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INDICES OF LINDELÖF FUNCTIONS AND THEIR DERIVATIVES

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1. **Introduction.** A transcendental entire function $f(z)$ is said to be of bounded index if there exists an integer N, independent of z, such that

$$\max_{0 \leqq k \leqq N} \left\{ \frac{|f^{(k)}(z)|}{k!} \right\} \geqq \frac{|f^{(j)}(z)|}{j!} \tag{1.1}$$

holds for all z and j. The least such integer N is called the index of f (cf. [3], [8]). It is known [11] that a function of bounded index is at most exponential type but all functions of exponential type need not be of bounded index (see [11], [13]). Lee and Shah [6], [7] have shown that if $\{a_n\}$ is any sequence of positive numbers such that $a_{n+1}/a_n \geqq \gamma > 1$, and a and b are any complex numbers, then

$$F(z) = e^{az+b} \prod_1^\infty \{1 - z/a_n\}$$

and all successive derivatives $F^{(k)}(z)$ are of bounded index. Further if $\{a_n\}$ is any sequence of complex numbers such that $|a_{n+1}| \geqq 5^n|a_n|$, $|a_1| \geqq 5$, then $\psi(z) = \prod_1^\infty (1 - z/a_n)$ and all derivatives $\psi^{(k)}(z)$ are of bounded index [10]. (The first author has proved this result with "5" replaced by "4" in her doctoral dissertation.)

In this paper we investigate the index of the Lindelöf function, f, [9], [4] defined by

$$f(z) = \prod_{n=1}^\infty (1 - z/n^\alpha), \qquad \alpha > 1. \tag{1.2}$$

Pugh (cf. [10, p. 192]) has shown that if $\alpha \geqq 8$, then f is of bounded index. We prove here

THEOREM 1. *Let* $f(z) = f(z, \alpha) = \prod_1^\infty (1 - z/n^\alpha)$, $\alpha > 1$; *then* $f(z)$ *is of bounded index. It is of index one if* $\alpha \geqq 3$.

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In general the derivative of a function of bounded index need not be of bounded index [14]. However for the Lindelöf function we have

THEOREM 2. *Let $f(z) = f(z, \alpha)$, $\alpha > 1$, be the function defined in Theorem 1; then all successive derivatives $f^{(k)}(z)$, $k > 1$, are of bounded index.*

REMARK. In Theorems 1 and 2, $\alpha > 1$ is a fixed number and index N will depend on α. If $\alpha = 2$, then $f(z, 2) = (\sin \pi\sqrt{z})/\pi\sqrt{z}$ is of bounded index. (For another proof see [11].) However, a direct computation shows that

$$|f''(1/16)|/2! \nleqq \max \{|f(1/16)|, |f'(1/16)|\},$$

so that $f(z, 2)$ is of index $N > 1$.

2. **Lemmas.** We require several lemmas. The first gives information about the location of the zeros $\{b_n\}$ of f'. Here, and in what follows, we define $\{a_n\}$ as the zeros n^α of f. It is known that b_n are all real [2, pp. 23–24], and $a_1 < b_1 < a_2 < b_2 < \cdots$.

LEMMA 1. *Let* $\alpha > 1$, $k_1 = 3(\alpha + 1)/(\alpha - 1) + 1$, $k_2 = 2^{\alpha-2} + \alpha + 2$, *then for* $n \geqq n_0 = n_0(\alpha) \geqq 3$,

$$\frac{k_1 n^\alpha + (n + 1)^\alpha}{k_1 + 1} < b_n < \frac{n^\alpha + k_2(n + 1)^\alpha}{k_2 + 1}. \tag{2.1}$$

PROOF. Taking logarithmic derivatives we note that

$$g(x) \equiv \frac{f'(x)}{f(x)} = \sum_{j=1}^{\infty} \frac{1}{x - j^\alpha}.$$

Thus $g(x) = 1/(x - n^\alpha) + \sum_{j \neq n} 1/(x - j^\alpha)$, and for $n^\alpha < x < (n + 1)^\alpha$, $g'(x) < 0$, i.e., $g(x)$ is a decreasing function in this interval. Therefore if $f'(x)/f(x) > 0$, then $b_n > x$, and likewise if $f'(x)/f(x) < 0$, then $b_n < x$.

Write $d = (n^\alpha + k_2(n + 1)^\alpha)/(k_2 + 1)$. We will prove that, for all sufficiently large n,

$$\frac{f'(d)}{f(d)} \equiv \sum_{j=1}^{\infty} \frac{1}{d - j^\alpha} < 0; \tag{2.2}$$

and this will give the inequality on the right-hand side of (2.1). We use Euler's summation formula [1, pp. 201–202] to estimate

$$\Sigma_1 \equiv \sum_1^{n-1} \frac{1}{d - j^\alpha}$$

$$= \int_1^{n-1} \frac{dx}{d - x^\alpha} + \frac{1}{2}\left\{\frac{1}{d - 1^\alpha} + \frac{1}{d - (n-1)^\alpha}\right\}$$

$$+ \int_1^{n-1} \left(x - [x] - \frac{1}{2}\right) \frac{\alpha x^{\alpha-1}}{(d - x^\alpha)^2}\, dx.$$

Here $[x]$ denotes the integer part of x. Note that $-\frac{1}{2} \leqq x - [x] - \frac{1}{2} < \frac{1}{2}$, and that if we substitute $d - x^\alpha = t$ we find

$$\left| \int_1^{n-1} \left(x - [x] - \frac{1}{2}\right) \frac{\alpha x^{\alpha-1}}{(d - x^\alpha)^2}\, dx \right|$$

$$\leqq \frac{1}{2}\left[\frac{1}{d - (n-1)^\alpha} - \frac{1}{d - 1^\alpha}\right].$$

Hence we have

$$(2.3) \quad \int_1^{n-1} \frac{dx}{d - x^\alpha} + \frac{1}{d - 1^\alpha} < \Sigma_1 < \int_1^{n-1} \frac{dx}{d - x^\alpha} + \frac{1}{d - (n-1)^\alpha}.$$

Now

$$n^\alpha < d < (n+1)^\alpha \quad \text{and so}$$

$$(2.4) \quad \int_1^{n/2} \frac{dx}{d - x^\alpha} < \frac{(n/2) - 1}{d - (n/2)^\alpha} < \frac{2^\alpha}{2(2^\alpha - 1)n^{\alpha-1}}.$$

By using the binomial expansion in the definition of d, we see that

$$(2.5) \quad d = n^\alpha + \frac{\alpha k_2 + o(1)}{k_2 + 1} n^{\alpha-1} \quad \text{as } n \to \infty;$$

and

$$\frac{1}{d - (n-1)^\alpha} = \frac{k_2 + 1 + o(1)}{\alpha(2k_2 + 1)n^{\alpha-1}}.$$

The inequality on the right of (2.3) now gives

$$(2.6) \quad \Sigma_1 < \int_{n/2}^{n-1} \frac{dx}{d - x^\alpha} + \frac{2^\alpha}{2(2^\alpha - 1)n^{\alpha-1}} + \frac{k_2 + 1 + o(1)}{\alpha(2k_2 + 1)n^{\alpha-1}}.$$

Further putting in the value for d and simplifying we obtain

$$(2.7)\ \Sigma_2 \equiv \left(\frac{1}{d - n^\alpha} + \frac{1}{d - (n+1)^\alpha}\right) = -\frac{(k_2{}^2 - 1)}{k_2} \frac{1}{(n+1)^\alpha - n^\alpha}.$$

In addition, with the help of the integral test we may verify that

$$\Sigma_3 \equiv \sum_{j=n+2}^{\infty} \frac{1}{d - j^\alpha} < - \sum_{j=n+2}^{3n+1} \frac{1}{j^\alpha - d} < - \int_{n+2}^{3n} \frac{dx}{x^\alpha - d}. \tag{2.8}$$

We shall denote by I the integral in (2.6) and by J the integral in (2.8). Let $t = d^{1/\alpha}$. Then

$$I - J = \int_{n/2}^{n-1} \frac{dx}{t^\alpha - x^\alpha} - \int_{n+2}^{3n} \frac{dx}{x^\alpha - t^\alpha}. \tag{2.9}$$

Putting $y = x/t$ in the first integral and $y = t/x$ in the second integral, we get

$$I - J = \frac{1}{t^\alpha - 1} \left\{ \int_{n/2t}^{(n-1)/t} \frac{dy}{1 - y^\alpha} - \int_{t/3n}^{t/(n+2)} \frac{y^{\alpha-2}dy}{1 - y^\alpha} \right\}$$

$$= \frac{1}{t^{\alpha-1}} \left\{ \int_{n/2t}^{(n-1)/t} \frac{1 - y^{\alpha-2}}{1 - y^\alpha} dy - \int_{t/3n}^{n/2t} \frac{y^{\alpha-2}dy}{1 - y^\alpha} - \int_{(n-1)/t}^{t/(n+2)} \frac{y^{\alpha-2}dy}{1 - y^\alpha} \right\}. \tag{2.10}$$

Combining (2.4) and (2.5) we have $t \equiv d^{1/\alpha} > n$ and

$$t = n + \frac{k_2 + o(1)}{k_2 + 1}. \tag{2.11}$$

Hence there exists n_1 such that for $n \geqq n_1$, we have $t > n$ and $t/3n < n/2t < (n-1)/t < t/(n+2) < 1$. Consequently,

$$I - J < \frac{1}{t^{\alpha-1}} \int_{n/2t}^{(n-1)/t} \frac{1 - y^{\alpha-2}}{1 - y^\alpha} dy. \tag{2.12}$$

Thus if $1 < \alpha \leqq 2$, then $I - J \leqq 0$. If $2 < \alpha$, then the integrand in (2.12) is less than 1, and so

$$I - J < \frac{1}{t^{\alpha-1}} \left\{ \frac{n-1}{t} - \frac{n}{2t} \right\} < \frac{n}{2t^\alpha} < \frac{n}{2n^\alpha} = \frac{1}{2n^{\alpha-1}}.$$

Let

$$h(\alpha) = 0, \qquad 1 < \alpha \leqq 2,$$
$$= \tfrac{1}{2}, \qquad 2 < \alpha.$$

From (2.2), (2.3), and (2.6) — (2.12) we obtain

$$\frac{f'(d)}{f(d)} < \frac{1}{n^{\alpha-1}} \left\{ \frac{2^\alpha}{2(2^\alpha - 1)} + \frac{(k_2+1)}{\alpha(2k_2+1)} - \frac{(k_2^2-1)}{\alpha k_2} + h(\alpha) + o(1) \right\}. \tag{2.13}$$

Now

$$- \left[\frac{(k_2+1)}{\alpha(2k_2+1)} - \frac{(k_2^2-1)}{\alpha k_2} \right]$$
$$= \frac{(k_2+1)[2k_2^2 - 2k_2 - 1]}{\alpha k_2(2k_2+1)} > \frac{1}{2\alpha k_2}(2k_2^2 - 2k_2 - 1)$$
$$= \frac{1}{\alpha}\left(k_2 - 1 - \frac{1}{2k_2}\right) = \frac{1}{\alpha}\left(2^{\alpha-2} + \alpha + 1 - \frac{1}{2k_2}\right).$$

Also,

$$\frac{2^\alpha}{2(2^\alpha-1)} + h(\alpha) = \frac{1}{2} + \frac{1}{2(2^\alpha-1)} + h(\alpha) < \frac{1}{2} + \frac{1}{2\alpha} + h(\alpha).$$

It is easy to verify that in both cases

$$2^{\alpha-2}/\alpha + \tfrac{1}{2} + 1/(2\alpha) > h(\alpha) + 1/(2\alpha k_2).$$

Hence the expression on the right side of (2.13) is negative provided $n \geqq n_2(\alpha)$; and so the inequality on the right side of (2.1) follows if we take $n_0 \geqq \max(n_1, n_2)$.

The proof of the remaining part of (2.1) follows in a somewhat analogous manner. We write

$$D = \frac{k_1 n^\alpha + (n+1)^\alpha}{k_1 + 1}.$$

Then $n < D^{1/\alpha} \equiv p < n + 1$ and

$$D = n^\alpha + \frac{\alpha + o(1)}{k_1 + 1} n^{\alpha-1}, \qquad D^{1/\alpha} = n + \frac{1 + o(1)}{k_1 + 1}. \tag{2.14}$$

As in (2.3) we have, for $n \geqq 3$,

$$\sum_1^{n-1} \frac{1}{D - j^\alpha} > \int_{n/2}^{n-1} \frac{dx}{D - x^\alpha}. \tag{2.15}$$

Also

$$\frac{1}{D-a_n}+\frac{1}{D-a_{n+1}}=\frac{k_1^2-1+o(1)}{\alpha k_1 n^{\alpha-1}}. \tag{2.16}$$

Further

$$\sum_{j=n+2}^{\infty}\frac{1}{a_j-D}<\frac{1}{a_{n+2}-D}+\int_{n+2}^{\infty}\frac{dx}{x^\alpha-D}. \tag{2.17}$$

Denote the integral in (2.15) by I^* and the integral in (2.17) by J^*. Then (cf. (2.9)–(2.10))

$$I^*-J^*=\int_{n/2}^{n-1}\frac{dx}{p^\alpha-x^\alpha}-\int_{n+2}^{\infty}\frac{dx}{x^\alpha-p^\alpha}$$

$$=\frac{p}{D}\left\{\int_{n/2p}^{(n-1)/p}\frac{dy}{1-y^\alpha}-\int_{(n+2)/p}^{2^{1/\alpha}}\frac{dy}{y^\alpha-1}-\int_{2^{1/\alpha}}^{\infty}\frac{dy}{y^\alpha-1}\right\} \tag{2.18}$$

$$=\frac{p}{D}\{I_1-I_2-I_3\},\quad \text{say.}$$

In I_2 we take $y=1/x$ and, in I_3, we use the inequality $y^\alpha-1\geqq y^\alpha/2$. Hence

$$I_2=\int_{2^{-1/\alpha}}^{p/(n+2)}\frac{x^{\alpha-2}}{1-x^\alpha}\,dx,\qquad I_3<\int_{2^{1/\alpha}}^{\infty}\frac{2\,dy}{y^\alpha}=\frac{2^{1/\alpha}}{\alpha-1}.$$

From (2.14) we see that, for $n\geqq n_3(\alpha)$,

$$\frac{n}{2p}<2^{-1/\alpha}<\frac{p}{n+2}<\frac{n-1}{p}<1;$$

and, from (2.18),

$$I^*-J^*>\frac{p}{D}\left[\int_{n/2p}^{2^{-1/\alpha}}\frac{dy}{1-y^\alpha}+\int_{2^{-1/\alpha}}^{p/(n+2)}\frac{1-y^{\alpha-2}}{1-y^\alpha}\,dy\right.$$

$$\left.+\int_{p/(n+2)}^{(n-1)/p}\frac{dy}{1-y^\alpha}-\frac{2^{1/\alpha}}{\alpha-1}\right].$$

If $\alpha\geqq 2$ then

$$I^*-J^*>\frac{-2^{1/\alpha}}{(\alpha-1)}\frac{p}{D}=\frac{-2^{1/\alpha}+o(1)}{(\alpha-1)n^{\alpha-1}}.$$

If $1<\alpha<2$ then

$$I^* - J^* > \frac{p}{D}\left[\frac{-2^{1/\alpha}}{\alpha - 1} - \int_{2^{-1/\alpha}}^{p/(n+2)} \frac{y^{\alpha-2} - 1}{1 - y^\alpha}\, dy\right]$$
$$= \frac{1 + o(1)}{n^{\alpha-1}}\left\{\frac{-2^{1/\alpha}}{\alpha - 1} - I_4\right\}$$

where

$$I_4 = \int_{2^{-1/\alpha}}^{p/(n+2)} \frac{y^{\alpha-2} - 1}{1 - y^\alpha}\, dy.$$

Since the integrand, in I_4, is less than $((2 - \alpha)/\alpha)y^{-2}$ [**5**, p. 39] we have, for $1 < \alpha < 2$,

$$I_4 < \frac{2 - \alpha}{\alpha}\left(2^{1/\alpha} - \frac{n+2}{p}\right) = \frac{2 - \alpha}{\alpha}(2^{1/\alpha} - 1 + o(1)).$$

Let

$$h(\alpha) = \frac{-2^{1/\alpha}}{\alpha - 1}, \qquad \alpha \geqq 2,$$
$$= \frac{-2^{1/\alpha}}{\alpha - 1} - \frac{2 - \alpha}{\alpha}(2^{1/\alpha} - 1), \qquad 1 < \alpha \leqq 2.$$

Then, since $p/D = n^{1-\alpha}(1 + o(1))$,

$$I^* - J^* > \frac{1}{n^{\alpha-1}}\{h(\alpha) + o(1)\}. \tag{2.19}$$

Further

$$(a_{n+2} - D)^{-1} = \frac{k_1 + 1 + o(1)}{\alpha(2k_1 + 1)n^{\alpha-1}}\,; \tag{2.20}$$

and we have, from (2.15)–(2.20),

$$\sum_1^\infty \frac{1}{D - j^\alpha} > \frac{1}{n^{\alpha-1}}\left\{\frac{k_1^2 - 1}{\alpha k_1} - \frac{k_1 + 1}{\alpha(2k_1 + 1)} + h(\alpha) + o(1)\right\}$$
$$> \frac{1}{n^{\alpha-1}}\left\{\frac{1}{\alpha}\left(k_1 - 1 - \frac{1}{2k_1}\right) + h(\alpha) + o(1)\right\} \tag{2.21}$$
$$= \frac{1}{n^{\alpha-1}}\left\{\frac{3}{\alpha}\left(\frac{\alpha + 1}{\alpha - 1}\right) - \frac{1}{2\alpha k_1} + h(\alpha) + o(1)\right\}.$$

If $\alpha \geqq 2$ then we show that

$$\frac{3}{\alpha}\left(\frac{\alpha + 1}{\alpha - 1}\right) - \frac{1}{2\alpha k_1} - \frac{2^{1/\alpha}}{\alpha - 1} > 0, \tag{2.22a}$$

that is,

$$\alpha\left(3 - 2^{1/\alpha} - \frac{1}{2k_1}\right) + 3 + \frac{1}{2k_1} > 0.$$

But $k_1 > 4$ and $2^{1/\alpha} \leqq 2^{1/2}$. Hence the expression on the left is positive. If $1 < \alpha < 2$ then we show that

$$(2.22b) \quad 3(\alpha + 1) - (\alpha - 1)/2k_1 > \alpha 2^{1/\alpha} + (\alpha - 1)(2 - \alpha)(2^{1/\alpha} - 1),$$

that is,

$$6\alpha + 1 - \alpha^2 > (4\alpha - \alpha^2 - 2)2^{1/\alpha} + (\alpha - 1)/2k_1 .$$

The expression on the left is greater than 6, $k_1 > 4$, $(\alpha - 1)/2k_1 < 1/8$ and

$$(4\alpha - \alpha^2 - 2)2^{1/\alpha} \leqq \max_{1 \leqq \alpha \leqq 2} (4\alpha - \alpha^2 - 2) \max_{1 \leqq \alpha \leqq 2} 2^{1/\alpha} = 4.$$

This proves (2.22b) when $1 < \alpha < 2$ and so the sum on the left of (2.21) is positive for $\alpha > 1$ and n sufficiently large. The proof of the lemma is complete.

LEMMA 2. *Let* $\alpha \geqq 3$. *Then*

$$(2.23) \quad \frac{3.6 + 2^\alpha}{3} < b_1 < \frac{1 + 2^{\alpha+1}}{4},$$

$$(2.24) \quad 1 + 2^{\alpha+1} < b_2 < \frac{2^\alpha + 3^{\alpha+1}}{4},$$

and, for $n \geqq 3$,

$$(2.25) \quad \frac{n^\alpha(n + 1) + (n + 1)^\alpha}{(n + 2)} < b_n < \frac{n^\alpha + (n + 1)^{\alpha+1}}{(n + 2)}.$$

The proof is similar to that of Lemma 1 and is omitted.

LEMMA 3. *Let* $\alpha > 1$ *and* $|z - a_j| \geqq 3/2$ *for all* j. *Then there exists a number* $R = R(\alpha) > 0$ *such that for* $|z| \geqq R$, $|z - a_j| \geqq 3/2$ $(j \geqq 1)$,

$$(2.26) \quad S(z) \equiv \sum_{1}^{\infty} \frac{1}{|z - a_j|} < 0.9.$$

PROOF. We shall first prove the following: Let $x > 0$, $|x - a_j| \geqq 3/2$ for all j. Then there exists an integer $N_0 = N_0(\alpha)$ such that, for $n \geqq N_0$, $a_n < x < a_{n+1}$ and $|x - a_j| \geqq 3/2$,

$$(2.26a) \quad S(x) = \sum_{j=1}^{\infty} \frac{1}{|x - a_j|} < 0.9.$$

PROOF. Let $N_1(\alpha)$ be such that all $j \geqq N_1$, $a_j - a_{j-1} > 4$. Suppose $n > N_1$ and consider

$$\Sigma_1 \equiv \sum_1^{n-1} \frac{1}{x - j^\alpha}.$$

Then

$$\Sigma_1 < \int_1^{n-2} \frac{dt}{x - t^\alpha} + \frac{1}{x - (n-2)^\alpha} + \frac{1}{x - (n-1)^\alpha}.$$

Since

$$a_n + \tfrac{3}{2} \leqq x \leqq a_{n+1} - \tfrac{3}{2}, \tag{2.27}$$

the sum of the last two terms on the right is $(3 + o(1))/(2\alpha n^{\alpha-1})$. The integral is less than

$$\frac{(n/2) - 1}{x - (n/2)^\alpha} + \int_{n/2}^{n-2} \frac{dt}{x - t^\alpha}.$$

We now use the inequality $x - t^\alpha > \alpha t^{\alpha-1}(x^{1/\alpha} - t)$ [5, p. 39] and obtain

$$I \equiv \int_{n/2}^{n-2} \frac{dt}{x - t^\alpha} < \int_{n/2}^{n-2} \frac{dt}{\alpha t^{\alpha-1}(x^{1/\alpha} - t)}$$
$$< \frac{2^{\alpha-1}}{\alpha n^{\alpha-1}} \log \frac{x^{1/\alpha} - (n/2)}{x^{1/\alpha} - (n-2)}.$$

By (2.27) we have

$$n + \frac{3 + o(1)}{2\alpha n^{\alpha-1}} \leqq x^{1/\alpha} \leqq n + 1 - \frac{3 + o(1)}{2\alpha n^{\alpha-1}} \tag{2.28}$$

and consequently

$$I < \frac{2^{\alpha-1}}{\alpha n^{\alpha-1}}(\log n + o(1)), \qquad \frac{(n/2) - 1}{x - (n/2)^\alpha} < \frac{n/2}{n^\alpha - (n/2)^\alpha},$$

and

$$\Sigma_1 < \frac{1}{n^{\alpha-1}} \left\{ \frac{2^{\alpha-1}(\log n + o(1))}{\alpha} + \frac{2^{\alpha-1}}{2^\alpha - 1} + \frac{3 + o(1)}{2\alpha} \right\}. \tag{2.29}$$

Further

$$\max\{|x - a_n|, |x - a_{n+1}|\} \geqq \frac{|x - a_n| + |x - a_{n-1}|}{2} \geqq \frac{a_{n+1} - a_n}{2},$$

and $|x - a_j| \geqq \frac{3}{2}$ for every j. Hence

$$(2.30)\qquad \frac{1}{|x-a_n|} + \frac{1}{|x-a_{n+1}|} \leqq \frac{2}{3} + \frac{2}{a_{n+1}-a_n} = \frac{2}{3} + \frac{2(1+o(1))}{\alpha n^{\alpha-1}}$$

and

$$(2.31)\qquad \sum_{n+2}^{\infty} \frac{1}{j^\alpha - x} < \frac{1}{(n+2)^\alpha - x} + \int_{n+2}^{\infty} \frac{dt}{t^\alpha - x}\ .$$

Let J denote the last integral and write $p = x^{1/\alpha}$. Taking $t = py$ we get

$$J = \frac{p}{x}\int_{(n+2)/p}^{\infty} \frac{dy}{y^\alpha - 1}\,.$$

We now split the interval of integration from $(n+2)/p$ to $2^{1/\alpha}$ and $2^{1/\alpha}$ to ∞, and note that $n < p < n+1$ and $(n+2)/p < 2^{1/\alpha}$ for $n \geqq N_2(\alpha) = 2/(2^{1/\alpha}-1)$. Let $n > N_2(\alpha)$. In the first integral, we note $y^\alpha - 1 > \alpha(y-1)$ and in the second integral $y^\alpha - 1 \geqq y^\alpha/2$. Thus

$$J < \frac{p}{x}\left[\int_{(n+2)/p}^{2^{1/\alpha}} \frac{dy}{\alpha(y-1)} + \int_{2^{1/\alpha}}^{\infty} \frac{2dy}{y^\alpha}\right].$$

Integrating and using $p < n+1$, $p/x < 1/n^{\alpha-1}$, we obtain

$$(2.32)\qquad J < \frac{1}{n^{\alpha-1}}\left\{\frac{1}{\alpha}\log(2^{1/\alpha}-1) + \frac{2^{1/\alpha}}{\alpha-1} + \log(n+1)\right\}\ .$$

From (2.27) we have

$$(n+2)^\alpha - x > \alpha n^{\alpha-1}(1+o(1)).$$

The inequalities (2.29)–(2.32) now show that we can choose $N_0 > \max(N_1, N_2)$ such that, for $n \geqq N_0$,

$$S(x) < \frac{2}{3} + \left(\frac{9}{10} - \frac{2}{3}\right) = \frac{9}{10}\,.$$

This proves (2.26a).

We now consider $S(z)$. Let $R = 2 + a_{N_0}$. Then $|R - a_j| > 3/2$ for every j and so, by (2.26a),

$$\sum_{j=1}^{\infty} \frac{1}{|R-a_j|} < 0.9.$$

Let $|z| \geqq R$ and $|z - a_j| \geqq 3/2$. Then if $x = \operatorname{Re} z$, and $x \leqq R$ we have $|z - a_j|^2 \geqq R^2 + a_j^2 - 2xa_j \geqq |R - a_j|^2$, and so

$$\sum_{j=1}^{\infty} \frac{1}{|z - a_j|} \leqq \sum_{j=1}^{\infty} \frac{1}{|R - a_j|} < 0.9.$$

When $x > R$ we estimate $S(z)$ directly. Let $a_n \leqq x \leqq a_{n+1}$, $n \geqq 2$, $x > R$. Then $a_k - a_{k-1} > 4$ for $k \geqq n$ and

$$S(z) = \sum_{1}^{n-1} \frac{1}{|z - a_j|} + \left(\frac{1}{|z - a_n|} + \frac{1}{|z - a_{n+1}|}\right) + \sum_{n+2}^{\infty} \frac{1}{|z - a_j|}$$

$$\leqq \sum_{1}^{n-1} \frac{1}{|x - a_j|} + \frac{2}{3} + \frac{2}{a_{n+1} - a_n} + \sum_{n+2}^{\infty} \frac{1}{|x - a_j|}.$$

By the argument for $S(x)$, we see that the last expression is less than 0.9. This completes the proof of the lemma.

LEMMA 4. *Let $|z - b_j| \geqq 3/2$ for all j. Then there exists a number $R_1 = R_1(\alpha) > 0$ such that, for $|z| \geqq R_1$, $|z - b_j| \geqq 3/2$,*

$$S_1(z) \equiv \sum_{1}^{\infty} \frac{1}{|z - b_j|} < 0.9. \tag{2.33}$$

The proof of this lemma is similar to that of Lemma 3 and is omitted.

LEMMA 5. *Let $\alpha \geqq 3$,*

$$D_n(\rho, z) = \bigcup_{j=n}^{\infty} \{z : |z - a_j| \leqq \rho\}, \qquad D_n'(\rho, z) = \bigcup_{j=n}^{\infty} \{z : |z - b_j| \leqq \rho\},$$

$$S(z) = \sum_{j=1}^{\infty} \frac{1}{|z - a_j|}, \quad z \neq a_j, \qquad S_1(z) = \sum_{j=1}^{\infty} \frac{1}{|z - b_j|}, \quad z \neq b_j.$$

Then $S(z) < 1$ in each of the following cases:

(a) $e_1 = \{z : 1^\alpha + 1.7 \leqq |z| < 2^\alpha - 1.7\}$,
(b) $e_2 = \{z : 2^\alpha + 1.7 \leqq |z| < 3^\alpha - 1.7\}$,
(c) $e_3 = \{z : |z| > 3^\alpha, z \notin D_3(3.6, z)\}$.

Also $S_1(z) < 1$ in each of the following cases:

(d) $e_{11} = \{z : 0 \leqq |z| < 1^\alpha + 1.7\}$,
(e) $e_{12} = \{z : 2^\alpha - 1.7 \leqq |z| < 2^\alpha + 1.7\}$,
(f) $e_{13} = \{z : 3^\alpha - 1.7 \leqq |z| < 3^\alpha + 1.7\}$,
(g) $e_{14} = \{z : |z| > 3^\alpha, z \notin D_3'(3.6, z)\}$.

PROOF. We shall prove part (a). The remaining parts can be similarly proved. For parts (d)–(g) we utilize the inequalities for b_1, b_2, and b_n ($n \geqq 3$), of Lemma 2.

(a) Either z satisfies $1^\alpha + 1.7 \leqq |z| \leqq 2^{\alpha-1} + \frac{1}{2}$ or $2^{\alpha-1} + \frac{1}{2} < |z| < 2^\alpha - 1.7$. In both cases we have

$$\frac{1}{|z - 1^\alpha|} + \frac{1}{|z - 2^\alpha|} \leqq \frac{1}{1.7} + \frac{2}{2^\alpha - 1}.$$

Now $n^\alpha - 2^\alpha \uparrow$ as $\alpha \uparrow$, provided $n > 2$. Hence

$$S(z) < \frac{1}{1.7} + \frac{2}{2^3 - 1} + \sum_{n=3}^{\infty} \frac{1}{n^3 - (2^3 - 1.7)} < 1,$$

and (a) is proved.

Note that for $n \geqq 3$ we have

$$\begin{aligned} b_n - a_n &> \frac{n^\alpha(n+1) + (n+1)^\alpha}{(n+2)} - n^\alpha \\ &= \frac{(n+1)^\alpha - n^\alpha}{(n+2)} > 2\rho, \qquad \rho = 3.6, \end{aligned} \tag{2.34}$$

and

$$a_{n+1} - b_n > \frac{(n+1)^\alpha - n^\alpha}{(n+2)} > 2\rho, \qquad \rho = 3.6. \tag{2.35}$$

These inequalities together with (a)—(g) show that, for all z, either $S(z) < 1$ or $S_1(z) < 1$.

LEMMA 6 (SHAH [12]). *Let $f(z) \not\equiv 0$ be an entire function and T a given positive number. Then there exists an integer P such that for every z, $|z| \leqq T$,*

$$\max_{0 \leqq k \leqq P} \left\{ \frac{|f^{(k)}(z)|}{k!} \right\} \geqq \frac{|f^{(j)}(z)|}{j!}, \qquad j = P+1, P+2, \cdots.$$

3. **Proof of Theorem 1.** (i) We prove first that $f(z, \alpha)$, $\alpha > 1$, is of bounded index. By Lemmas 1, 3, and 4 we can choose a number $T = T(\alpha) > 0$ such that

(1) $S(z) < 1$ for $|z| \geqq T$, $|z - a_j| \geqq 3/2$ $(j \geqq 1)$;
(2) $S_1(z) < 1$ for $|z| \geqq T$, $|z - b_j| \geqq 3/2$ $(j \geqq 1)$;
(3) $\{|z| = T\} \cap \{\bigcup_{j=1}^{\infty} |z - a_j| \leqq 3/2\} = \emptyset$;
(4) $\{|z| = T\} \cap \{\bigcup_{j=1}^{\infty} |z - b_j| \leqq 3/2\} = \emptyset$;

and if $a_m > T$,

(5) $$\left\{\bigcup_{j=m}^{\infty} |z-a_j| \leqq 3/2 \right\} \cap \left\{\bigcup_{j=m}^{\infty} |z-b_j| \leqq 3/2 \right\} = \emptyset.$$

This relation (5) is possible for $b_n - a_n > \alpha n^{\alpha-1}/(k_1+1) \to \infty$, $a_{n+1} - b_n > \alpha n^{\alpha-1}/(k_2+1) \to \infty$ as $n \to \infty$. Consider now the set of points

$$E = \{z : |z| \geqq T, |z - a_j| \geqq 3/2, j = 1, 2, 3, \cdots\},$$

and write

(3.1) $$G(z) = \sum_{j=1}^{\infty} \frac{1}{z-a_j}, \quad z \neq a_j; \quad g(z) = \sum_{j=1}^{\infty} \frac{1}{z-b_j}, \quad z \neq b_j.$$

Then for $z \in E$,

$$|G(z)| < 1, \qquad |G'(z)| = \left| \sum_{1}^{\infty} \frac{1}{(z-a_j)^2} \right| < S^2 < 1,$$

and in general

(3.2) $$|G^{(n)}(z)| < n! S^{n+1} < n!\ .$$

Now $f'/f = G$ and so for $n = 0, 1, 2, \cdots, z \in E$, we have

$$\left| \frac{f^{(n+1)}(z)}{(n+1)!} \right| = \left| \frac{1}{(n+1)} \sum_{j=0}^{n} \frac{G^{(j)}(z)}{j!} \frac{f^{(n-j)}(z)}{(n-j)!} \right|$$

(3.3) $$\leqq \frac{1}{n+1} \left\{ \left(\max_{0 \leqq i \leqq n} \frac{|f^{(i)}(z)|}{i!} \right) \sum_{j=0}^{n} \frac{|G^{(j)}(z)|}{j!} \right\}$$

$$< \max_{0 \leqq i \leqq n} \frac{|f^{(i)}(z)|}{i!}\ .$$

Consider now the set of points $E_1 = \{z : |z| \geqq T,\ |z - b_j| \geqq 3/2, j = 1, 2, 3, \cdots\}$, we write $f' = \psi$. Then we have

(3.4) $$\frac{f''(z)}{f'(z)} = \sum_{1}^{\infty} \frac{1}{z-b_j} = \frac{\psi'(z)}{\psi(z)} = g(z),$$

and for $n \geqq 0$, $z \in E_1$, $|g^{(n)}(z)| < n!$. Hence, for $n \geqq 0$, and $z \in E_1$,

(3.5) $$\frac{|\psi^{(n+1)}(z)|}{(n+1)!} = \left| \frac{1}{n+1} \sum_{j=0}^{n} \frac{g^{(j)}(z)}{j!} \frac{\psi^{(n-j)}(z)}{(n-j)!} \right|$$

$$\leqq \frac{1}{n+1} \sum_{j=0}^{n} \frac{|g^{(j)}(z)|}{j!} \max_{0 \leqq i \leqq n} \frac{|\psi^{(i)}(z)|}{i!} < \max_{0 \leqq i \leqq n} \left\{ \frac{|\psi^{(i)}(z)|}{i!} \right\}.$$

Consequently for $n \geqq 0$, $z \in E_1$,

$$(3.6)\qquad \begin{aligned}\frac{|f^{(n+2)}(z)|}{(n+2)!} &< \frac{1}{n+2}\max_{0\leqq i\leqq n}\left\{\frac{|f^{(i+1)}(z)|}{(i+1)!}(i+1)\right\}\\ &< \max_{0\leqq i\leqq n}\frac{|f^{(i+1)}(z)|}{(i+1)!},\end{aligned}$$

that is, for $n \geqq 1$, $z \in E_1$,

$$(3.7)\qquad \frac{|f^{(n+1)}(z)|}{(n+1)!} < \max_{1\leqq i\leqq n}\left\{\frac{|f^{(i)}(z)|}{i!}\right\}.$$

Since $E \cup E_1 = \{z : |z| \geqq T\}$, we have for $n \geqq 1$ and $|z| \geqq T$,

$$\frac{|f^{(n+1)}(z)|}{(n+1)!} < \max_{0\leqq i\leqq n}\left\{\frac{|f^{(i)}(z)|}{i!}\right\}.$$

Hence, by induction on n, we have for $j \geqq 2$ and $|z| \geqq T$,

$$(3.8)\qquad \frac{|f^{(j)}(z)|}{j!} < \max\{|f(z)|, |f'(z)|\}.$$

Lemma 6 and (3.8) show that $f(z, \alpha)$, $\alpha > 1$, is of bounded index.

(ii) We now show that $f(z, \alpha)$ is of index one if $\alpha \geqq 3$. Let $\alpha \geqq 3$ and $E = e_1 \cup e_2 \cup e_3$. Then for $z \in E$, $n \geqq 0$,

$$|G^{(n)}(z)| < n!.$$

Hence we have, as in (3.3),

$$(3.9)\qquad \frac{|f^{(n+1)}(z)|}{(n+1)!} < \max_{0\leqq i\leqq n}\left\{\frac{|f^{(i)}(z)|}{i!}\right\},\quad z \in E.$$

Let $E_1 = e_{11} \cup e_{12} \cup e_{13} \cup e_{14}$. Then for $z \in E_1$, $n \geqq 0$,

$$|g^{(n)}(z)| < n!,$$

and

$$(3.10)\qquad \frac{|f^{(n+2)}(z)|}{(n+2)!} < \max_{0\leqq i\leqq n}\left\{\frac{|f^{(i+1)}(z)|}{(i+1)!}\right\}.$$

Since every $z \in E \cup E_1$, we see from (3.9) and (3.10), that for all z and $j \geqq 2$,

$$(3.11)\qquad \frac{|f^{(j)}(z)|}{j!} < \max\{|f(z)|, |f'(z)|\}.$$

Since f has zeros, its index N is greater than or equal to one. This with (3.11) completes the proof.

4. **Proof of Theorem 2.** We note that the argument given in Lemmas 1, 3, 4 and Theorem 1, first part, can be used to prove Theorem 2. The details are similar and omitted.

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References

1. T. M. Apostol, *Mathematical analysis: a modern approach to advanced calculus,* Addison-Wesley, Reading, Mass., 1957. MR **19**, 398.

2. R. P. Boas, Jr., *Entire functions,* Academic Press, New York, 1954. MR **16**, 914.

3. Fred Gross, *Entire functions of bounded index,* Proc. Amer. Math. Soc. **18** (1967), 974-980. MR **36** #1649.

4. G. H. Hardy, *On the function* $P_\rho(x)$, Quart. J. Math. **37** (1906), 146-172.

5. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities,* 2nd ed., Cambridge Univ. Press, New York, 1952. MR **13**, 727.

6. B. S. Lee and S. M. Shah, *On the growth of entire functions of bounded index,* Indiana Univ. Math. J. **20** (1970/71), 81-87. MR **41** #3766.

7. ———, *Inequalities satisfied by entire functions and their derivatives,* Trans. Amer. Math. Soc. **149** (1970), 109-117. MR **41** #2006.

8. B. Lepson, *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index,* Proc. Sympos. Pure Math., vol. 11, Amer. Math. Soc., Providence, R. I., 1968, pp. 298-307. MR **38** #6069.

9. E. Lindelöf, *Memoire sur la théorie des fonctions entières de genre fini,* Acta. Soc. Sci. Fennicae **31** (1902), no. 1.

10. W. J. Pugh and S. M. Shah, *On the growth of entire functions of bounded index,* Pacific J. Math. 33 (1970), 191-201. MR **41** #3767.

11. S. M. Shah, *Entire functions of bounded index,* Proc. Amer. Math. Soc. **19** (1968), 1017-1022. MR **38** #6070.

12. ———, *Entire functions satisfying a linear differential equation,* J. Math. Mech. **18** (1968/69), 131-136. MR **37** #2994.

13. ———, *Entire functions of unbounded index and having simple zeros,* Math. Z. **118** (1970), 193-196.

14. ———, *On entire functions of bounded index whose derivatives are of unbounded index,* Proc. London Math. Soc. (2) 4 (1971), 127-139.

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