existence of terms  $G_{ji}(x), j = 1, \cdots, m, i = 1, \cdots, n$ , satisfying

$$\begin{aligned} (1) \quad G_{ji}(x) &= F_i(x)(A_j), \\ G_{j1}(x) &= \cdots = G_{jn}(x)(B). \end{aligned}$$

If  $A' = \text{def} = A_1 \cup \cdots \cup A_m$  is equal to  $A$ , we select any term  $T(x, x_1, \cdots, x_m)$  of  $m + 1$  arguments which satisfies for each  $a \in A$  the following noncontradictory conservative conditions:

$$T(a, G_{1i}(a), \cdots, G_{mi}(a)) = F_i(a), \quad i = 1, \cdots, n.$$

To see this, suppose that  $F_i(a) \neq F_k(a)$  for some pair  $i \neq k$ . But then, since  $a \in A_j$  for some  $j$ , it follows from (1) that  $G_{ji}(a) \neq G_{jk}(a)$ . Letting

$$G_i(x) = T(x, G_{1i}(x), \cdots, G_{mi}(x))$$

and applying the conditions of (1) we observe that

$$\begin{aligned} G_i(x) &= F_i(x)(A), \\ G_1(x) &= \cdots = G_n(x)(B). \end{aligned}$$

Next assume that  $A - A'$  is nonempty. Since  $A$  and  $B$  have non-isomorphic structures and as semiprimal algebras are simple, it follows from the subdirect factorization theorems of [5] that they do not satisfy the same identities. Hence, there exist terms  $H_k(x_1, \cdots, x_s)$ ,  $k = 1, 2$ , such that  $H_1 = H_2(B)$  and  $H_1 \neq H_2(A)$ . Let  $a_1, \cdots, a_s$  be elements of  $A$  for which

$$c_1 = \text{def} = H_1(a_1, \cdots, a_s) \neq H_2(a_1, \cdots, a_s) = \text{def} = c_2.$$

Choose terms  $K_1(x), \cdots, K_s(x)$  which satisfy in  $A$  the property  $K_t(x) = a_t(A - A')$ ,  $t = 1, \cdots, s$ . Letting

$$H'_k(x) = H_k(K_1(x), \cdots, K_s(x))$$

we see that  $H'_1(x) = H'_2(x)(B)$  and  $H'_k(x) = c_k(A - A')$ . Now choose any term  $Q(x_1, \cdots, x_{n+1})$  which satisfies, for  $i = 1, \cdots, n$ , the conservative conditions

$$Q(x, \underbrace{c_1, \cdots, c_1}_{i \text{ terms}}, c_2, \cdots, c_2) = F_i(x)(A).$$

Defining

$$K_i(x) = Q(x, \underbrace{H'_1(x), \cdots, H'_1(x)}_{i \text{ terms}}, H'_2(x), \cdots, H'_2(x))$$

we see that

$$\begin{aligned} (2) \quad K_i(x) &= F_i(x)(A - A'), \\ K_1(x) &= \cdots = K_n(x)(B). \end{aligned}$$

Finally, select any term  $T(x, x_1, \cdots, x_m, y)$  of  $m + 2$  arguments satisfying for each  $a \in A$  and each  $i = 1, \cdots, n$ , the noncontradictory conservative conditions

$$T(a, G_{1i}(a), \cdots, G_{mi}(a), K_i(a)) = F_i(a).$$

Letting

$$G_i(x) = T(x, G_{1i}(x), \cdots, G_{mi}(x), K_i(x))$$

and applying conditions (1) and (2) we have that

$$\begin{aligned} G_i(x) &= F_i(x)(A), \\ G_1(x) &= \cdots = G_n(x)(B). \end{aligned}$$

This completes the induction step and with it the lemma.

**LEMMA 4.** *If an algebra  $A$  is strongly surjective and  $K$  is any term in which no indeterminate occurs twice, then  $K$  is itself a strongly surjective  $A$ -function.*

If  $K$  is an indeterminate symbol, then trivially it is strongly surjective in  $A$ . The proof of Lemma 4 then follows in a straightforward manner by induction on the number of primitive operation symbols of which  $K$  is composed. We leave the details to the reader.

We now establish the following notation in an algebra  $A$ :

$$(3) \quad A_0 = \bigcap (A' : A' \text{ is a subalgebra of } A).$$

In an algebra  $A$  satisfying condition (c) of Theorem 1,  $A_0$  will always be nonempty.

The characterization theorems of [4] provide large classes of strongly surjective semiprimal algebras of the type under consideration here. Other examples can be found in [13].

**LEMMA 5.** *Let  $K = \{A, B\}$  be a two-element class of semiprimal algebras satisfying (a)-(c) of Theorem 1. Then, for each  $m \in A_0$  (see (3)) and term  $F(x)$ , we can find a term  $H(x)$  for which*

$$\begin{aligned} H(x) &= m(A), \\ &= F(x)(B). \end{aligned}$$

**PROOF.** Since  $A$  is semiprimal, there exists a unary term  $G$  for which  $G(x) = m(A)$ . Following O'Keefe [10], we denote by  $G'(x_1, \dots, x_n)$  the term derived from  $G$  by distinguishing each occurrence of  $x$ , i.e.  $G'(x, \dots, x) = G(x)$ . Since the factorization property holds in  $B$  (Lemma 1), there exist terms  $G_i(x)$ ,  $i = 1, \dots, n$ , such that

$$G'(G_1(x), \dots, G_n(x)) = F(x)(B).$$

But by Lemma 3 we can find terms  $F_i(x)$ ,  $i = 1, \dots, n$ , satisfying

$$\begin{aligned} F_i(x) &= G_i(x)(B), \\ F_1(x) &= \cdots = F_n(x)(A). \end{aligned}$$

The term

$$H(x) = G'(F_1(x), \dots, F_n(x))$$

then suffices for the lemma since it certainly reduces to  $F(x)$  in  $B$  and in  $A$ ,

$$G'(F_1(x), \dots, F_n(x)) = G'(F_1(x), \dots, F_1(x)) = G(F_1(x)) = m.$$

We now obtain terms which partially reflect the criteria of independence.

**LEMMA 6.** *Let  $A$  and  $B$  be as in Lemma 5 and assume that each maximal subalgebra of  $A$  is pairwise independent with  $B$ . Then, there exists an element  $c \in A_0$ , and corresponding to each  $a \in A$  a binary term  $F_a(x, y)$  satisfying*

$$\begin{aligned} F_a(x, y) &= y(B), \\ (4) \quad F_a(a, y) &= a, \quad y \in A, \\ F_a(x, y) &= c, \quad x, y \in A, x \neq a. \end{aligned}$$

**PROOF.** Associated with each of the distinct maximal subalgebras  $A_1, \dots, A_n$  of  $A$  is a binary term  $T_i, i = 1, \dots, n$ , satisfying the independence criteria

$$\begin{aligned} T_i(x, y) &= x(A_i), \\ &= y(B). \end{aligned}$$

We construct  $F_a(x, y)$  according as  $a \in A$  does or does not belong to  $A' = \text{def} = A_1 \cup \dots \cup A_n$ .

First, we consider the case in which  $a \in A - A'$ , provided this set is nonempty. Let  $d \in B_0$  be fixed and  $G$  be a binary term which satisfies in  $B$  the symmetric identities

$$\begin{aligned} (5) \quad G(d, y) &= y, \\ G(x, d) &= x. \end{aligned}$$

Denote by  $G'(x_1, \dots, x_r, y_1, \dots, y_s)$  the term derived from  $G(x, y)$  by distinguishing each occurrence of  $x$  as  $x_1, \dots, x_r$  and  $y$  as  $y_1, \dots, y_s$ . Since  $A$  is strongly surjective, there exist elements  $a_1, \dots, a_r, b_1, \dots, b_s \in [a] = A$  such that  $G'(a_1, \dots, a_r, b_1, \dots, b_s) = a$ . Moreover, this choice can be made to satisfy the following "normalized" conditions: either  $[a_j] \neq [a]$ , all  $j = 1, \dots, r$ , or  $[b_k] \neq [a]$ , all  $k = 1, \dots, s$ . Assume, without loss of generality, the former alternative. If the latter holds, we proceed similarly to the remarks below taking into account the symmetry of (5). Then, each  $a_j$  belongs to a maximal subalgebra  $A_{i(j)}$  of  $A$ . Pick terms  $F_j(x), j = 1, \dots, r$ , and  $G_k(x), k = 1, \dots, s$ , so that the conservative conditions

$$\begin{array}{ll} F_j(a) = a_j, & G_k(a) = b_k, \\ F_j(x) = m, & x \neq a, \quad \text{and} \quad G_k(x) = m, \quad x \neq a, \end{array}$$

hold in  $A$ , where  $m$  is a fixed element of  $A_0$ . Applying Lemma 5, we can obtain terms  $H_j(y)$  and  $G_k'(x)$  for which

$$\begin{array}{ll} H_j(y) = m(A), & G_k'(x) = G_k(x)(A), \\ = y(B), & \text{and} \quad = d(B). \end{array}$$

Letting  $K_j(x, y) = T_{i(j)}(F_j(x), H_j(y))$  for  $j = 1, \dots, r$  we verify that

$$F_a(x, y) = \text{def} = G'(K_1(x, y), \dots, K_r(x, y), G_1'(x), \dots, G_s'(x))$$

is a term associated with  $a \in A$  which satisfies (4), since

$$\begin{aligned} F_a(a, y) &= G'(a_1, \dots, a_r, b_1, \dots, b_s) = a, \quad y \in A, \\ (6) \quad F_a(x, y) &= G'(m, \dots, m, m, \dots, m) = \text{def} = c \in A_0, \\ & \quad x, y \in A, x \neq a, \\ F_a(x, y) &= G'(y, \dots, y, d, \dots, d) = G(y, d) = y(B). \end{aligned}$$

Finally, suppose  $a \in A_i$  for some  $i = 1, \dots, n$ , and let  $H(x)$  be a term so that the properties  $H(a) = a$  and  $H(x) = c$ ,  $x \neq a$ , hold in  $A$  where  $c \in A_0$  is as in (6). Again, invoking Lemma 5 we select a term  $G(y)$  for which  $G(y) = y(B)$  and  $G(y) = c(A)$ . Then,

$$F_a(x, y) = \text{def} = T_i(H(x), G(y))$$

possesses the desired properties of (4). This concludes the proof.

We are now in a position to prove our final lemma.

**LEMMA 7.** *Let  $K = \{A, B\}$  be a pair of semiprimal algebras satisfying the hypotheses of Theorem 1. If each maximal subalgebra of  $A$  is pairwise independent with  $B$ , then  $A$  and  $B$  are independent.*

**PROOF.** Let  $a_1, \dots, a_n$  be an enumeration of the elements of  $A$  and  $c \in A_0$  be as in (4). Since  $A$  is semiprimal there exists a term  $K(x_1, \dots, x_n)$  which satisfies in  $A$  the conditions

$$K(\underbrace{c, \dots, c}_{j-1 \text{ factors}}, a_j, c, \dots, c) = a_j, \quad j = 1, \dots, n.$$

Denote by  $K'(x_{11}, x_{12}, \dots, x_{j1}, x_{j2}, \dots, x_{n1}, x_{n2}, \dots)$  the term obtained from  $K$  by distinguishing each occurrence of  $x_j$  as  $x_{j1}, x_{j2}, \dots$  for  $j = 1, \dots, n$ . Applying Lemma 1 to  $B$ , we get unary terms  $G_{jk} = G_{jk}(y)$  such that

$$K'(G_{11}, G_{12}, \dots, G_{j1}, G_{j2}, \dots, G_{n1}, G_{n2}, \dots) = y(B).$$

For each  $a_j \in A$ , let  $F_{a_j}(x, y)$  be a term satisfying the conditions (4) of Lemma 6 and set

$$F_{jk} = F_{jk}(x, y) = F_{a_j}(x, G_{jk}(y)).$$

Then, letting  $F(x, y) = K'(F_{11}, F_{12}, \dots, F_{21}, F_{22}, \dots, F_{n1}, F_{n2}, \dots)$  we verify that in  $A$ ,

$$\begin{aligned} F(a_j, y) &= K'(c, c, \dots, a_j, a_j, \dots, c, c, \dots) \\ &= K(c, \dots, c, a_j, c, \dots, c) = a_j, \end{aligned}$$

for  $j = 1, \dots, n$ , and consequently that  $F(x, y) = x(A)$ . It is readily checked that  $F(x, y) = y(B)$ , thus establishing the independence of  $A$  and  $B$ .

**THEOREM 2.** *Let  $K = \{A, B\}$  be a pair of semiprimal algebras satisfying the conditions of Theorem 1. Then  $A$  and  $B$  are independent.*

**PROOF.** This result was established in [10] for the case of  $A$  and  $B$  both primal. We conclude, therefore, that  $A_0$  and  $B_0$  (see (3)) are independent. Holding  $A_0$  fixed, a many-fold application of Lemma 7 up through the lattice of subalgebras of  $B$  leads to the independence of  $A_0$  and  $B$ . Now fixing  $B$  and again applying Lemma 7 within the subalgebra structure of  $A$ , we finally obtain the independence of  $A$  and  $B$  themselves.

Upon applying Theorem 2, and Lemma 2 of §3, we obtain the principal Theorem 1 of this paper.

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