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FIELDS AND PROJECTIVE PLANES: A CATEGORY EQUIVALENCE

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1. **Introduction.** The well-known correspondence between fields and Pappian projective planes [1, p. 144] suggests a category equivalence if appropriate morphisms can be found. In this paper we shall construct a category of Pappian planes equivalent to the category of fields and places. As a consequence, we obtain a generalization for Pappian planes of the notion of collineation [4, p. 71] together with a characterization in terms of places (cf. [3, p. 154]).

The first problem is the choice of morphisms between planes. We begin with a general definition.

DEFINITION. A point-transformation $f: \pi \rightarrow \rho$ between projective planes is called a *lineation* if and only if points which are collinear in π have images under f which are collinear in ρ.

By this definition, for example, a collineation becomes a bijective lineation from a projective plane to itself. But a lineation is too general, so two specializations are considered.

A lineation will be called *proper* in case its image contains a quadrangle. By a *based plane* we mean a Pappian plane together with a distinguished ordered quadrangle, called a *base*. A lineation between based planes is called *basic* in case the base of the domain is carried in order onto the base of the codomain.

It is the category of based planes and basic lineations which proves to be equivalent to the category of fields and places. It will follow from the equivalence that to every proper lineation between Pappian planes there corresponds a place between the coordinatizing fields.

2. **The category equivalence.** It is necessary at this point to introduce some notation. If K is a field, we will denote by $\pi(K)$ the Pappian plane whose points and lines are respectively the one- and two-dimensional subspaces of a three-dimensional K-vector space. We will denote by $\beta(K)$ the based plane $\pi(K)$ together with the base $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$.

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A place $\varphi : K \to L$ will be treated as a partial ring homomorphism whose domain A_φ is a valuation subring of K. We also treat L as an extension of $A_\varphi/\ker \varphi$.

Our first result is due in part to D. K. Harrison.

PROPOSITION 1. *If A is a valuation subring of field K, with maximal ideal J, and $x_1, \cdots, x_n \in K$ are not all zero, then there exists $\alpha \in K$ such that $\alpha x_1, \cdots, \alpha x_n$ are all in A but not all in J.*

PROOF. If v is the valuation on K induced by A, then $\max \{v(x_i)\} > 0$. Let $v(\delta) = \max \{v(x_i)\}$, and set $\alpha = \delta^{-1}$. Then $\max \{v(\alpha x_i)\} = v(\alpha)v(\delta) = 1$, and the result follows.

Note that the choice of α in Proposition 1 is not unique. In fact, α_1 and α_2 both satisfy Proposition 1 for $x_1, \cdots, x_n$ relative to A if and only if $\alpha_1\alpha_2^{-1}$ is a unit of A.

If $\varphi : K \to L$ is a place, then a point P of $\pi(K)$ has coordinates (x_1, x_2, x_3) such that the choice $\alpha = 1$ satisfies Proposition 1 for x_1, x_2, x_3 relative to A_φ. Such a set of coordinates of P will be called a normal form for P relative to φ.

THEOREM 1. *If $\varphi : K \to L$ is a place and $\beta(\varphi) : \beta(K) \to \beta(L)$ is defined by $\beta(\varphi)(P) = (\varphi(x_1), \varphi(x_2), \varphi(x_3))$ where (x_1, x_2, x_3) is a normal form for P relative to φ, then $\beta(\varphi)$ is a basic lineation.*

PROOF. Since two normal forms for P are proportional by a unit u of A_φ, the images of the two forms under $\beta(\varphi)$ are proportional by $\varphi(u)$. Thus $\beta(\varphi)$ is well defined. Let P_j $(j = 1, 2, 3)$ be three collinear points in $\beta(K)$ with normal forms (x_{1j}, x_{2j}, x_{3j}). Then $\det \|(x_{ij})\| = \det \|x_{ij} + \ker \varphi\| = \det \|x_{ij}\| + \ker \varphi = 0 \in L$, so $\beta(\varphi)$ is a lineation. That $\beta(\varphi)$ is basic is immediate.

THEOREM 2. *If $f : \beta(K) \to \beta(L)$ is a basic lineation, then there is a unique place $\varphi_f : K \to L$ such that $\beta(\varphi_f) = f$.*

PROOF. Since f is basic and preserves collinearity, it follows that, for $i, j, k \in \{0, 1\}$, $f(i, j, k) = (i, j, k)$. For example, since $(1, 1, 0)$ is collinear with $(1, 0, 0)$ and $(0, 1, 0)$, and also with $(0, 0, 1)$ and $(1, 1, 1)$, $f(1, 1, 0)$ must be $(1, 1, 0)$. This type of argument is used repeatedly in the proof.

Let $J = \{a \in K \mid f(1, a, 0) = (1, 0, 0)\}$. If $a, b \in J$, we use collinearities (displayed in Fig. 1) to show that $f(1, a, 1) = (1, 0, 1)$, $f(0, a, 1) = (0, 0, 1)$, $f(1, a + b, 1) = (1, 0, 1)$, and $f(1, a + b, 0) = (1, 0, 0)$. Thus $a + b \in J$. Similarly (see Fig. 2), $f(0, 1 - a, 1) = (0, 1, 1)$, $f(1, b, b) = (1, 0, 0)$, and $f(1, ab, 0) = (1, 0, 0)$, so $ab \in J$.

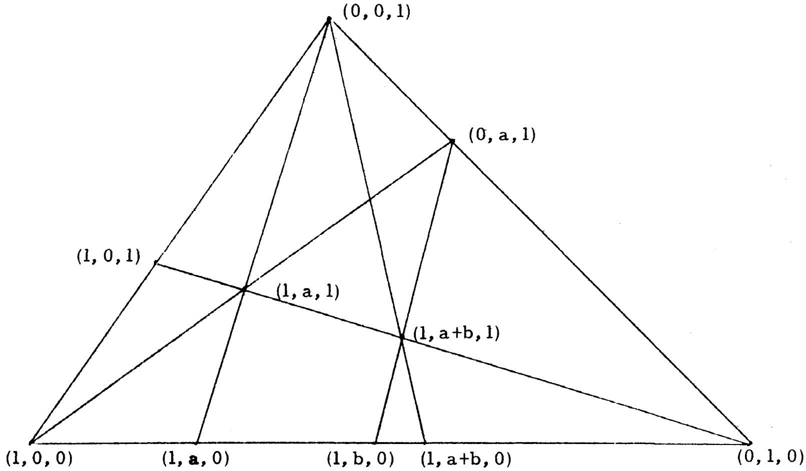

FIGURE 1

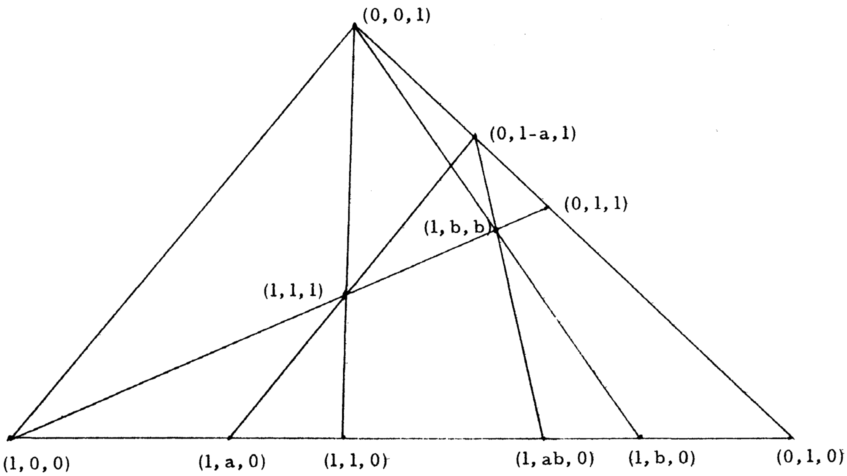

FIGURE 2

Now let $A = \{a \in K \mid aJ \subseteq J\}$, the idealizer of J. If $a \in K$ and $a \notin A$, then there is $b \in J$ such that $ab \notin J$. Hence (see Fig. 3) $f(1, b, b) = (1, 0, 0)$, $f(0, 1 - a, 1) = (0, 1, 0)$, $f(a, 1, 1) = (1, 0, 0)$, and $f(1, a^{-1}, 0) = (1, 0, 0)$. Thus $a^{-1} \in J$, so A is a valuation subring of K with maximal ideal J.

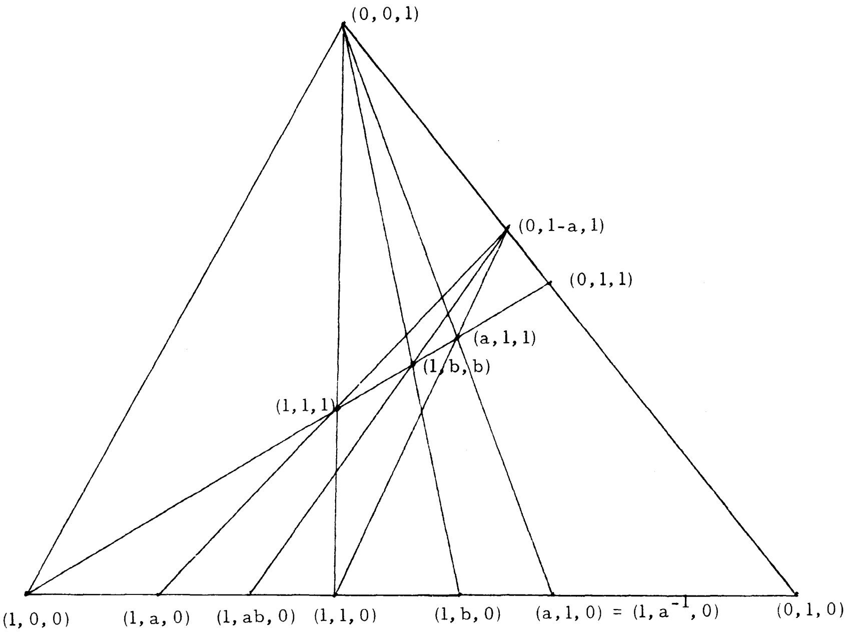

FIGURE 3

Using the same technique as above, we can show that for each $a \in A$ there is a unique $a' \in L$ such that $f(1, a, 0) = (1, a', 0)$. Therefore we define $\varphi_f: A \to L$ by $f(1, a, 0) = (1, \varphi_f(a), 0)$. It now follows that φ_f is a place with domain A, and that $\beta(\varphi_f) = f$. Since uniqueness is clear, the proof is complete.

We now establish the category equivalence; our terminology will follow Mitchell [2, pp. 51–52]. The function β of Theorem 1 proves to be a functor from the category of fields and places to the category of based planes and basic lineations. To show that β is faithful, consider places $\varphi, \theta: K \to L$, with $\beta(\varphi) = \beta(\theta)$. Let $a \in \ker \varphi$; then $\beta(\varphi)(1, a, 0) = (1, 0, 0)$. If $a \notin A_\theta$, then $a^{-1} \in \ker \theta$, and $\beta(\theta)(1, a, 0) = \beta(\theta)(a^{-1}, 1, 0) = (0, 1, 0)$, contradicting $\beta(\varphi) = \beta(\theta)$. Thus $a \in A_\theta$, so $\beta(\theta)(1, a, 0) = (1, \theta(a), 0) = (1, 0, 0)$, implying $\theta(a) = 0$ and $a \in \ker \theta$. Thus $\ker \varphi \leqq \ker \theta$. In a similar fashion, $\ker \theta \leqq \ker \varphi$. We conclude $A_\varphi = A_\theta$ and $\varphi \doteq \theta$, by properties of valuation rings. That β is full follows from Theorem 2. That β is representative follows from the fact that any based plane can be homogeneously coordinatized with a field, using the base as reference quadrangle. Thus β is an equivalence, and we have our main result.

THEOREM 3. *The category of based planes and basic lineations is equivalent to the category of fields and places.*

3. **Consequences.** While it is not possible to deal with proper lineations in a category setting (the composition of proper lineations need not be proper), the category equivalence nonetheless gives a characterization of proper lineations.

Let $f: \pi \to \rho$ be a proper lineation between Pappian planes. Then $f(\pi)$ contains a quadrangle $B_1B_2B_3B_4$. Let A_i be a point of π such that $f(A_i) = B_i$ $(i = 1, 2, 3, 4)$. Then $A_1A_2A_3A_4$ is a quadrangle of π. Coordinatizing π and ρ suitably with fields K and L, we may regard f as a basic lineation between the based planes $\beta(K)$ and $\beta(L)$. Thus by the category equivalence, there is a unique place $\varphi_f: K \to L$ associated with f. We restate this result as a theorem.

THEOREM 4. *To each proper lineation between Pappian planes, there corresponds a unique place between the coordinatizing fields.*

The corresponding place can be used to characterize certain types of proper lineations.

THEOREM 5. *A proper lineation between Pappian planes is injective (surjective) if and only if the corresponding place is injective (surjective).*

PROOF. We will treat f as a basic lineation between $\beta(K)$ and $\beta(L)$, with $\varphi: K \to L$ the corresponding place.

If f is injective, let $\varphi(a) = 0$. Then $f(1, a, 0) = (1, \varphi(a), 0) = f(1, 0, 0)$, so $a = 0$ and φ is injective. If φ is injective, let $f(x_1, x_2, x_3) = f(y_1, y_2, y_3)$ where (x_1, x_2, x_3) and (y_1, y_2, y_3) are normal forms. Then there is $c \in L$, $c \neq 0$, such that $\varphi(x_i) = c\varphi(y_i)$ $(i = 1, 2, 3)$. Now $\varphi(y_j) \neq 0$ for some j, so $c = \varphi(x_jy_j^{-1})$, and $\varphi(x_i) = \varphi(x_jy_j^{-1})\varphi(y_i) = \varphi(x_jy_j^{-1}y_i)$. Thus $x_i = x_jy_j^{-1}y_i$, so $(x_1, x_2, x_3) = (y_1, y_2, y_3)$. Thus f is injective.

Finally, φ is surjective if and only if L is isomorphic to $A_\varphi/\ker\varphi$, if and only if for each $(x_1, x_2, x_3) \in (L)$, $(x_1, x_2, x_3) = (\varphi(y_1), \varphi(y_2), \varphi(y_3)) = f(y_1, y_2, y_3)$ for some $(y_1, y_2, y_3) \in (K)$, if and only if f is surjective.

Using the characterization of projective collineations found in [3, p. 154], we have immediately the following result.

COROLLARY. *A proper lineation from a Pappian plane to itself is a collineation if and only if the corresponding place is an automorphism; it is a projective collineation if and only if the corresponding place is the identity automorphism.*

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