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# REAL REPRESENTATIONS OF SPLIT METACYCLIC GROUPS

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The number of real absolutely irreducible representations of a rather special class of metacyclic groups was determined in [2]. The results of [2] are extended here to the class of all split metacyclic groups. This class includes, for example, all groups having every Sylow subgroup cyclic (see [1, p. 112]).

Suppose $G$ is a split metacyclic group having cyclic subgroups $A = \langle a \rangle$ and $B = \langle b \rangle$, with $A \lhd G$, $AB = G$, and $A \cap B = 1$. Suppose $|A| = m$, $|B| = s$, and $b^{-1}ab = a^r$, with $0 < r < m$. Then $(m, r) = 1$ and $r^s \equiv 1 \pmod{m}$. We shall assume throughout that $r > 1$, for otherwise $G$ is abelian. Denote by $u$ the order of $r$ modulo $m$, i.e., $u$ is the least positive integer such that $r^u \equiv 1 \pmod{m}$. Then $u \mid s$.

If $\zeta \in \mathbf{C}$ is a primitive $m$th root of unity set $\varphi_i(a) = \zeta^i$. Then $\hat{A} = \{\varphi_0, \varphi_1, \cdots, \varphi_{m-1}\}$ is the set of all irreducible complex characters of $A$. It will be convenient to utilize a method due to Mackey (see [3]) for constructing all the irreducible characters of $G$. Mackey's construction is given in the context of locally compact groups, but it is not difficult to verify his results directly for finite groups.

Observe that $B$ acts as a permutation group on $\hat{A}$, via $\varphi_i{}^b(a) = \varphi_i(b^{-1}ab) = \varphi_i(a^r) = \zeta^{ir}$. For each $\varphi_i \in \hat{A}$ let us denote by $\mathcal{O}_i$ the $B$-orbit of $\varphi_i$ and by $B_i$ the stabilizer in $B$ of $\varphi_i$. Thus $|\mathcal{O}_i| = |B : B_i|$, the index of $B_i$ in $B$, for each $i$. If we set $u_i = |\mathcal{O}_i|$ it is easy to see that $u_i$ is the least positive integer such that $m \mid i(r^{u_i} - 1)$, and that $u_i \mid u$. Furthermore $B_i = \langle b^{u_i} \rangle$ and $|B_i| = s/u_i$.

If $\xi \in \mathbf{C}$ is a primitive $s$th root of unity then the irreducible characters of $B_i$ are given by $\{\psi_{ij} : 0 \leqq j \leqq s/u_i - 1\}$, where $\psi_{ij}(b^{u_i}) = \xi^{ju_i}$. Define characters $\chi_{ij}$ of $B_iA$ by setting

$$\chi_{ij}(b^{u_i\beta}a^\alpha) = \psi_{ij}(b^{u_i\beta})\varphi_i(a^\alpha) = \xi^{u_ij\beta}\zeta^{i\alpha}.$$

If one representative $\varphi_i$ is chosen from each orbit $\mathcal{O}_i$, and if the resulting characters $\chi_{ij}$ are induced up to characters of $G$, then the set $\{\chi_{ij}^G\}$ of induced characters is the full set of inequivalent irreducible complex characters of $G$.

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It will at times be typographically convenient in the following computations to write powers, both in $G$ and in $\mathbf{C}$, in the form $x \uparrow y$ rather than $x^y$. As an illustration observe that if $\alpha$, $\beta$, $\gamma$, and $\delta$ are arbitrary integers then

$$(b^\gamma a^\delta)^{-1}(b^{u_i\beta}a^\alpha)(b^\gamma a^\delta) = (b \uparrow u_i\beta)(a \uparrow (\delta(1 - r^{u_i\beta}) + \alpha r^\gamma)).$$

PROPOSITION 1. *Suppose $0 \leqq i \leqq m - 1$ and $\alpha, \beta$ are integers. Then*

$$\chi_{ij}^G(b^{u_i\beta}a^\alpha) = (u_i/u)(\xi \uparrow u_i\beta j) \sum_{\gamma=0}^{u-1} \zeta \uparrow i\alpha r^\gamma.$$

*If $x \in G \backslash B_iA$ then $\chi_{ij}^G(x) = 0$.*

PROOF. By the observation preceding the proposition

$$\chi_{ij}((b^\gamma a^\delta)^{-1}b^{u_i\beta}a^\alpha(b^\gamma a^\delta)) = (\xi \uparrow u_i\beta j)(\zeta \uparrow i(\delta(1 - r^{u_i\beta}) + \alpha r^\gamma)).$$

Thus

$$\begin{aligned}\chi_{ij}^G(b^{u_i\beta}a^\alpha) &= |B_iA|^{-1} \sum_{\gamma=0}^{s-1} \sum_{\delta=0}^{m-1} (\xi \uparrow u_i\beta j)(\zeta \uparrow i(\delta(1 - r^{u_i\beta}) + \alpha r^\gamma)) \\ &= (u_i/sm)(\xi \uparrow u_i\beta j) \sum_\delta (\zeta \uparrow i\delta(1 - r^{u_i\beta})) \sum_\gamma (\zeta \uparrow i\alpha r^\gamma).\end{aligned}$$

Repeated application of the fact that $ir^{u_i} \equiv i \pmod m$ shows that $\zeta \uparrow i\delta(1 - r^{u_i\beta}) = 1$ for all $\beta$ and $\delta$, so that $\sum_\delta(\zeta \uparrow i\delta(1 - r^{u_i\beta})) = m$. The set $\{1, r, r^2, \cdots, r^{s-1}\}$ is, modulo $m$, just the set $\{1, r, \cdots, r^{u-1}\}$ with each element repeated $s/u$ times. Thus

$$\sum_{\gamma=0}^{s-1} (\zeta \uparrow i\alpha r^\gamma) = (s/u) \sum_{\gamma=0}^{u-1} (\zeta \uparrow i\alpha r^\gamma),$$

and the formula above reduces to $(u_i/u)(\xi \uparrow u_i\beta j)\sum_{\gamma=0}^{u-1}\zeta \uparrow i\alpha r^\gamma$. Finally $B_iA \lhd G$, so $t^{-1}xt \notin B_iA$ if $x \notin B_iA$, and so $\chi_{ij}^G(x) = 0$.

Set $\nu(\chi_{ij}^G) = |G|^{-1}\sum\{\chi_{ij}^G(x^2) : x \in G\}$. Then by the theorem of Frobenius and Schur [1, p. 21] $\chi_{ij}^G$ is the character of a representation over the real field $\mathbf{R}$ if and only if $\nu(\chi_{ij}^G) = 1$.

PROPOSITION 2. *Suppose that either $i = 0$ or, if $m$ is even, $i = m/2$. Then $u_i = 1$ and $\chi_{ij}^G = \chi_{ij}$ is linear for all $j$. Also, $\nu(\chi_{i0}) = 1$ and if $s$ is even then $\nu(\chi_{i,s/2}) = 1$, but $\nu(\chi_{ij}) = 0$ for all other values of $j$.*

PROOF. It is obvious that $u_0 = 1$. If $m$ is even then $r$ is odd, so $r - 1$ is even, and $m \mid (m/2)(r - 1)$. Thus $u_{m/2} = 1$. In both cases $B_iA = G$ so $\chi_{ij}^G$ is linear. Since the values of $\chi_{ij}$ are roots of unity, $\nu(\chi_{ij}) = 1$ if and only if $\chi_{ij}(b^\mu a^\lambda)^2 = \xi \uparrow 2\mu j = 1$ for all $\mu$. Taking

$\mu = 1$ we see that $\nu(\chi_{ij}) = 1$ if and only if $s \mid 2j$. Since $0 \leqq j \leqq s - 1$ it follows that $j = 0$ or $s$ is even and $j = s/2$.

Suppose in general that $0 \leqq \mu \leqq s - 1$ and $0 \leqq \lambda \leqq m - 1$. Then $(b^\mu a^\lambda)^2 = b^{2\mu}(a \uparrow \lambda(r^\mu + 1)) \in B_i A$ if and only if $u_i \mid 2\mu$. Thus $(b^\mu a^\lambda)^2 \in B_i A$ if and only if

$$\begin{aligned} \mu &= \rho u_i, && 0 \leqq \rho \leqq (s/u_i) - 1, \text{ if } u_i \text{ is odd,} \\ &= \rho u_i/2, && 0 \leqq \rho \leqq (2s/u_i) - 1, \text{ if } u_i \text{ is even.} \end{aligned}$$

**Theorem 1.** *Suppose $u_i$ is odd but $i \neq 0$, $i \neq m/2$. Then $\nu(\chi_{ij}^G) = 0$, all $j$.*

**Proof.** By Proposition 1 and the observation above,

$$\begin{aligned} \nu(\chi_{ij}^G) &= (ms)^{-1} \sum_{\rho=0}^{s/u_i - 1} \sum_{\lambda=0}^{m-1} \chi_{ij}^G((b^{\rho u_i} a^\lambda)^2) \\ &= (ms)^{-1} \sum_{\rho,\lambda} \chi_{ij}^G((b \uparrow 2\rho u_i)(a \uparrow \lambda(r^{\rho u_i} + 1))) \\ &= (ms)^{-1} \sum_{\rho,\lambda} (u_i/u)(\xi \uparrow 2u_i j\rho) \sum_{\gamma=0}^{u-1} (\zeta \uparrow i(r^{\rho u_i} + 1) r^\gamma \lambda) \\ &= (u_i/msu) \sum_{\rho,\gamma} (\xi \uparrow 2u_i j\rho) \sum_{\lambda} (\zeta \uparrow i(r^{\rho u_i} + 1) r^\gamma)^\lambda. \end{aligned}$$

As above, $m \mid i(r^{\rho u_i} - 1)$, all $\rho$, and $i(r^{\rho u_i} + 1) = i(r^{\rho u_i} - 1) + 2i$, so $m \mid i(r^{\rho u_i} + 1)$ if and only if $m \mid 2i$. But that means $i = 0$ or $i = m/2$, and those cases have been ruled out. Since $(m, r) = 1$ it follows that $\zeta \uparrow i(r^{\rho u_i} + 1) r^\gamma \neq 1$ for all $\rho$ and $\gamma$, and so $\sum_\lambda (\zeta \uparrow i(r^{\rho u_i} + 1) r^\gamma)^\lambda = 0$ by the formula for the sum of a geometric progression. As a result, $\nu(\chi_{ij}^G) = 0$.

As a consequence of Theorem 1 the only real linear characters of $G$ are those discussed in Proposition 2. Thus $G$ has either 1, 2, or 4 real linear characters depending on oddness or evenness of $m$ and $s$.

**Theorem 2.** *If $i \neq 0$, $m/2$ then $\nu(\chi_{ij}^G) = 1$ if and only if $j = 0$, $u_i$ is even, and $m \mid i(r^{u_i/2} + 1)$.*

**Proof.** Because of Theorem 1 we need only consider the case where $u_i$ is even. By Proposition 1,

$$\begin{aligned} \nu(\chi_{ij}^G) &= (ms)^{-1} \sum \{\chi_{ij}^G((b \uparrow \rho u_i)(a \uparrow \lambda(r^{\rho u_i/2} + 1))) : \\ &\qquad\qquad 0 \leqq \rho \leqq 2s/u_i - 1, 0 \leqq \lambda \leqq m - 1\} \\ &= (u_i/msu) \sum_{\rho,\gamma} (\xi \uparrow \rho u_i j) \sum_{\lambda} (\zeta \uparrow i(r^{\rho u_i/2} + 1) r^\gamma)^\lambda. \end{aligned}$$

When $\rho$ is even, say $\rho = 2\eta$, then

$$i(r^{\rho u_i/2} + 1) = i(r^{\eta u_i} + 1) \equiv 2i \pmod m$$

as in the proof of Theorem 1. But $2i \not\equiv 0 \pmod m$, so

$$(\zeta \uparrow i(r^{\rho u_i/2} + 1)r^\gamma) \neq 1,$$

and so $\sum_\lambda (\zeta \uparrow i(r^{\rho u_i/2} + 1)r^\gamma)^\lambda = 0$. Suppose then that $\rho$ is odd, say $\rho = 2\eta + 1$. In that case

$$i(r^{\rho u_i/2} + 1) = i(r^{\eta u_i} r^{u_i/2} + 1) \equiv i(r^{u_i/2} + 1) \pmod m.$$

Thus $\zeta \uparrow i(r^{\rho u_i/2} + 1)r^\gamma = 1$ if and only if $m \mid i(r^{u_i/2} + 1)$. If

$$m \nmid i(r^{u_i/2} + 1)$$

then $\sum_\lambda(\zeta \uparrow i(r^{\rho u_i/2} + 1)r^\gamma)^\lambda = 0$, as before, so suppose $m \mid i(r^{u_i/2} + 1)$. Then

$$\begin{aligned}\nu(\chi_{ij}^G) &= (u_i/su) \sum_\gamma \sum\{(\xi \uparrow (2\eta + 1)u_i j) : 0 \leqq \eta \leqq s/u_i - 1\} \\ &= (u_i/s)(\xi \uparrow u_i j) \sum_\eta (\xi \uparrow 2u_i j)^\eta.\end{aligned}$$

Again we see that $\nu(\chi_{ij}^G) = 0$ unless $\xi \uparrow 2u_i j = 1$, i.e., unless $s \mid 2u_i j$, or $(s/u_i) \mid 2j$. Since $0 \leqq j \leqq s/u_i - 1$ we have $\nu(\chi_{ij}^G) = 0$ unless $j = 0$ or $2j = s/u_i$. When $j = 0$ it is immediate that $\nu(\chi_{ij}^G) = 1$. If $j = s/2u_i$, then $\nu(\chi_{ij}^G) = \xi^{s/2} = -1$.

COROLLARY (OF THE PROOF). *$\nu(\chi_{ij}^G) = -1$ if and only if $u_i$ is even, $2j = s/u_i$, and $m \mid i(r^{u_i/2} + 1)$.*

The corollary describes the circumstances under which $\chi_{ij}^G$ is the character of a matrix representation $T$ that is not similar to a real representation, although it is similar to its own complex conjugate $\bar{T}$.

For each even divisor $2v$ of $u$ set $d_v = (m, r^v + 1)$, and let $M_v$ be the set of divisors $w$ of $u$ that are maximal with respect to $w < 2v$. Thus $w \in M_v$ if and only if $w \mid u$, $w < 2v$, and $w = w'$ whenever $w' \mid u$ and $w \leqq w' < 2v$. Next set

$$\begin{aligned}d_v^{(0)} &= d_v, \\ d_v^{(1)} &= \sum\{(d_v, r^w - 1) : w \in M_v\}, \\ d_v^{(2)} &= \sum\{(d_v, r^{(w,z)} - 1) : w, z \in M_v, w \neq z\},\end{aligned}$$

and in general set

$$d_v^{(k)} = \sum\{(d_v, r^{(w_1, \cdots, w_k)} - 1) : w_i \in M_v, w_i \neq w_j \text{ if } i \neq j\}.$$

We agree that $d_v^{(k)} = 0$ if $k > |M_v|$.

THEOREM 3. *The split metacyclic group G has exactly*

$$\sum\{(2v)^{-1}\sum\{(-1)^k d_v{}^{(k)} : k = 0, 1, \cdots\} : 2v \mid u\}$$

*distinct nonlinear absolutely irreducible representations over the real field* **R**.

PROOF. The proof is basically an application of the combinatorial principle of inclusion-exclusion. Suppose $2v \mid u$. Using Theorem 2, we wish to determine which $i$, $0 \leqq i \leqq m-1$, satisfy $u_i = 2v$ and also $m \mid i(r^v + 1)$. Observe that $m \mid i(r^v+1)$ if and only if $(m/d_v) \mid i((r^v+1)/d_v)$, hence if and only if $(m/d_v) \mid i$, or $i = 0$, $(m/d_v)$, $\cdots$, $(d_v - 1)(m/d_v)$. These $d_v = d_v{}^{(0)}$ values of $i$ correspond to characters of real representations provided that $u_i = 2v$, so we must eliminate the values for which $u_i \neq 2v$, which means that $u_i < 2v$, since $u_i$ is minimal such that $m \mid i(r^{u_i} - 1)$. Thus we must eliminate those $i$ for which there is a divisor $w$ of $u$, with $w < 2v$, such that $m \mid i(r^w - 1)$. Observe though that if $z \mid w$ and $m \mid i(r^z - 1)$ then also $m \mid i(r^w - 1)$, since $r^z - 1 \mid r^w - 1$. As a result, if certain values of $i$ are eliminated by way of the elements $w$ of $M_v$ then there will be no further eliminations.

Suppose then that $i = k(m/d_v)$ and $w \in M_v$. Then

$$m \mid k(m/d_v)(r^w - 1)$$

if and only if $d_v \mid k(r^w - 1)$, which is if and only if $(d_v/(d_v, r^w - 1)) \mid k$, or $k = n(d_v/(d_v, r^w - 1))$, $0 \leqq n \leqq (d_v, r^w - 1) - 1$. Eliminating those $(d_v, r^w - 1)$ values of $i$ for each $w \in M_v$ results in the portion $d_v{}^{(0)} - d_v{}^{(1)}$ of the formula.

The process is still too crude, however, for some values of $i$ may have been eliminated more than once. If $z, w \in M_v$, $z \neq w$, and if $d_v$ divides both $k(r^z - 1)$ and $k(r^w - 1)$, write $(w, z) = \alpha w + \beta z$, $\alpha, \beta \in \mathbf{Z}$. Modulo $d_v$ we may assume that $\alpha$ and $\beta$ are both positive. Then $k(r^{(w,z)} - 1) = k(r^{w\alpha} r^{z\beta} - 1) \equiv 0 \pmod{d_v}$, and we see that $d_v \mid k(r^{(w,z)} - 1)$ if and only if $d_v$ divides both $k(r^w - 1)$ and $k(r^z - 1)$. But $d_v \mid k(r^{(w,z)} - 1)$ if and only if $(d_v/(d_v, r^{(w,z)} - 1)) \mid k$, or $k = n(d_v/(d_v, r^{(w,z)} - 1))$, $0 \leqq n \leqq (d_v, r^{(w,z)} - 1) - 1$. Thus if we restore each value of $i$ once for each pair of times it was eliminated, the formula grows to $d_v{}^{(0)} - d_v{}^{(1)} + d_v{}^{(2)}$.

The process continues, with analogous arguments at each step, yielding finally

$$d_v{}^{(0)} - d_v{}^{(1)} + d_v{}^{(2)} - d_v{}^{(3)} + \cdots$$

values of $i$ for which $u_i = 2v$ and $m \mid i(r^v + 1)$. Since these values must be distributed among orbits of size $2v$, the total number of distinct real representations of degree $2v$ is

$$(2v)^{-1}\sum\{(-1)^k d_v^{(k)} : k = 0, 1, \cdots\}.$$

The theorem follows.

COROLLARY. *A split metacyclic group $G \neq 1$ has all its absolutely irreducible representations real if and only if $G$ is a dihedral group $D_m$ of order $2m$.*

PROOF. If $G$ is abelian it is immediate that $G$ must be either cyclic of order 2 or Klein's four group, i.e., $G = D_1$ or $G = D_2$. Assume then that $G$ is not abelian, and suppose all representations are real. Then $s$ is even by Theorem 1. The commutator subgroup $G'$ of $G$ has index $s(m, r - 1)$, so $G$ has $s(m, r - 1)$ linear characters. Thus

$$\begin{aligned} s(m, r - 1) &= 2 \quad \text{if } m \text{ is odd,} \\ &= 4 \quad \text{if } m \text{ is even,} \end{aligned}$$

by Proposition 2 and Theorem 1. In the first case $s = u = 2$ and $(m, r - 1) = 1$. In the second case $s = u = (m, r - 1) = 2$. In either case the formula in Theorem 3 is just $\frac{1}{2}((m, r + 1) - (m, r + 1, r - 1))$, which is thus the number of nonlinear irreducible characters of $G$. Each has degree 2, so equating the sum of the squares of their degrees with $|G| - [G : G']$ we have $(m, r + 1) - (m, r + 1, r - 1) = m - (m, r - 1)$. But $(m, r + 1, r - 1) = (m, r - 1)$ in either case, so $m = (m, r + 1)$, and $m \mid r + 1$. Since $1 < r < m$ we conclude that $r = m - 1$, and so $G$ is dihedral. The converse is well known; it also follows easily from the criteria above.

Let us illustrate Theorem 3 with two concrete examples.

1. Suppose $r = m - 1$, as, for example, when $G$ is dihedral. Then $u = 2$ since $r^2 - 1 = m^2 - 2m$, so $s$ is even. We have $M_1 = \{1\}$, $d_1 = (m, r + 1) = m$, and

$$\begin{aligned} d_1^{(1)} = (m, m - 2) &= 1 \quad \text{if } m \text{ is odd,} \\ &= 2 \quad \text{if } m \text{ is even.} \end{aligned}$$

Thus the number of absolutely irreducible real representations is $(m - 1)/2 + 2$ if $m$ is odd, $(m - 2)/2 + 4$ if $m$ is even.

2. Suppose $m = p$ is an odd prime. Then $u$ divides $\varphi(p) = p - 1$, and all $u_i = u$ except that $u_0 = 1$. If $u$ is odd then only the principal character $1_G$ is real if $s$ is odd, and there is one additional real linear character if $s$ is even, by Proposition 2. Suppose $u = 2v$ is even. Then $d_v = (p, r^v + 1) = p$ since $p \mid r^{2v} - 1$ but $p \nmid r^v - 1$. For each $w \in M_v$ we have $(d_v, r^w - 1) = 1$ since $p \nmid r^w - 1$. If $|M_v| = n$

then $d_v{}^{(1)} = n = \binom{n}{1}$, $d_v{}^{(2)} = \binom{n}{2}$, et cetera, and the number of real absolutely irreducible representations of $G$ is

$$\frac{1}{u}\left(p + \sum_{k=1}^{n} (-1)^k \binom{n}{k}\right) + 2 = \frac{p-1}{u} + 2.$$

## References

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