

INTRODUCTION TO GENERAL THEORY OF REPRODUCING KERNELS

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1. **Introduction.** The theory of reproducing kernels is of fairly recent origin. The beginnings go back to the work of G. Szegö (1921) and S. Bergman (1922). We shall give formal definitions later but at this stage a rough description may be helpful.

Consider a class F of functions $P \rightarrow f(P)$ defined on some set S . A function of two arguments $K(P, Q)$ is a reproducing kernel for the class F if for each $f \in F$ we have

$$(1.1) \quad f(P) = \int f(Q)K(P, Q) dQ$$

where the integral is taken over S or over some proper subset of S .

This formulation is a little too general for our purposes, but it gives the idea. With this formulation it is easy to give examples of reproducing kernels some of which are of quite an old vintage. Cauchy's integral is such a case. Here F is the class of all functions holomorphic in a domain D bounded by a simple closed rectifiable oriented curve C . We assume every $f \in F$ to be continuous in $C \cup D$ and can then write for z in D

$$(1.2) \quad f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t - z} dt.$$

This is of the form (1.1) if we take

$$(1.3) \quad K(z, t) = \frac{1}{2\pi i} \frac{1}{t - z}.$$

There is of course no need to assume that D is simply-connected; if not, C will have to be the total boundary.

Various generalizations of Cauchy's integral are known. Thus we can let f be matrix-valued, P an n by n matrix, and $f(P)$ be given by a resolvent integral. This would already be a deviation from the pattern set by (1.1).

Another formula which comes to mind is the following:

$$(1.4) \quad f(s) = \int_0^1 f(t) dX(s, t).$$

Here F is the class $C[0, 1]$ and $X(s, t)$ is the characteristic function of the interval $[s, 1]$, i.e. $X(s, t) = 0$ for $0 \leq t < s$ and $= 1$ for $s \leq t \leq 1$. This is a Riemann-Stieltjes integral and thus a little more general than the prototype. It is mentioned here because

$$(1.5) \quad \int_0^1 f(t) dg(t)$$

with $f \in C[0, 1]$, $g \in BV[0, 1]$ is the general form of a linear bounded functional on $C[0, 1]$. As we shall see later, reproducing kernels and linear functionals are closely related.

These formulas may be interesting but they do not suggest the actual development. The canonical theory calls for a Hilbert space and the kernel should belong to the space for each fixed value of one of the variables. In a Hilbert space we have orthogonal systems and the early work on reproducing kernels was associated with such systems, the kernel being defined by a series of the form

$$(1.6) \quad K(P, Q) = \sum_n \overline{\omega_n(P)} \omega_n(Q)$$

where $\{\omega_n(P)\}$ is a complete orthonormal system for the space. It turns out that if one system will do, then every system works and the kernel is independent of the system chosen.

If we use (1.6) as a tentative definition, the series must be meaningful and, in particular, the series should converge for $Q = P$ so that $K(P, P) \geq 0$. This requirement would throw out the Cauchy kernel right away, and not every Hilbert space will possess an orthogonal system for which the series

$$(1.7) \quad \sum |\omega_n(P)|^2 < \infty.$$

Let us have a look at some special cases. The simplest example of a Hilbert space that we can think of is $L_2(a, b)$. If a and b are finite real numbers, we can reduce the discussion to $L_2(-\pi, \pi)$ and here the standard orthonormal system is

$$(1.8) \quad \omega_n(t) = (2\pi)^{-1/2} e^{nit}, \quad n = 0, \pm 1, \pm 2, \dots$$

We see right away that the corresponding series

$$(1.9) \quad \sum_{-\infty}^{\infty} \overline{\omega_n(s)} \omega_n(t)$$

diverges for all values of s and t , since the terms of the series do not go to zero as $|n| \rightarrow \infty$. For this space there can be no kernel function, at least if we try to define the latter by the series (1.6).

Now it may be argued that we have given up the struggle too easily; the speaker is supposed to be familiar with the theory of summability, and the Abel-Poisson method applies to the divergent series (1.9). Here for $0 < r < 1$

$$(1.10) \quad P(u; r) = \sum_{-\infty}^{\infty} r^{|n|} e^{niu} = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r \cos u + r^2}$$

and when $r \rightarrow 1$ the limit is $+\infty$ for $u = 0$, 0 for $0 < |u| \leq \pi$. In other words, the Abel-Poisson limit of $P(u; r)$ is the Dirac function $\delta(u)$. This does not strike me as an acceptable reproducing kernel.

On the other hand, we have

$$(1.11) \quad f(s; r) = \sum_{-\infty}^{\infty} f_n r^{|n|} e^{nis} = \int_{-\pi}^{\pi} f(t) P(s - t; r) dt$$

with obvious notation and

$$(1.12) \quad \lim_{r \rightarrow 1} f(s; r) = f(s) \quad \text{almost all } s$$

and

$$(1.13) \quad \lim \|f(\cdot; r) - f(\cdot)\|_2 = 0.$$

These formulas take the place of the missing

$$(1.14) \quad f(s) = \int_{-\pi}^{\pi} f(t) K(s, t) dt.$$

Let us now consider a Hilbert space with kernel function. We take $L_2H(D)$, the space of all functions holomorphic in the domain D which are quadratically integrable over D . Here we restrict ourselves to the special case where D is the unit disk $D : (z; |z| < 1)$. The integrability condition is now

$$(1.15) \quad \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^2 r dr d\theta < \infty.$$

The square root of this expression is taken as the norm of f in the space and the inner product is

$$(1.16) \quad (f, g) = \int_0^1 \int_0^{2\pi} f(re^{i\theta}) \overline{g(re^{i\theta})} r dr d\theta.$$

If $f(z) = \sum_0^\infty a_n z^n$, then

$$(1.17) \quad \|f\|_2^2 = \frac{1}{\pi} \sum_{n=0}^\infty (n+1)|a_n|^2.$$

The convergence of this series is necessary and sufficient for f to belong to $L_2H(D)$. It is an easy matter to find an orthogonal system in the space: the powers of z , z^n , $n = 0, 1, 2, \dots$, meet the requirement. For

$$(1.18) \quad \int_0^1 \int_0^{2\pi} [re^{i\theta}]^n [re^{-i\theta}]^m r dr d\theta = \frac{2\pi\delta_{mn}}{m+n+2}.$$

This means that we can take

$$(1.19) \quad \omega_n(z) = (1/\pi)^{1/2}(n+1)^{1/2}z^n$$

as our orthonormal system. It is evidently complete in $L_2H(D)$ since every element of this space can be represented in series of these functions which is absolutely convergent for $|z| < 1$.

We now set

$$(1.20) \quad \begin{aligned} K(z, t) &= \sum_{n=0}^\infty \overline{\omega_n(z)} \omega_n(t) = \frac{1}{\pi} \sum_{n=0}^\infty (n+1)(\bar{z}t)^n \\ &= \frac{1}{\pi}(1 - \bar{z}t)^{-2}. \end{aligned}$$

This is simple enough but is it a reproducing kernel? There are, as we shall see, two requirements to be met. (1) For each fixed z with $|z| < 1$ we should have $K(z, t)$ as an element of the space as a function of t . (2) For each $f \in L_2H(D)$

$$(1.21) \quad f(z) = (f(\cdot), K(z, \cdot)).$$

As to (1) we see that if $|z| < 1$ then $K(z, t)$ is continuous and bounded for $|t| \leq 1$ and certainly quadratically integrable over D . Property (2) requires some computation. The inner product is given by (1.16) and, as will be shown later, if

$$(1.22) \quad f(t) = \sum_0^\infty f_n \omega_n(t), \quad g(t) = \sum g_n \omega_n(t)$$

then

$$(1.23) \quad (f, g) = \sum f_n \bar{g}_n,$$

where the series is absolutely convergent. We apply this with

$$g(t) = \frac{1}{\pi} (1 - \bar{z}t)^{-2} = \sum_0^\infty \overline{\omega_n(z)} \omega_n(t).$$

Thus $f_n = f_n, g_n = \overline{\omega_n(z)}$ and (1.23) becomes

$$(1.24) \quad (f, g) = \left(f(t), \frac{1}{\pi} (1 - \bar{z}t)^{-2} \right) = \sum_0^\infty f_n \omega_n(z).$$

Since this is $f(z)$ we have got hold of a reproducing kernel for $L_2H(D)$ and, as we shall see later, a Hilbert space has at most one reproducing kernel.

This is one of the oldest examples of a reproducing kernel and it goes back to S. Bergman. A closely related orthogonal system was considered by G. Szegö for the space $HL_2(D)$. The “ H ” here refers to G. H. Hardy and the members of the space are functions holomorphic in D which again we take as $D: (z; |z| < 1)$. The integrability condition is different, however. It is required that

$$(1.25) \quad M_2[r; f] = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right\}^{1/2}$$

be a bounded function of r . The least upper bound of $M_2[r; f]$ is taken as the norm of f . If $f(z) = \sum_0^\infty a_n z^n$ it is seen that

$$(1.26) \quad \|f\| = \left(\sum_0^\infty |a_n|^2 \right)^{1/2}$$

so the convergence of this series is necessary and sufficient for $f \in HL_2(D)$. If $g(z) = \sum_0^\infty b_n z^n$ we define the inner product by

$$(1.27) \quad (f, g) = \sum_0^\infty a_n \bar{b}_n.$$

For powers of z we have $(z^m, z^n) = \delta_{mn}$ so the powers form an orthonormal system for this space which is evidently a complete system since every element has a unique Fourier (power) series expansion. Here the reproducing kernel is

$$(1.28) \quad K(z, t) = (1 - \bar{z}t)^{-1}$$

and we have clearly

$$(1.29) \quad (f(t), K(z, t)) = \sum_0^\infty a_n z^n = f(z)$$

for every element of the space.

Thus we know at least two Hilbert spaces with reproducing kernel to make up for the one example without a bona fide kernel.

2. Hilbert space. In the first lecture there was much talk about Hilbert space and the time has come to be more precise and to refresh our memories.

An inner-product space or a pre-Hilbert space is one satisfying the following postulates. X is a linear space over C .

DEFINITION 2.1. X is an inner-product space if, for any ordered pair of vectors, $x, y \in X$, an inner product (x, y) is defined satisfying the following conditions:

- (1) (x, y) is a complex number.
- (2) $(x, x) \geq 0$ and $(x, x) = 0$ iff $x = 0$.
- (3) $(y, x) = \overline{(x, y)}$.
- (4) $(ax + by, z) = a(x, z) + b(y, z)$.

Various facts follow from these postulates. Thus

$$(2.1) \quad (z, ax + by) = \bar{a}(z, x) + \bar{b}(z, y)$$

so the inner product is *bilinear*. Since

$$(x, 0) = (x, 0 + 0) = (x, 0) + (x, 0) = 2(x, 0)$$

we have

$$(2.2) \quad (x, 0) = 0, \quad \forall x.$$

LEMMA 2.1. *If $(x, y) = 0$ for a fixed y and all x , then $y = 0$.*

For, in particular, $(y, y) = 0$ so $y = 0$. ■

LEMMA 2.2. *For all $x, y \in X$,*

$$(2.3) \quad |(x, y)|^2 \leq (x, x)(y, y)$$

with equality iff y is a constant multiple of x .

PROOF. We have

$$0 \leq (y - ax, y - ax) = (y, y) - a(x, y) - \bar{a}(y, x) + |a|^2(x, x).$$

The desired inequality (Cauchy-Bouniakovski-Schwarz) is obtained by choosing $a = (y, x)/(x, x)$. This presupposes $x \neq 0$, but for $x = 0$ there is nothing to prove. ■

LEMMA 2.3. *The convention*

$$(2.4) \quad \|x\| = (x, x)^{1/2}$$

defines a norm in X and a normed topology by

$$(2.5) \quad d(x, y) = \|x - y\|.$$

PROOF. It is clear that $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$. Further $\|ax\| = |a| \|x\|$. To prove the triangle inequality consider

$$\begin{aligned} \|x + y\|^2 &= (x + y, x + y) = (x, x) + (x, y) + (y, x) + (y, y) \\ &\leq \|x\|^2 + 2|(x, y)| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2. \quad \blacksquare \end{aligned}$$

LEMMA 2.4. For a fixed y in X the inner product (x, y) defines a linear bounded functional on X with norm $\|y\|$.

This follows from Definition 2.1 plus (2.3). \blacksquare

LEMMA 2.5 (PARALLELOGRAM LAW). For any two vectors $x_1, x_2 \in X$,

$$(2.6) \quad \|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 = 2[\|x_1\|^2 + \|x_2\|^2].$$

LEMMA 2.6 (EXTENDED PARALLELOGRAM LAW). For any n vectors

$$(2.7) \quad \sum_{1 \leq j < k \leq n} \|x_j - x_k\|^2 + \left\| \sum_1^n x_j \right\|^2 = n \sum_1^n \|x_j\|^2.$$

Left as homework!

DEFINITION 2.2. Vectors x and y are called orthogonal or perpendicular if

$$(2.8) \quad (x, y) = 0.$$

More homework:

LEMMA 2.7. If x and y are orthogonal then the Pythagorean theorem holds, i.e.

$$(2.9) \quad \|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

DEFINITION 2.3. An inner-product space is called a Hilbert space if it is complete in the normed metric.

We use H to designate a Hilbert space. Euclidean, real or complex, spaces are Hilbert, so are l_2 and $L_2(a, b)$. It will be shown later that $L_2H(D)$ and $HL_2(D)$ are Hilbert spaces. We suppose in the following that H is infinite-dimensional. Mostly H will be separable, i.e. there is a countable subset of elements dense in the space.

If $\dim H = +\infty$ then for each n there is a set of n linearly independent vectors. If v_1, v_2, \dots, v_n is such a set, we can find a set u_1, u_2, \dots, u_n which is orthonormal by the so-called *Gram-Schmidt process*. The assumption of linear independence implies and is implied

by

$$(2.10) \quad G = \det((v_j, v_k)) \neq 0.$$

This determinant is known as the *Gramian*. Its geometrical meaning is the volume of the parallelepiped with the vectors v_1, v_2, \dots, v_n as edges. The new u -vectors are obtained by taking linear combinations of the v 's. Thus u_k is a combination of v_1, v_2, \dots, v_k with coefficients which are quotients of $k-1$ by $k-1$ minors of G . Here u_k has to be a unit vector, furthermore it has to be orthogonal to u_1, u_2, \dots, u_{k-1} which will be the case iff

$$(2.11) \quad (u_k, v_m) = 0, \quad m = 1, 2, \dots, k-1.$$

These conditions determine the u 's uniquely but we shall not write down the resulting expressions.

Now suppose that $x \in H$ and form the numbers

$$(2.12) \quad \hat{x}_k = (x, u_k),$$

where $S = \{u_k\}$ is an infinite orthonormal system for H . These are the Fourier coefficients with respect to S and lead to the Fourier series for x

$$(2.13) \quad x \sim \sum \hat{x}_k u_k.$$

In what sense is this series convergent and does it converge to x ? If both queries are answered in the affirmative for all x in H , we say that S is *complete* in H .

At any rate the partial sums of the series have certain extremal properties which are basic.

LEMMA 2.8. *The partial sums of (2.13) form a sequence of best approximation to x in H by the system S in the sense that for any n and any choice of numbers c_1, c_2, \dots, c_n*

$$(2.14) \quad \left\| x - \sum_{k=1}^n c_k u_k \right\|^2 \geq \|x\|^2 - \sum_1^n |\hat{x}_k|^2$$

with equality iff $c_k = \hat{x}_k$ for all k .

The proof follows from the observation that the left member of (2.14) equals

$$(2.15) \quad \|x\|^2 - \sum_{k=1}^n |\hat{x}_k|^2 + \sum_{k=1}^n |c_k - \hat{x}_k|^2. \quad \blacksquare$$

COROLLARY (BESSEL'S INEQUALITY). *We have*

$$(2.16) \quad \sum_{k=1}^{\infty} |\hat{x}_k|^2 \leq \|x\|^2.$$

LEMMA 2.9. *The partial sums of (2.13) form a Cauchy sequence.*

For we have

$$\left\| \sum_{k=m}^n \hat{x}_k u_k \right\|^2 = \sum_m^n |\hat{x}_k|^2$$

and as $m \rightarrow \infty$ the right member goes to zero by virtue of (2.14). ■

Since H is a complete metric space, the Cauchy sequence converges to an element of H say \tilde{x} . What is now the relation between \tilde{x} and x ? We have the following result.

LEMMA 2.10. *The following assertions are equivalent:*

- (1) *The set of all finite linear combinations of the u_k 's is dense in H .*
- (2) $\sum_1^{\infty} |\hat{x}_k|^2 = \|x\|^2$ holds for all x .
- (3) $(x, y) = \sum_{k=1}^{\infty} \hat{x}_k \hat{y}_k$ holds for all x and y .
- (4) $\hat{x}_k = 0$ for all k iff $x = 0$.
- (5) $\tilde{x} = x$ for all x .

PROOF. We give brief indications of the argument. If (1) holds, then for a given $x \in H$ we can make the left member of (2.14) as small as we please by choosing n large and suitable c 's. This makes the right member small for large n so that (2) holds. If (2) holds then we apply the identity to

$$\sum_{k=0}^3 i^k \|x + i^k y\|^2 = 4(x, y)$$

and obtain (3). If (3) holds and all \hat{x}_k are 0, then $(x, y) = 0$ for all y and, by Lemma 2.1, $x = 0$. If (4) holds, we observe that the Fourier coefficients of x and of \tilde{x} are identical and this requires that $\tilde{x} = x$. If (5) holds then the partial sums of (2.13) approximate x arbitrarily closely and thus (1) holds. ■

It is perhaps in order to remark that every separable Hilbert space possesses complete orthonormal systems. This we can see by a combination of the sieve method and the Gram-Schmidt method. Since there exists a countable dense set of elements, we can arrange them in a sequence $\{w_j\}$. Suppose $w_1 \neq 0$; we then take $w_1 = v_1 = u_1$. A particular w_j is omitted if it is a linear combination of the preceding w_1, w_2, \dots, w_{j-1} . If the space is infinite dimensional this weeding-

out process leaves an infinite sequence of linearly independent vectors $\{v_k\}$ the linear combinations of which are dense in H . These we orthogonalize and obtain an infinite sequence $\{u_k\}$ which forms an orthonormal system S the linear combinations of which again are dense in H so that condition (1) of Lemma 2.10 is satisfied and S is complete in H .

DEFINITION 2.4. $M \subset H$ is a closed linear subspace of H if (1) M is a linear, (2) every Cauchy sequence of elements of M converges to an element of M .

LEMMA 2.11. *Let M be a linear closed subspace of H . Let $x \in H$. Then there exist two uniquely defined elements $y \in M$ and $z \in H$ such that (1)*

$$(2.17) \quad x = y + z.$$

(2) z is orthogonal to every element of M .

(3) For all elements u of M we have $\|x - u\|$ is a minimum for $u = y$.

PROOF. We restrict ourselves to a separable Hilbert space H . In this case we have a complete orthonormal system $S = \{u_k\}$ and

$$x = \sum (x, u_k)u_k.$$

Here we split S into two subsystems S_1 and S_2 . A vector u_k belongs to S_1 if it $\in M$, otherwise to S_2 . We now set

$$(2.18) \quad y = \sum (x, u_k)u_k, \quad z = \sum (x, u_m)u_m$$

where in the first series we sum over S_1 and in the second series over S_2 . We have then clearly $x = y + z$ and z is orthogonal to all of M . Suppose now that

$$u = \sum a_k u_k \in M.$$

Then

$$\|x - u\|^2 = \sum |(x, u_k) - a_k|^2 + \sum |(x, u_m)|^2$$

where again the first sum extends over S_1 and the second over S_2 . This will be a minimum iff $a_k = (x, u_k)$ for all k , i.e. for $u = y$. The minimum distance then is $\|z\|$. ■

The set of all elements of the form

$$(2.19) \quad z = \sum b_m u_m, \quad u_m \in S_2,$$

also forms a Hilbert space. It is known as the orthogonal complement of M and is often denoted by M^\perp . We have

$$(2.20) \quad H = M \oplus M^\perp, \quad M^\perp = H \ominus M.$$

We come next to linear functionals. They have already figured in Lemma 2.4 where it was stated that for fixed $y \in H$ the inner product (x, y) defines a linear bounded functional. We have to prove the converse proposition that every linear bounded functional is of this form.

DEFINITION 2.5. A mapping from H into C is called a functional. It is linear if (1) $D[f]$, the domain of definition of f , is a linear subspace of H , (2) for $x, y \in D[f]$ and any constants a, b

$$(2.21) \quad f(ax + by) = af(x) + bf(y).$$

It is bounded if there is a constant $B > 0$ such that

$$(2.22) \quad |f(x)| \leq B\|x\|, \quad x \in D[f].$$

LEMMA 2.12. *If f is a linear bounded functional with $D[f] = H$ then there exists a unique $y \in H$ such that $f(x) = (x, y)$.*

PROOF. Consider first the null space of f , i.e. the set

$$N = \{z; f(z) = 0\}.$$

This is a linear subspace of H and it is clearly closed. There is an orthogonal complement N^\perp . If N^\perp should reduce to the zero element, then $N = H$ and $f(x) = (x, 0)$ is the trivial representation of the zero functional as an inner product. Suppose next that N^\perp contains an element $y \neq 0$. We shall show that there is a constant c such that $f(x) = (x, cy)$. To this end consider the decomposition

$$x = (x - by) + by$$

where b is a number that will depend upon x . We choose b so that $x - by \in N$. This requires $b = f(x)[f(y)]^{-1}$ where we note that $f(y) \neq 0$ since $y \in N^\perp$. We have then $(x - by, cy) = 0$ for all c . Now choose c so that

$$f(by) = (by, cy) = b\bar{c}\|y\|^2.$$

This gives

$$(2.23) \quad \bar{c}\|y\|^2 = f(y)$$

which determines c uniquely. Hence

$$f(x) = f(x - by) + bf(y) = (x - by, cy) + b(x, cy) = (x, cy).$$

The uniqueness of the representation of the functional by an inner

product follows from the fact that $(x, cy) = (x, u)$ for all x requires $u = cy$. ■

A set S is convex if $x_1, x_2 \in S$ and $0 < t < 1$ implies that $tx_1 + (1 - t)x_2 \in S$. In other words, if the two points are in S , so is the line segment joining them.

LEMMA 2.13. *A nonempty closed convex set S in H contains a point whose distance from the origin is less than that of any other point in S .*

PROOF. If the origin is in S , there is nothing to prove. If not, then

$$d(0, S) = \inf_{x \in S} \|x\| \equiv \delta > 0.$$

By the definition of the infimum, there exists a sequence $\{x_n\} \subset S$ such that $\lim \|x_n\| = \delta$. It is claimed that this is a Cauchy sequence. The assertion follows from the identity

$$(2.24) \quad \left\| \frac{1}{2}(x_n - x_m) \right\|^2 = \frac{1}{2} \|x_n\|^2 + \frac{1}{2} \|x_m\|^2 - \left\| \frac{1}{2}(x_n + x_m) \right\|^2.$$

Here $\frac{1}{2}(x_n + x_m) \in S$ by the convexity of S . Hence its norm is at least δ . If $m < n$ and $m \rightarrow \infty$

$$\limsup_{m \rightarrow \infty} \left\| \frac{1}{2}(x_n - x_m) \right\|^2 \leq \frac{1}{2} \delta^2 + \frac{1}{2} \delta^2 - \delta^2 = 0$$

whence it follows that $\{x_n\}$ is a Cauchy sequence and has a limit denoted by x_0 . We have then $x_0 \in S$ since S is closed and $\|x_0\| = \delta$. Suppose there is another point $y_0 \in S$ with $\|y_0\| = \delta$. We have then from (2.23) with appropriate replacements

$$\begin{aligned} \left\| \frac{1}{2}(x_0 - y_0) \right\|^2 &= \frac{1}{2} \|x_0\|^2 + \frac{1}{2} \|y_0\|^2 - \left\| \frac{1}{2}(x_0 + y_0) \right\|^2 \\ &\leq \frac{1}{2} \delta^2 + \frac{1}{2} \delta^2 - \delta^2 = 0 \end{aligned}$$

so that $y_0 = x_0$ and there is one and only one point at the minimum distance. Here we have again used convexity to conclude that $\frac{1}{2}(x_0 + y_0) \in S$ and hence of norm $\geq \delta$. ■

3. The spaces $L_2H(D)$ and $HL_2(D)$. We shall now consider the spaces $L_2H(D)$ and $HL_2(D)$ of the Introduction but in a more general setting. We allow D to be any bounded domain of finite connectivity and consider the class of all functions $z \rightarrow f(z)$ which are (1) defined and holomorphic in D and (2) quadratically integrable over D . We set

$$(3.1) \quad \|f\| = \left\{ \int_D \int |f(x + iy)|^2 dx dy \right\}^{1/2}.$$

The restrictions made on D are really unnecessarily restrictive but

have the advantage that there can be no doubt about the existence of members of the class. We also define an inner product. If f and $g \in X$ then

$$(3.2) \quad (f, g) = \int \int_D f(z)\overline{g(z)} \, dx \, dy.$$

This definition makes sense and the inner product has the properties postulated in Definition 2.1.

Since $f \in L_2H(D)$ is holomorphic in D , f is bounded in any subdomain D_0 such that $\bar{D}_0 \subset D$. In fact it is possible to get local bounds on f from the metric of the space.

LEMMA 3.1. *If $f \in L_2H(D)$ and if $z_0 \in D$ is at the distance R from ∂D , then*

$$(3.3) \quad |f(z_0)| \leq [\pi R^2]^{-1/2} \|f\|.$$

PROOF. The following argument is due to B. Epstein. Let C denote the circle $|z - z_0| = r$ where r is so small that the closed disk bounded by C lies in D . We apply Cauchy's theorem to the holomorphic function $[f(z)]^2$ (in the mean value form) to obtain

$$(3.4) \quad f^2(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f^2(z_0 + re^{i\theta}) \, d\theta.$$

We multiply both sides of (3.4) by $r \, dr$ and integrate from 0 to R . This gives

$$\begin{aligned} \frac{1}{2} R^2 f^2(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^R f^2(z_0 + re^{i\theta}) r \, dr \, d\theta \\ &= \frac{1}{2\pi} \int \int_{|z-z_0| < R} f^2(z) \, dx \, dy. \end{aligned}$$

Hence

$$\begin{aligned} \pi R^2 |f(z_0)|^2 &\leq \int \int_{|z-z_0| < R} |f|^2 \, dx \, dy \\ &\leq \int \int_D \|f\|^2 \, dx \, dy = \|f\|^2 \end{aligned}$$

as asserted. ■

With the aid of this inequality we can prove that $L_2H(D)$ is complete.

LEMMA 3.2. *$L_2H(D)$ is a Hilbert space.*

PROOF. Consider a Cauchy sequence $\{f_n\}$ and let D_0 be a subdomain of D at the distance $\delta > 0$ from ∂D . We have then by the preceding lemma for $z \in D_0$ that

$$\sqrt{\pi}d(z, \partial D)|f_m(z) - f_n(z)| \leq \|f_m - f_n\|.$$

Here $d(z, \partial D) \geq \delta$ so that

$$(3.5) \quad |f_m(z) - f_n(z)| \leq (1/\sqrt{\pi}\delta)\|f_m - f_n\|.$$

This shows that $\{f_n(z)\}$ is a Cauchy sequence in the space $H(D_0)$ of functions holomorphic in D_0 and continuous in $\overline{D_0}$. Its limit $f(z)$ is then an element of the same space, moreover

$$\int_{D_0} |f(z)|^2 dx dy < \sup \|f_n\|^2.$$

This holds for every D_0 so we infer that $f(z)$ exists in D , is holomorphic there and $\|f\|$ is finite. Moreover, if $m, n \geq N_\epsilon$ we have $\|f_m - f_n\| < \epsilon$ and passing to the limit with m we get $\|f - f_n\| \leq \epsilon$. Since ϵ is arbitrary we have proved that every Cauchy sequence in $L_2H(D)$ has a limit in the space. Thus the inner-product space is complete and hence a Hilbert space. ■

The space $L_2H(D)$ is separable. This is fairly easy to see if D is simply-connected as well as bounded. For then the family of polynomials is dense in the space. But a direct argument for the general case is difficult. The fact will come out later from properties of the kernel function.

At this stage we shall assume the existence of a *complete* orthonormal system in $L_2H(D)$. We know that we can find any number of orthonormal systems which are countably infinite since H is infinite dimensional. The only assumption then is that the system is complete. Let $\{\omega_n(z)\}$ be such a system and use Lemma 2.10. This says that to each f in $L_2H(D)$ there is a unique Fourier series

$$(3.6) \quad f(z) \sim \sum_{n=1}^{\infty} (f, \omega_n)\omega_n(z)$$

which converges to f in the sense of the metric. But here we have a much stronger assertion.

LEMMA 3.3. *The Fourier series (3.6) converges to the sum $f(z)$ pointwise in D , uniformly in any domain D_0 .*

PROOF. This follows from Lemma 3.1. For if $z \in D_0$ and the distance of D_0 from the boundary of D is δ then

$$\left| f(z) - \sum_{k=1}^n (f, \omega_k) \omega_k(z) \right| \leq (\pi \delta^2)^{-1} \left\| f - \sum_{k=1}^n (f, \omega_k) \omega_k \right\|$$

and here the right member goes to zero as $n \rightarrow \infty$. So does the left and we have uniform convergence in D_0 . ■

Actually we have also absolute convergence in D_0 but this requires more inequalities. The assertion would follow from the convergence of the series

$$(3.7) \quad \sum_{n=1}^{\infty} |\omega_n(z)|^2$$

for each z in D_0 . This would follow from the boundedness of the partial sums which is implied by

LEMMA 3.4. *If $t \in D$ and the distance of t from the boundary of D is R , then*

$$(3.8) \quad \sum_{n=1}^{\infty} |\omega_n(t)|^2 \leq (\pi R^2)^{-1}.$$

PROOF. This is an application and sharpening of Lemma 2.13. The latter says that every nonempty closed convex subset S of H has a point nearest to the origin. We now define a subset S of H as follows. S consists of all elements of the form

$$(3.9) \quad f(z) = c_1 \omega_1(z) + c_2 \omega_2(z) + \dots + c_n \omega_n(z)$$

with the condition that $f(t) = 1$, where t is a fixed point in D of distance R from the boundary. This set S is obviously nonempty and convex. Any Cauchy sequence with elements in S has a limit which is also in S so S is also closed. There is then a unique point of S nearest to the origin. Note that the zero element cannot belong to S since $f(t) = 1$ for all f in S . From (3.9) we get

$$(3.10) \quad \|f\|^2 = \sum_{k=1}^n |c_k|^2, \quad \sum_{k=1}^n c_k \omega_k(t) = 1.$$

By Cauchy's inequality

$$1 = \left| \sum_{k=1}^n c_k \omega_k(t) \right|^2 \leq \sum_{k=1}^n |c_k|^2 \sum_{k=1}^n |\omega_k(t)|^2$$

with equality iff \bar{c}_k is a constant multiple of $\omega_k(t)$. It is seen that the multiplier is

$$(3.11) \quad \lambda_n = \left[\sum_{k=1}^n |\omega_k(t)|^2 \right]^{-1}.$$

The point nearest the origin is then at the distance $(\lambda_n)^{1/2} = \|f\|_{\min}$. From Lemma 3.1 we then get

$$1 \leq (\pi R^2)^{-1/2} (\lambda_n)^{1/2}$$

and (3.8) follows as well as the absolute convergence of the Fourier series. ■

LEMMA 3.5. *The set of elements of $L_2H(D)$ coincides with the set of series*

$$(3.12) \quad \sum_{n=1}^{\infty} c_n \omega_n(z) \quad \text{with} \quad \sum_1^{\infty} |c_n|^2 < \infty.$$

PROOF. We know that every f has such a series expansion and the converse follows from the Riesz-Fischer theorem. ■

We can now define the kernel function as

$$(3.13) \quad K(z, t) = \sum_{n=1}^{\infty} \overline{\omega_n(z)} \omega_n(t).$$

Note that the sign of conjugation is often put over the second factor instead. In that case formula (3.16) below figures without a conjugation sign. It is the definition of the inner product which introduces the conjugate.

LEMMA 3.6. *The series (3.13) is absolutely convergent and*

$$(3.14) \quad |K(z, t)|^2 \leq K(z, z)K(t, t),$$

$$(3.15) \quad \|K(\cdot, t)\|^2 = K(t, t).$$

Further, for all $f \in L_2H(D)$,

$$(3.16) \quad f(z) = \int_D \int f(t) \overline{K(z, t)} dt.$$

PROOF. The convergence of

$$(3.17) \quad \sum_{n=1}^{\infty} |\omega_n(t)|^2 = K(t, t)$$

has already been established whence it follows that (3.13) is absolutely convergent. Cauchy's inequality gives (3.14). From the Riesz-Fischer

theorem plus (3.17) we see that $K(z, t)$ for fixed z is an element of $L_2H(D)$ as a function of t while for fixed t the conjugate of $K(z, t)$ is an element of $L_2H(D)$ as a function of z . We can then evaluate the right member of (3.16) using Lemma 2.10(3). This gives

$$\sum (f, \omega_n) \omega_n(z) = f(z)$$

since the orthonormal system is complete. ■

COROLLARY. *We have*

$$(3.18) \quad f(z) = (f(\cdot), K(z, \cdot)).$$

This we shall later adopt as the definition of the reproducing kernel. It does not involve orthonormal systems and adherent complications.

LEMMA 3.7. *The function*

$$(3.19) \quad z \rightarrow K(z, t)/K(t, t) \equiv f(z, t)$$

has the least norm of all elements of $L_2H(D)$ which at $z = t$ take the value 1.

PROOF. We know by Lemma 2.13 that the minimum exists and is unique. Further

$$\begin{aligned} 1 &= |f(t)|^2 = |(f(\cdot), K(t, \cdot))|^2 \\ &\leq \|f\|^2 \|K(t, \cdot)\|^2 = \|f\|^2 K(t, t) \end{aligned}$$

by (3.14). Here we have equality iff $f(u) = \mu K(t, u)$ for almost all u and where μ is independent of u . This requires that $\mu = [K(t, t)]^{-1}$ and the lemma is proved. ■

The function $f(z, t)$ is closely related to the problem of conformal mapping of D , supposed to be simply-connected, onto a circular disk: this will be taken up in §6 and in more detail in the lectures by Professor George Springer.

We shall return to this important space in later lectures using slightly different methods.

We have to give a second installment of the theory of the Hardy-Lebesgue spaces $HL_2(D)$ as developed by Gabor Szegö in 1921. In §1 the case of $D = \{z; \|z\| < 1\}$ was developed. Strictly speaking, the unit disk or more generally, a circular disk is the only case which is associated with Hardy's name. Other cases can be reduced to this case by conformal mapping provided D is simply-connected and has a sufficiently smooth boundary.

Suppose that D is a bounded domain with a boundary consisting of p simple closed analytic curves (analytic in the sense of having local

representations by analytic functions of a real parameter). Consider the class $H(D)$ of functions holomorphic (single-valued!) in D . In this class we can find a system S of functions $\omega_n(z)$ satisfying the following conditions. (1) Each $\omega_n(z) \in H(D)$ and is also holomorphic on ∂D , the boundary of D . (2) S is an orthonormal system on ∂D so that

$$(3.20) \quad \int_{\partial D} \omega_m(z) \overline{\omega_n(z)} ds = \delta_{mn}$$

where s is arclength. (3) S is complete with respect to the corresponding class $L_2H(D)$.

The existence of such a system is not obvious. Again it follows from classical theorems on approximation of holomorphic functions by rational ones (polynomials if $p = 1$) due to Runge, Hilbert, Walsh and others. This means that linear combinations of powers and rational functions (with poles outside of D !) are dense in $HL_2(D)$. Since the poles may be placed at points with rational coordinates we have a countable set which we may orthonormalize with respect to ∂D by the Gram-Schmidt process.

Once we have such a system S we can define $HL_2(D)$ as the class of all functions f which are representable by orthogonal series of the form

$$(3.21) \quad f(z) \sim \sum_{n=1}^{\infty} (f, \omega_n) \omega_n(z)$$

where

$$(3.22) \quad (f, \omega_n) = \int_{\partial D} f(z) \overline{\omega_n(z)} ds \equiv c_n$$

and

$$(3.23) \quad \sum_1^{\infty} |c_n|^2 < \infty.$$

This class is obviously not vacuous. Whether or not it is a Hilbert space under suitable definition of the inner product remains to be seen.

Our first task is to derive analogues of the lemmas which were useful in the $L_2H(D)$ case.

LEMMA 3.8. *If f is holomorphic in $D \cup \partial D$ and if $z_0 \in D$ and is at the distance R from ∂D , then*

$$(3.24) \quad \int_{\partial D} |f(z)|^2 ds \geq 2\pi R |f(z_0)|^2.$$

PROOF. Again we have

$$[f(z_0)]^2 = \frac{1}{2\pi i} \int_{\partial D} \frac{[f(z)]^2 dz}{z - z_0}$$

and this gives

$$|f(z_0)|^2 \leq \frac{1}{2\pi R} \int_{\partial D} |f(z)|^2 ds$$

which is (3.24). ■

Let us now consider the n th partial sum of the prospective kernel series:

$$(3.25) \quad K_n(z, t) = \sum_{k=1}^n \overline{\omega_k(z)} \omega_k(t).$$

If one of the variables is held fixed in D this sum as a function of the other variable is a member of $HL_2(D)$ so that (3.24) applies.

LEMMA 3.9. For $t \in D$ and at the distance R from ∂D

$$(3.26) \quad \sum_{k=1}^n |\omega_k(t)|^2 \leq (2\pi R)^{-1}.$$

PROOF. For we have

$$2\pi R [K_n(t, t)]^2 \leq \int_{\partial D} |K_n(z, t)|^2 ds$$

whence

$$\begin{aligned} 2\pi R \left[\sum_{k=1}^n |\omega_k(t)|^2 \right]^2 &\leq \int_{\partial D} \left| \sum_{k=1}^n \overline{\omega_k(z)} \omega_k(t) \right|^2 ds \\ &= \sum_1^n \sum_1^n \overline{\omega_j(t)} \omega_k(t) \int_{\partial D} \overline{\omega_k(z)} \omega_j(z) ds \\ &= \sum_1^n |\omega_k(t)|^2. \end{aligned}$$

Cancellation of a common factor gives (3.26). ■

We now set

$$(3.27) \quad K(z, t) = \sum_1^\infty \overline{\omega_k(z)} \omega_k(t)$$

Using (3.26) and Cauchy's inequality we see that the series is absolutely convergent for z and t in D and this holds uniformly for z and t in a domain D_0 of positive distance from ∂D .

Again $K(z, t)$ is an element of $HL_2(D)$ as a function of t for fixed z in D and the conjugate of $K(z, t)$ as a function of z for fixed t is also an element.

We now return to a closer examination of formula (3.21) which was taken as the temporary definition of the class $HL_2(D)$. Consider a partial sum of the series

$$(3.28) \quad f_n(z) = \sum_{k=1}^n c_k \omega_k(z).$$

This function is holomorphic in $D \cup \partial D$ so Lemma 3.8 applies and gives the inequalities

$$(3.29) \quad \int_{\partial D} |f_n(z)|^2 ds \geq 2\pi R |f_n(z_0)|^2,$$

$$(3.30) \quad \int_{\partial D} |f_n(z) - f_m(z)|^2 ds \geq 2\pi R |f_n(z_0) - f_m(z_0)|^2.$$

Here z_0 lies in D at the distance R from the boundary and $1 \leq m < n$. On the other hand, the left members in these inequalities are

$$(3.31) \quad \sum_{k=1}^n |c_k|^2 \quad \text{and} \quad \sum_{k=m}^n |c_k|^2,$$

respectively.

Let $L_2(\partial D)$ be the class of functions $f(z)$ defined almost everywhere on ∂D and such that

$$(3.32) \quad \int_{\partial D} |f(z)|^2 ds < \infty.$$

This becomes an inner-product space if

$$(3.33) \quad (f, g) = \int_{\partial D} f(z) \overline{g(z)} ds.$$

It is clearly a Hilbert space.

Now (3.31) shows that $\{f_n(z)\}$ which belongs to $L_2(\partial D)$ is a Cauchy sequence in this space and hence has a limit function which we denote by $f^*(z)$, z on ∂D .

Before going further in this direction, let us note that (3.29) shows that the series (3.21) converges in D , uniformly in any D_0 . Further,

$$\begin{aligned} \sum_{n=1}^{\infty} |c_n \omega_n(z)| &\leq \left\{ \sum_1^{\infty} |c_k|^2 \sum_1^{\infty} |\omega_k(z)|^2 \right\}^{1/2} \\ &\leq \left[\sum |c_k|^2 \right]^{1/2} [2\pi d(z, \partial D)]^{-1/2} \end{aligned}$$

so we have absolute convergence in D , uniformly in D_0 .

Our assumption is that f is defined by the series (3.21). Since the series converges in D its sum then is $f(z)$. Moreover, any such f has boundary values on ∂D denoted above by f^* where

$$\lim_{n \rightarrow \infty} \int_{\partial D} |f^*(z) - f_n(z)|^2 ds = 0.$$

The series (3.21) is still meaningful on ∂D where it is the Fourier series of f as a function in $L_2(\partial D)$. This serves as motivation for considering f^* as the definition of f on ∂D . If f should turn out to be continuous in the closure of D , a fortiori holomorphic in the closure, then of course $f^* = f$ on ∂D without further ado.

We see that $HL_2(D)$ is a Hilbert space with norm and inner product defined by (3.32) and (3.33) where it is understood that the values of f on ∂D are given by those of f^* .

The case where D is a bounded simply-connected domain leads to simple formulas. Suppose D is bounded by a simple closed analytic curve. Then the Riemann mapping theorem says that there exists a function $w(z)$, holomorphic in the unit disk $|z| < 1$, which maps the disk conformally onto D in such a manner that the origin goes into a preassigned point w_0 of D and $\arg w'(0)$ has a given value. Let $z(w)$ be the inverse mapping function. We now consider the orthonormal basis for the unit disk as given by the discussion at the end of §1. With our present definition of the inner product, the orthonormal system for the unit circle is

$$(3.34) \quad (2\pi)^{-1/2} z^n, \quad n = 0, 1, 2, \dots$$

Under the conformal mapping, this system goes over into a corresponding orthonormal system for ∂D , namely

$$(3.35) \quad (2\pi)^{-1/2} [z(w)]^n [z'(w)]^{1/2}$$

where w describes ∂D . For we have

$$\begin{aligned} (2\pi)^{-1} \int_{\partial D} [z(w)]^m [z'(w)]^{1/2} [\overline{z(w)}]^m [\overline{z'(w)}]^{1/2} |dw| \\ = (2\pi)^{-1} \int_{|z|=1} z^n \bar{z}^m |dz(w)| = \delta_{mn}. \end{aligned}$$

Hence (3.24) is an orthonormal system for $HL_2(D)$ and it is complete since (3.23) is complete for the unit circle.

The expression for the reproducing kernel is rather interesting in this case. Observing that under conformal mapping of a domain D_1 onto a domain D , the reproducing kernel for $HL_2(D_1)$ goes into the reproducing kernel of $HL_2(D)$ we can use (1.27) to find the reproducing kernel for $HL_2(D)$. Introducing a factor $(2\pi)^{-1}$ for the modified definition of the inner product we see that

$$\begin{aligned} K(w, w^*) &= \frac{1}{2\pi} \sum [\overline{z(w)}]^n [z(w^*)]^n [\overline{z'(w)} z'(w^*)]^{1/2} \\ (3.36) \qquad &= \frac{1}{2\pi} [1 - \overline{z(w)} z(w^*)]^{-1} [\overline{z'(w)} z'(w^*)]^{1/2} \end{aligned}$$

which is the natural generalization of (1.27).

It was assumed that in the mapping $z = 0$ corresponds to $w = w_0$. If we set $w^* = w_0$ in (3.36), it reduces to

$$(3.37) \qquad K(w, w_0) = \frac{1}{2\pi} [z'(w_0)]^{1/2} [\overline{z'(w)}]^{1/2}.$$

It follows that for a simply-connected domain the square of the conjugate kernel function yields the derivative of the function which maps the domain on the interior of a circle.

4. The functional approach. We have repeatedly encountered relations between our theory and that of linear bounded functionals on a Hilbert space. This suggests making a fresh attack on the problem via the theory of such functionals. This line of attack was opened up by N. Aronszajn.

Suppose that H is a Hilbert space of complex-valued functions f defined on some set S . Each linear bounded functional on H is of the form

$$(4.1) \qquad (f, g)$$

for some fixed $g \in H$. Let P be a fixed point in S and consider the mapping from H to C defined by

$$(4.2) \qquad f \rightarrow f(P).$$

This is a linear mapping but not necessarily bounded. Such a situation arose in §1 in the case of the space $L_2(a, b)$ where the mapping $f \rightarrow f(s)$ is not defined for all s and not bounded when it is meaningful. In fact, this fact caused our lack of success in trying to find a reproducing kernel for the space.

On the other hand, our success with the space $L_2H(D)$ is due to the fact that for any z in D the mapping is bounded and this also lies behind our success in the $HL_2(D)$ case.

DEFINITION 4.1. H is a kernel space if the mapping (4.2) is bounded for each P in S .

If H is a kernel space, then for a fixed P we can find an element g_P of H such that

$$(4.3) \quad f(P) = (f, g_P).$$

As the notation indicates, g_P depends upon P . Now the mapping

$$(4.4) \quad P \rightarrow g_P$$

is from S to H and we write

$$(4.5) \quad g_P(Q) = K(P, Q)$$

and this is for each fixed P an element of H .

DEFINITION 4.2. $K(P, Q)$ is the kernel function of H .

This is justified by the relation

$$(4.6) \quad f(P) = (f(Q), K(P, Q)), \quad \forall P \in S.$$

LEMMA 4.1. *A Hilbert space has at most one kernel function.*

PROOF. If H is a kernel space in the sense of Definition 4.1 then there exists a kernel function. And by Lemma 2.12, g_P is uniquely defined, i.e. there is only one function $K(P, Q)$ which satisfies (4.6). ■

LEMMA 4.2. *The reproducing kernel has the following properties:*

$$(4.7) \quad K(P, P) \geq 0, \quad K(P, Q) = \overline{K(Q, P)},$$

$$(4.8) \quad |K(P, Q)|^2 \leq K(P, P)K(Q, Q).$$

PROOF. The first inequality follows from applying (4.6) to the kernel function. We have

$$(4.9) \quad K(U, P) = (K(U, Q), K(P, Q))$$

for all $U \in S$. In particular for $U = P$ we get

$$(4.10) \quad K(P, P) = \|K(P, \cdot)\|^2 > 0.$$

For the skew symmetry note that we can write (4.9)

$$K(U, P) = (K(U, V), K(P, V))$$

whence

$$K(Q, P) = (K(Q, V), K(P, V))$$

which is the conjugate of

$$(4.11) \quad (K(P, V), K(Q, V)) = K(P, Q).$$

Cauchy's inequality applied to (4.11) and using (4.10) gives (4.8). ■

LEMMA 4.3. *For any set of complex numbers $\{\lambda_j\}$ and any countable set of points $P_j \in S$ we have for each n*

$$(4.12) \quad H_n = \sum_{j=1}^n \sum_{k=1}^n \lambda_j \bar{\lambda}_k K(P_j, P_k) \geq 0.$$

PROOF. We have

$$(4.13) \quad 0 \leq \left(\sum_{j=1}^n \lambda_j K(P, P_j), \sum_{k=1}^n \lambda_k K(P, P_k) \right) = H_n. \quad \blacksquare$$

Here H_n is an Hermitian form in the λ 's. It is positive by (4.12). We shall return to such positive matrices $(a_{jk}) = (K(P_j, P_k))$ in the next section.

We shall now generalize some of the results derived in earlier sections for the special space $L_2H(D)$. We shall vary the notation somewhat.

LEMMA 4.4. *The Hilbert space H with elements $f(s)$, $s \in S$, has a reproducing kernel iff $f(s)$ is a linear bounded functional on H for each fixed $s \in S$.*

PROOF. We know that the condition is sufficient. To prove that it is necessary, suppose there exists a $K(s, t)$ with the properties (i) $K(s, t) \in H$ as a function of t for each $s \in S$, (ii) $0 \leq K(s, s) < \infty$ for all $s \in S$, and (iii)

$$(4.14) \quad f(s) = (f(\cdot), K(s, \cdot))$$

for all $f \in H$, $s \in S$. This is clearly a linear functional on H and by Cauchy's inequality

$$(4.15) \quad |f(s)| \leq \|f\| \|K(s, \cdot)\| = [K(s, s)]^{1/2} \|f\|,$$

since (4.10) is a consequence of (4.14). This shows that the functional is bounded as well as linear. Thus the condition is necessary as well as sufficient. ■

We note that $[K(s, s)]^{1/2}$ is the norm of the functional for equality is reached in (4.15) for a fixed s iff $f(t)$ is proportional to $K(s, t)$ for

all t in S with the factor of proportionality a function of s . Since

$$[K(s, s)]^{1/2} = K(s, s)/\|K(s, \cdot)\|$$

we get

$$(4.16) \quad |f(s)|/\|f\| \leq K(s, s)/\|K(s, \cdot)\|$$

or, equivalently,

$$(4.17) \quad \|K(s, \cdot)\|/K(s, s) \leq \|f\|/|f(s)|.$$

Again these inequalities are sharp and equality is reached iff

$$(4.18) \quad u \rightarrow f(u) = K(u, s)/K(s, s).$$

This leads to the following generalization of Lemma 3.7.

LEMMA 4.5. *If H is a Hilbert kernel space and H_0 is the subset of elements of H which take the value 1 at $s = t$, then H_0 has a unique point nearest to the zero element and this is*

$$(4.19) \quad s \rightarrow f(s) = K(s, t)/K(t, t).$$

On the other hand, all elements of H of norm 1 assume values at $s = t$ which are in absolute value at most equal to

$$(4.20) \quad K(t, t)/\|K(t, \cdot)\|.$$

PROOF. Use (4.16)-(4.18). ■

We can also extend Lemma 3.6 in the following manner.

LEMMA 4.6. *If H is a Hilbert kernel space and $\{f_n\}$ is a Cauchy sequence in H then $\{f_n(s)\}$ is a Cauchy sequence in C for each $s \in S$. Moreover, the convergence of $\{f_n(s)\}$ to its limit $f(s)$ is uniform on any subset of S where $K(s, s)$ is bounded.*

PROOF. Suppose that $f_n \rightarrow f$ in H so that $s \rightarrow f(s)$ is an element of H . We have then

$$|f(s) - f_n(s)| = |(f(\cdot) - f_n(\cdot), K(s, \cdot))| \leq \|f - f_n\| [K(s, s)]^{1/2}$$

which proves the assertion. ■

It should be noted that ordinary pointwise convergence can be concluded already if $\{f_n\}$ converges weakly rather than strongly to f , for this implies $(f_n, g) \rightarrow (f, g)$ for all $g \in H$, in particular, for the kernel.

LEMMA 4.7. *If H is a Hilbert kernel space and if there is a complete orthonormal system $\{\omega_n\}$ in H then for each $f \in H$*

$$(4.21) \quad f(s) = \sum_{n=1}^{\infty} (f, \omega_n) \omega_n(s)$$

where the series converges for all $s \in S$ and uniformly on any subset S_0 of S where $K(s, s)$ is uniformly bounded.

PROOF. The series in (4.21) is the Fourier series of f with respect to the system $\{\omega_n\}$ and its partial sums converge to f so that Lemma 4.6 applies. ■

We come now to the question of the existence of a complete orthonormal system in a Hilbert kernel space. This question has caused us plenty of trouble in the special cases studied so far so it is time to get rid of this bug-bear.

LEMMA 4.8. *Suppose that H is a Hilbert kernel space. Then there exists a complete orthonormal system $\{\omega_n\}$ in terms of which the reproducing kernel $K(s, t)$ is represented by the series*

$$(4.22) \quad K(s, t) = \sum_{n=1}^{\infty} \overline{\omega_n(s)} \omega_n(t)$$

iff H is separable.

PROOF. Suppose that H is separable. H is of course infinite dimensional. There exists then a countable set of elements f_n in H which are dense there. Out of this set we can select by the sieve process an infinite subset of linearly independent elements whose linear combinations are still dense in H . Applying the Gram-Schmidt process to this set we get an orthonormal system. The linear combinations of finite subsets of this system are still dense in H so the system is complete.

By assumption we have a reproducing kernel $K(s, t)$ which for fixed s is a function of t belonging to H . Hence it has a Fourier expansion

$$(4.23) \quad K(s, t) \sim \sum_{n=1}^{\infty} c_n(s) \omega_n(t).$$

Here

$$c_n(s) = (K(s, u), \omega_n(u)) = \overline{(\omega_n(u), K(s, u))} = \overline{\omega_n(s)}$$

which gives (4.22).

Conversely, if (4.22) holds then, for any $f \in H, s \in S$,

$$f(s) = \left(f(u), \sum_{n=1}^{\infty} \overline{\omega_n(s)} \omega_n(u) \right) = \sum (f, \omega_n) \omega_n(s)$$

where the series is absolutely convergent. Since every $f \in H$ is representable in this manner, the finite linear combinations of the ω_k 's are dense in H and H is separable. ■

Under slight restrictions on the set S we can show that a non-separable Hilbert space cannot have a kernel function.

LEMMA 4.9. *Let H be a nonseparable Hilbert space the elements of which are continuous functions of s defined on a separable topological space E . Then H cannot have a reproducing kernel.*

PROOF. Suppose contrariwise that $K(s, t)$ is a kernel function of this space. Then $K(s, t)$ for fixed s in E is an element of H and a continuous function of t . Now E is separable so we can find a countable set $\{s_j\}$ dense in E . Form the set $\{K(s_j, t)\}$. This is a countable set of elements of H . If their finite linear combinations are not dense in H , there exists a linear bounded functional on H which is annihilated by every $K(s_j, t)$. Hence there exists an element g of H such that

$$(K(s_j, \cdot), g(\cdot)) = 0, \quad \forall j.$$

But then we have also

$$0 = (g(\cdot), K(s_j, \cdot)) = g(s_j).$$

But g is continuous and the set $\{s_j\}$ is dense in E . Hence g is the zero functional and we conclude that the finite linear combinations of the elements $K(s_j, \cdot)$ are dense in H . This makes H separable against the assumption. Hence such a space H can have no kernel function. ■

We conclude from this that if a nonseparable Hilbert space of functions f defined on a separable topological space E has a kernel function then H must have discontinuous functions among its elements and these must have discontinuities of the second kind. We also get the following condition of separability:

LEMMA 4.10. *If H is a Hilbert space of continuous functions defined on a separable topological space E and if for each $f \in H$ and each $s \in E$ there is a finite number $M(s)$ such that*

$$(4.24) \quad |f(s)| \leq M(s)\|f\|,$$

then H is separable and has a kernel function.

PROOF. The assumption (4.24) says that, for each fixed s , $f(s)$ is a linear bounded functional on H , i.e. there exists a function $K(s, t)$ which for each fixed s in E is an element of H and such that $f(s) = (f(\cdot), K(s, \cdot))$. Here $K(s, t)$ is a continuous function of t for each fixed s . By (4.15) we see that we can take $M(s) = [K(s, s)]^{1/2}$ and no smaller

number is an admissible bound. We can repeat the above argument for the set $K(s, t)$ of elements in H and see that their finite linear combinations are dense in H so that H is separable. ■

Our last remarks in this section are concerned with linear operators on a Hilbert kernel space. Let T be a *linear bounded operator* on H and let T^* be the *adjoint operator*, i.e.

$$(4.25) \quad \langle Tf, f \rangle = \langle f, T^*f \rangle, \quad \forall f.$$

Since H has a reproducing kernel $K(s, t)$ which for fixed s belongs to H as a function of t , we can operate with T^* on $K(s, t)$ as a function of t . We set

$$(4.26) \quad L(s, t) = T_t^*[K(s, t)]$$

where the subscript t indicates that we operate on the second argument. This gives us the following representation of Tf .

LEMMA 4.11. *Let H be a Hilbert kernel space, T a linear bounded operator on H and $L(s, t)$ the kernel defined above. Then*

$$(4.27) \quad T[f](s) = \langle f(\cdot), L(s, \cdot) \rangle.$$

For

$$\langle f(\cdot), L(s, \cdot) \rangle = \langle f(u), T_u^*K(s, u) \rangle = \langle Tf(u), K(s, u) \rangle = T[f](s)$$

as asserted.

We can regard $L(s, t)$ as the kernel of the operation T .

LEMMA 4.12. *The kernel of the adjoint operator T^* is under the same assumptions given by*

$$(4.28) \quad L^*(s, t) = \overline{L(t, s)}.$$

PROOF. We observe that $(T^*)^* = T$ and use (4.27). Then

$$\begin{aligned} \langle T_u K(s, u), K(t, u) \rangle &= \langle T_u^{**} K(s, u), K(t, u) \rangle \\ &= \langle L^*(s, u), K(t, u) \rangle = L^*(s, t). \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle T_u K(s, u), K(t, u) \rangle &= \overline{\langle K(t, u), T_u K(s, u) \rangle} \\ &= \overline{\langle T_u^* K(s, u), K(t, u) \rangle} \\ &= \overline{\langle L(s, u), K(t, u) \rangle} = \overline{L(s, t)}. \end{aligned}$$

And this is the required relation.

The representation is particularly simple if $T = U$ is a unitary operator, i.e.

$$(4.29) \quad U^* = U^{-1}.$$

LEMMA 4.13. *If L is the kernel of a unitary operator U , then*

$$(4.30) \quad (L(s, u), L(t, u)) = K(s, t).$$

PROOF. By (4.27) the left member of this relation is

$$UL(s, t) = UU^*K(s, t) = K(s, t).$$

LEMMA 4.14. *If $L(s, t)$ is the kernel of a unitary transformation on a Hilbert kernel space then there exist two complete orthonormal systems $\{\varphi_n\}$ and $\{\omega_n\}$ in terms of which*

$$(4.31) \quad L(s, t) = \sum_{n=1}^{\infty} \overline{\varphi_n(s)} \omega_n(t).$$

PROOF. Since $L(s, t)$ for fixed s as a function of t is an element of H , it admits a representation of type (4.31) for any complete orthonormal system $\{\omega_n(t)\}$. We have to show that the coefficients themselves form a complete orthonormal system. From (4.30) and (4.31) we obtain

$$(4.32) \quad \begin{aligned} K(s, t) &= (L(s, u), L(t, u)) \\ &= \left(\sum \overline{\varphi_m(s)} \omega_m(u), \sum \overline{\varphi_n(t)} \omega_n(u) \right), \end{aligned}$$

$$K(s, t) = \sum_{n=1}^{\infty} \overline{\varphi_n(s)} \varphi_n(t)$$

the series being absolutely convergent for all s, t in S . Hence for each $f \in H$

$$(4.33) \quad \begin{aligned} f(s) &= (f(t), K(s, t)) = \left(f(t), \sum \overline{\varphi_n(s)} \varphi_n(t) \right) \\ &= \sum_{n=1}^{\infty} (f, \varphi_n) \varphi_n(s). \end{aligned}$$

We still do not know whether or not this is an orthogonal series. To clinch the argument note that

$$U[f](s) = (f(t), L(s, t))$$

and in particular

$$(4.34) \quad \begin{aligned} U[\omega_n](s) &= (\omega_n(t), L(s, t)) \\ &= \left(\omega_n(t), \sum_{m=1}^{\infty} \overline{\varphi_m(s)} \omega_m(t) \right) = \varphi_n(s). \end{aligned}$$

Since U is unitary,

$$(\varphi_m, \varphi_n) = (U\omega_m, U\omega_n) = (\omega_m, U^*U\omega_n) = (\omega_m, \omega_n) = \delta_{mn}$$

and $\{\varphi_n\}$ is indeed an orthonormal system. That it is complete follows from (4.33) which holds for all $f \in H$. ■

This discussion shows that to every unitary operator U acting in a Hilbert kernel space there are two reciprocal orthonormal systems $\{\omega_n\}$ and $\{\varphi_n\}$ where

$$(4.35) \quad U[\omega_n](s) = \varphi_n(s), \quad U[\varphi_n](s) = \omega_n(s)$$

such that for any $f \in H$, $g = U[f]$,

$$(4.36) \quad g(s) = (f(t), L(s, t)) = \sum_1^{\infty} (f, \varphi_n) \omega_n(s),$$

$$(4.37) \quad f(s) = (g(t), \overline{L(t, s)}) = \sum_1^{\infty} (g, \omega_n) \varphi_n(s),$$

and

$$(4.38) \quad L(s, t) = \sum_{n=1}^{\infty} \overline{\varphi_n(s)} \omega_n(t).$$

Such reciprocal transformations play an important role in analysis. See Bochner-Chandrasekharan, *Fourier transforms*.

5. Miscellaneous questions. In this section we shall discuss questions like extremal and minimal problems, and positive matrices, all connected with reproducing kernels.

Our point of departure is formed by Lemmas 2.13 and 4.5. They assert that of all the functions $f \in H$, a Hilbert kernel space of elements defined in a set S , such that $f(t) = 1$ for a fixed t , the mapping

$$(5.1) \quad s \rightarrow \frac{K(s, t)}{K(t, t)}$$

has the least norm, namely $[K(t, t)]^{-1/2}$. We generalize this problem as follows. Let t_1, t_2, \dots, t_n be n distinct points in S and let $a_1, a_2, \dots,$

a_n be n complex numbers distinct or not. It is now required to find the element of least norm for which

$$(5.2) \quad f(t_j) = a_j, \quad j = 1, 2, \dots, n.$$

The subset of H where the elements satisfy these conditions is evidently closed, nonvoid and convex. Thus Lemma 2.13 applies and shows the existence of a unique element of minimum norm. To determine this element and its norm we argue as follows.

H has a kernel function $K(s, t)$ such that $K(s, t)$ for fixed s and regarded as a function of t belongs to H . Now consider all elements of H of the form

$$(5.3) \quad \sum_{j=1}^n c_j K(t_j, t).$$

This is a linear finite-dimensional subspace M_n of H and has an orthogonal complement M_n^\perp . Hence for any $f \in H$ we have a unique decomposition

$$(5.4) \quad f = g + h, \quad g \in M_n, h \in M_n^\perp.$$

Since $(g, h) = 0$ the Pythagorean Theorem gives

$$\|f\|^2 = \|g\|^2 + \|h\|^2.$$

Further

$$h(t_j) = (h(\cdot), K(t_j, \cdot)) = 0$$

since $h(t) \in M_n^\perp, K(t_j, t) \in M_n$. Hence

$$f(t_j) = g(t_j) = a_j.$$

It follows that in finding the minimum we can restrict ourselves to elements of M_n since adding an h to a g only increases the norm. Thus the minimum function is of the form $\sum_{j=1}^n c_j K(t_j, t)$ for some particular choice of the c_j 's. There are n such numbers and we have n linear conditions at our disposal

$$(5.5) \quad \sum_{j=1}^n c_j K(t_j, t_k) = a_k, \quad k = 1, 2, \dots, n,$$

with determinant

$$(5.6) \quad \det [K(t_j, t_k)] \equiv \Delta_n.$$

Since $\overline{K(t_k, t_j)} = K(t_j, t_k)$ the value of Δ_n is real. But we can say more.

According to Lemma 4.3 the values taken on by the Hermitian form

$$(5.7) \quad \sum_{j=1}^n \sum_{k=1}^n \lambda_j \bar{\lambda}_k K(t_j, t_k)$$

for $\sum_{j=1}^n |\lambda_j|^2 = 1$ are real and nonnegative. It follows that $\Delta_n \geq 0$. The value zero can be excluded since this would mean that there would be some set $\{a_j\}$ for which the equations (5.5) could not be solved and hence the extremal problem has no solution.

Once the c 's have been found, say by Cramer's rule, we can find the minimum value of the norm

$$(5.8) \quad \sum_{j=1}^n \sum_{k=1}^n c_j \bar{c}_k K(t_j, t_k),$$

i.e. the value of the Hermitian form (5.7) for $\lambda_j = c_j$, $j = 1, 2, \dots, n$. Equations (5.5) allow us to write this as

$$(5.9) \quad \|f\|^2 = \sum_{k=1}^n \bar{c}_k a_k.$$

Another important minimum problem is the following.

LEMMA 5.1. *Let H be a Hilbert kernel space of functions holomorphic in a domain D . For each $n = 0, 1, 2, \dots$ let F_n be the subset of elements of H which at a given point t of D satisfy the conditions*

$$(5.10) \quad f^{(k)}(t) = 0, \quad k = 0, 1, 2, \dots, n-1, f^{(n)}(t) = 1.$$

Then F_n contains a unique element $g_n(z) = g_n(z, t)$ of minimum norm and the functions $g_n(z, t)$ form a complete orthogonal system.

PROOF. It should be noted that F_n is nonvoid. This is trivial if D is bounded for then any function of the form

$$(5.11) \quad \frac{1}{n!} (z-t)^n + \sum_{m=n+1}^{\infty} a_m (z-t)^m$$

belongs to F_n , if the radius of convergence of the power series exceeds the l.u.b. of the distances of t from points of the boundary of D . Here we are assuming that $f \equiv 1$ belongs to H . This is not necessarily true for an arbitrary definition of the inner product. So to avoid further ado, let us assume that F_n is nonvoid for any n . Since F_n is closed and convex, there is a unique element of minimum norm for each n and it cannot be the zero element of H .

Thus the existence of $g_n(z, t)$ is assured and in some neighborhood

of $z = t$ we have an expansion of type (5.11). To prove the orthogonality we note that for any n and any f we have a unique decomposition

$$f = g_n + h, \quad (g_n, h) = 0.$$

If $p > n$, then g_p satisfies $(g_p, g_n) = 0$. Thus we are dealing with an orthogonal system. To prove uniqueness we argue as follows. We form an orthonormal system

$$(5.12) \quad \omega_n(z) = \|g_{n-1}\|^{-1}g_{n-1}(z), \quad n = 1, 2, 3, \dots$$

Let $f \in H$ be arbitrary and form the series

$$(5.13) \quad g(z) = \sum_{k=1}^{\infty} (f, \omega_k)\omega_k(z) \equiv \sum_{k=1}^{\infty} a_k\omega_k(z).$$

The series converges absolutely in D , uniformly on compact subsets, and is an element of H . We aim to prove that $f(z) \equiv g(z)$.

To this end consider a sequence of functions $\{f_n(z)\}$ such that

$$(5.14) \quad f_n(z) = \sum_{k=1}^n b_{kn}\omega_k(z),$$

$$(5.15) \quad f_n^{(k)}(t) = f^{(k)}(t), \quad k = 0, 1, \dots, n - 1.$$

These conditions may be satisfied, uniquely at that. For the system

$$\sum_{k=1}^n b_{kn}\omega_k^{(j)}(t) = f^{(j)}(t), \quad j = 0, 1, \dots, n - 1,$$

has a triangular matrix and the elements along the main diagonal are different from zero so the b_{kn} 's may be determined successively. Since the function $f(z) - f_n(z)$ vanishes for $z = t$ together with its first $n - 1$ derivatives we see that

$$(f - f_n, \omega_k) = 0, \quad k = 1, 2, \dots, n.$$

Thus

$$(f, \omega_k) = (f_n, \omega_k) \quad \text{or} \quad b_{kn} = a_k$$

and

$$f_n(z) = \sum_{k=1}^n a_k\omega_k(z)$$

is the n th partial sum of the series (5.13). We now return to (5.15).

We note that $f_n^{(j)}(z)$ converges uniformly to $g^{(j)}(z)$ as $n \rightarrow \infty$. It follows that $f^{(j)}(t) = g^{(j)}(t)$ for all j and by the identity theorem for holomorphic functions this gives $g(z) \equiv f(z)$. Thus the system is complete.

We turn now to the theory of positive matrices. Given a function $K(s, t)$ of two variables defined for s and t in some set E . The question arises if $K(s, t)$ can be the kernel function of some Hilbert space H of elements f defined for s in E . We know a necessary condition: $K(s, s)$ must be a positive matrix in the sense defined above, i.e. for any choice of distinct points s_j in E and of complex numbers λ_j we must have for all n

$$(5.16) \quad \sum_{j=1}^n \sum_{k=1}^n \lambda_j \bar{\lambda}_k K(s_j, s_k) \geq 0, \quad K(s, t) = \overline{K(t, s)}.$$

According to H. Meschkowski this condition is also sufficient. We shall prove this imposing somewhat restrictive conditions on E and $K(s, t)$.

LEMMA 5.2. *Suppose that E is a domain (open connected set) in a complete metric space X and that $K(s, t)$ is a continuous function of (s, t) from $E \times E$ to C . Suppose that K satisfies (5.16) for any choice of $\{s_j\} \subset E$ and $\{\lambda_j\} \subset C$. Then there exists a Hilbert space H of elements f defined in E such that K is the kernel function of H .*

PROOF. Take a fixed sequence $\{s_j\} \subset E$ but dense in E in the sense of the metric. We are going to build H from the set $\{K(s_j, t)\}$ forming linear combinations and passing to the limit. This has to be governed by an appropriate use of (5.16). First let R be a closed connected subset of E and suppose that $M(R)$ is the maximum of $K(s, t)$ for $(s, t) \in R \times R$. Let us now consider any sequence of complex numbers $\{a_j\}$ such that if

$$(5.17) \quad 0 \leq \sum_{j=1}^n \sum_{k=1}^n a_j \bar{a}_k K(s_j, s_k) = A_n$$

then

$$(5.18) \quad \lim_{n \rightarrow \infty} A_n \equiv A$$

exists and is finite. Set

$$(5.19) \quad f_n(t) = \sum_{j=1}^n a_j K(s_j, t)$$

and define

$$\|f_n\|^2 = A_n, \quad n = 1, 2, 3, \dots,$$

and note that

$$\lim_{n \rightarrow \infty} \|f_n\|^2 = A.$$

We shall try to attach a meaning to $\lim f_n(t)$, to start with for any $t = s_j$.

Consider the sequence $\{f_n(s_k); n = 1, 2, 3, \dots\}$. If now

$$(5.20) \quad g_n(t) = \sum_{j=1}^n b_j K(s_j, t)$$

and B_n is what we obtain by replacing a_j by b_j for all j in (5.17), we assume that $\lim B_n \equiv B$ is finite. For two such functions we define an inner product by

$$(5.21) \quad (f_n, g_n) = \sum_{j=1}^n \sum_{k=1}^n a_j \bar{b}_k K(s_j, s_k).$$

A couple of special cases of this formula are basic for the following. If $n \geq \max(j, k)$ we get

$$(5.22) \quad (K(s_j, \cdot), K(s_k, \cdot)) = K(s_j, s_k).$$

We also get

$$(5.23) \quad f_n(s_j) = (f_n(\cdot), K(s_j, \cdot))$$

so that K is beginning to act as a kernel function, at least for finite linear combinations of the basis elements and values of s in the dense set $\{s_j\}$.

We now start with the limiting processes in a cautious manner. We have

$$(5.24) \quad \begin{aligned} |f_n(s_j) - f_n(s_k)| &= |(f_n(\cdot), K(s_j, \cdot)) - (f_n(\cdot), K(s_k, \cdot))| \\ &\leq \|f_n\| \|K(s_j, \cdot) - K(s_k, \cdot)\| \\ &= A_n^{1/2} [K(s_j, s_j) - K(s_j, s_k) - K(s_k, s_j) + K(s_k, s_k)]. \end{aligned}$$

Here $A_n^{1/2}$ is less than some finite M since $\lim A_n = A$ exists and is finite. We now have to use that $K(s, t)$ is uniformly continuous for $(s, t) \in R \times R$. This implies that given an $\epsilon > 0$ we can find a

$\delta > 0$ which goes to zero with ϵ such that

$$(5.25) \quad |K(s', t') - K(s'', t'')| \leq \epsilon$$

if $d(s', s'') \leq \delta$, $d(t', t'') \leq \delta$. Hence for any choice of $s_j \in R$ and any choice of $s_k \in R$ such that $d(s_j, s_k) \leq \delta$ we have

$$(5.26) \quad |f_n(s_j) - f_n(s_k)| \leq 2M\epsilon,$$

and this holds uniformly in j, k, n . If now $s_j \rightarrow s'$, $s_k \rightarrow s''$ where still $d(s', s'') \leq \delta$, then we have

$$(5.27) \quad |f_n(s') - f_n(s'')| \leq 2M\epsilon$$

uniformly in n . This asserts that the sequence $\{f_n(s)\}$ is equicontinuous for $s \in R$. Further we have by (5.23)

$$(5.28) \quad |f_n(s_j)| \leq \|f_n\| [K(s_j, s_j)]^{1/2} \leq MM(R).$$

This is uniformly bounded with respect to n for all s_j in R and hence for all s in R . Again by (5.23) for $1 \leq m < n$,

$$\begin{aligned} |f_n(s_j) - f_m(s_j)| &= |(f_n - f_m, K(s_j, \cdot))| \leq \|f_n - f_m\| [K(s_j, s_j)]^{1/2} \\ &= |A_n - A_m| [K(s_j, s_j)]^{1/2}. \end{aligned}$$

This expression goes to zero as $m \rightarrow \infty$, uniformly for $s_j \in R$.

Thus we have a sequence of functions $\{f_n(s)\}$, equicontinuous and equibounded in R , which moreover converges at all points $\{s_j\}$, a dense set in E . By the Ascoli-Arzelà theorem, there is a subsequence that converges uniformly to a limit $f(s)$ in R and since the whole sequence converges in a dense set, it will have to converge everywhere. The limit function $f(s)$ is continuous in E . The family of all such functions f is the first extension of the basic set $K(s_j, t)$. Note the reproducing property. From (5.23) we get for $n \rightarrow \infty$

$$f(s_j) = (f(\cdot), K(s_j, \cdot))$$

and then by continuity

$$(5.29) \quad f(s) = (f(\cdot), K(s, \cdot)), \quad \forall s.$$

We have a family of continuous functions obeying the reproducing property, but where is the Hilbert space or even an inner product? An inner product is defined for the finite linear combinations of the basis elements $K(s_j, t)$. If now

$$\lim f_n = f, \quad \lim g_m = g,$$

it is seen that

$$(5.30) \quad \lim (f_n, g_m) \equiv (f, g)$$

independent of how m and n go to infinity. This follows from the inequality

$$\begin{aligned} |(f_n, g_m) - (f, g)| &\leq \|f_p\| \|g_m - g_k\| + \|g_m\| \|f_p - f_n\| \\ &= A_p|B_m - B_k| + B_m|A_p - A_n|. \end{aligned}$$

Thus an inner product is defined for all f, g in our set. Moreover, this is a bona fide inner product and has all the required properties. Thus the functions f form an inner-product space. If this should not be complete, we can embed it in the set of all Cauchy sequences in the usual manner. This is now the desired space H . ■

Positive matrices have certain composition properties which we shall state without proof.

LEMMA 5.3. *If H_1 and H_2 are two Hilbert kernel spaces with kernels K_1 and K_2 and norms $\| \cdot \|_1$ and $\| \cdot \|_2$, respectively, then $K(s, t) = K_1 + K_2$ is the kernel of a Hilbert space H formed by all elements of the form $f = f_1 + f_2$ where $f_i \in H_i, i = 1, 2$, and the norm is defined by*

$$(5.31) \quad \|f\|^2 = \min[\|f_1\|_1^2 + \|f_2\|_2^2]$$

where the minimum is to be taken with respect to all decompositions $f = f_1 + f_2$ with $f_i \in H_i$.

If the decomposition $f = f_1 + f_2$ is unique, then

$$(5.32) \quad \|f\|^2 = [\|f_1\|_1^2 + \|f_2\|_2^2].$$

The lemma can be extended to finite sums of kernel functions. There are also corresponding results for a difference, provided $K_1(s, t) - K_2(s, t)$ is also a kernel function.

For a product of kernel functions K_1 and K_2 with Hilbert spaces H_1 and H_2 the direct product of two spaces $H_1 \otimes H_2$ comes into play. We form

$$(5.33) \quad f(x_1, x_2) = \sum_{j=1}^n f_{j1}(x_1)f_{j2}(x_2), \quad f_{ji} \in H_i.$$

With a suitable definition of inner product and norm, the set of all such elements forms the germ of an inner-product space which can be completed to a Hilbert space $H = H_1 \otimes H_2$. The kernel of this space is

$$(5.34) \quad K(s_1, s_2, t_1, t_2) = K_1(s_1, t_1)K_2(s_2, t_2).$$

For details we refer to H. Meschkowski, *Hilbertsche Räume mit Kernfunktion*, Chapter 5.

6. Conformal mapping, invariantive metrics, curvature. I shall do comparatively little with these interesting topics since they will be treated in detail by Professor George Springer. In conformal mapping I shall restrict myself to simply-connected bounded domains, leaving the multiply-connected ones entirely to Springer.

We have already had an inkling of the situation in connection with the discussion of the Bergman and Szegö kernels. We start the discussion of the Bergman case by recalling the notion of complex derivatives and giving a suitable form of the Theorem of Stokes.

Suppose that $f(x, y)$ is a mapping of a domain D in the complex plane into C . Restrictions will later be imposed on the boundary of C . The function f is supposed to have partial derivatives $f_x(x, y)$ and $f_y(x, y)$ in D .

DEFINITION 6.1. The complex derivatives f_z and $f_{\bar{z}}$ are

$$(6.1) \quad f_z = \frac{1}{2}(f_x - if_y), \quad f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

These objects obey the ordinary rules for differentiation of sums, products, differences, and quotients. If f is holomorphic in a neighborhood of a point $z = z_0$ then by the Cauchy-Riemann equations, $f_z(z_0) = f'(z_0)$ and $f_{\bar{z}}(z_0) = 0$. If $\overline{f(x, y)}$ is holomorphic in a neighborhood of z_0 , then the situation is reversed, $f_z(z_0) = 0$ while $f_{\bar{z}}(z_0) = g'(z_0)$ with $g(z) = \overline{f(x, y)}$. We also note the expression for the Laplacian

$$(6.2) \quad \Delta f = 4f_{z\bar{z}}.$$

Next we come to the Theorem of Stokes, first in the classical setting.

Suppose that two functions $u(z)$ and $v(z)$ are defined in a region $R (= D \cup \partial D)$ where they are continuous together with their partials of sufficiently high order. D is supposed to be bounded and $\partial D = \Gamma$ consists of a finite number of "sufficiently" smooth simple closed curves. Then

LEMMA 6.1. *We have*

$$(6.3) \quad \iint_D [u_x + v_y] dx dy = \int_{\Gamma} [u dy - v dx]$$

where the integration over Γ is performed in the positive sense with respect to D .

If here $v = iu$ and we take into account (6.1) the formula becomes

$$(6.4) \quad \iint_D u_z \, dx \, dy = -\frac{1}{2} i \int_{\Gamma} u \, dz.$$

Suppose that f and h are two functions holomorphic in D and continuous on the boundary. We take $u(z) = f(z)\overline{h(z)}$ and apply (6.1) to obtain

$$(6.5) \quad \iint_D f(z)\overline{h'(z)} \, dx \, dy = -\frac{1}{2} i \int_{\Gamma} f(z)\overline{h(z)} \, dz.$$

After this spade work we can return to the Bergman kernel and its bearing on the problem of conformal mapping. We recall that Riemann's Conformal Mapping Theorem states the following.

A given simply-connected domain D in the extended complex plane with at least two boundary points may be mapped conformally and one-to-one onto the unit disk. Further a given point $z = t$ in D may be made to map into the origin of the image plane and the argument of $f'(t)$ may be prescribed.

An equivalent formulation is the following.

There exists a unique element f of $H(D)$ such that

(1) $|f(z)| < 1$ everywhere in D .

(2) For each w , $|w| < 1$, the equation $f(z) = w$ possesses a unique solution in D .

(3) $f(t) = 0$ for a given $t \in D$.

(4) $\arg f'(t) = \theta$, a prescribed real number.

The mapping function is known to have several extremal properties. Thus of all the functions in $H(D)$ which satisfy (1)–(3) it is the one for which $|f'(t)|$ is a maximum. It will now be seen that the mapping function is expressible in terms of the Bergman kernel function for $L_2H(D)$. This gives another extremal property for the mapping.

Let $f(z)$ be the mapping function of the bounded domain D and let D_r be that part of D which is mapped by $w = f(z)$ onto the open disk $|w| < r$, $0 < r < 1$. Let Γ_r be the boundary of D_r . It is a simple closed analytic curve since it is given by an analytic correspondence between the z - and the w -planes. It is clear that $f \in L_2H(D)$ since both f and D are bounded. Let g be any other element of $L_2H(D)$ and apply (6.5) with h replaced by f and f by g . Then

$$(6.6) \quad \iint_{D_r} g(z)\overline{f'(z)} \, dx \, dy = -\frac{1}{2} i \int_{\Gamma_r} g(z)\overline{f(z)} \, dz.$$

This simplifies since $f(z)\overline{f(z)} = r^2$ for z on Γ_r . Hence the right member becomes

$$(6.7) \quad -\frac{1}{2}ir^2 \int_r \frac{g(z)}{f(z)} dz.$$

This contour integral can be worked out by the calculus of residues. The integrand has a single pole (at $z = t$ which is a simple zero of $f(z)$). The value of the right-hand side of (6.6) is then $\pi r^2 g(t)/f'(t)$. As $r \rightarrow 1$ the left side of (6.6) approaches a limit and so does the right. Thus

$$(6.8) \quad (g, f') = \pi g(t)/f'(t)$$

or

$$(6.9) \quad g(t) = (g(u), (1/\pi)\overline{f'(t)}f'(u)).$$

This shows that the function $(1/\pi)\overline{f'(t)}f'(s)$ has the reproducing property with respect to arbitrary elements of $L_2H(D)$. But the reproducing kernel, if it exists and it does for $L_2H(D)$, is unique. Hence

$$(6.10) \quad K(z, t) = (1/\pi)f'(t)\overline{f'(z)}.$$

This gives

$$(6.11) \quad |f'(t)| = (\pi K(t, t))^{1/2}$$

for the positive square root. Since $\arg f'(t) = \theta$ we get

$$(6.12) \quad f'(t) = e^{i\theta}[\pi K(t, t)]^{1/2},$$

and

$$(6.13) \quad f'(z) = e^{i\theta}\pi^{1/2}[K(t, t)]^{-1/2}K(z, t).$$

Integration gives

$$f(z) = e^{i\theta}\pi^{1/2}[K(t, t)]^{-1/2} \int_t^z K(u, t) du$$

as the expression for the mapping function in terms of the kernel for the class $L_2H(D)$.

Some geometrical comments are in order. Suppose that $g(z)$ is holomorphic and univalent in D and that $g' \in L_2H(D)$. Then

$$(6.14) \quad \iint_D |g'(z)|^2 dx dy = \text{Area of } g(D).$$

If g is not univalent in D , then the map of D is a portion of a Riemann surface and the integral represents the area of this portion of the

surface. Take, in particular, $g(z) = f(z)$, the mapping function of D onto the unit disk. Here by (6.11) we get

$$[K(t, t)]^{-1} \iint_D |K(z, t)|^2 dx dy = \pi,$$

the area of the unit disk, so that

$$(6.15) \quad \iint_D |K(z, t)|^2 dx dy = K(t, t).$$

The next observation deals with arc length under conformal mapping. Suppose that C is a smooth arc in D , say with piecewise continuously turning curve tangent. The Euclidean length of C is simply

$$(6.16) \quad \int_C |dz| = l(C).$$

The Euclidean length of its image $f(C)$ under the conformal mapping of D into the unit disk is

$$(6.17) \quad \int_C |f'(z)| |dz| = \pi^{1/2} \int_C [K(z, z)]^{1/2} |dz|.$$

Thus the Euclidean lengths of C and of its image $f(C)$ are normally distinct. This raises the question of the existence of a linear measure invariant under conformal mapping. The kernel functions provide such a possibility. In order to explore this let us consider the relationship between two classes $L_2H(D_0)$ and $L_2H(D)$ if there exists a holomorphic function $w = f(z)$ which maps D conformally and 1-1 into D_0 . If both domains are simply-connected such a function exists, if not the mapping problem imposes restrictions on D_0 (or on D , whichever is the primary object). Suppose now that $g(w) \in L_2H(D)$. Then

$$(6.18) \quad g(w) \rightarrow g[f(z)] f'(z)$$

are corresponding functions. Inner products are preserved under the mapping for

$$\iint_{D_0} g_1(w) \overline{g_2(w)} du dv = \iint_D g_1[f(z)] \overline{g_2[f(z)]} |f'(z)|^2 dx dy.$$

Note that $|f'(z)|^2$ is the Jacobian of the transformation. In order to see how the kernel function transforms, suppose that the functions $\{b_n(w)\}$ constitute an orthonormal basis for $L_2H(D_0)$. Then the functions $\{h_n(z) = b_n[f(z)] f'(z)\}$ form an orthonormal basis for $L_2H(D)$. We have then

$$\begin{aligned}
 K_D(z, t) &= \sum_{n=1}^{\infty} \overline{h_n(z)} h_n(t) \\
 &= \sum_{n=1}^{\infty} \overline{b_n[f(z)]} b_n[f(t)] \overline{f'(z)} f'(t) \\
 &= \sum_{n=1}^{\infty} \overline{b_n(w)} b_n(q) \overline{f'(z)} f'(t), \quad q = f(t).
 \end{aligned}$$

This shows that

$$(6.19) \quad K_D(z, t) = K_{D_0}(w, q) \overline{f'(z)} f'(t).$$

In particular,

$$(6.20) \quad K_D(z, z) = K_{D_0}(w, w) |dw/dz|^2$$

or

$$(6.21) \quad [K_D(z, z)]^{1/2} |dz| = [K_{D_0}(w, w)]^{1/2} |dw|.$$

This is the desired relation.

For if C is an arc in D and C_0 its image in D_0 then

$$(6.22) \quad \int_C [K(z, z)]^{1/2} |dz| = \int_{C_0} [K_{D_0}(w, w)]^{1/2} |dw|.$$

DEFINITION 6.2. The conformally invariant length for mappings from D to D_0 is

$$(6.23) \quad \int_C [K(z, z)]^{1/2} |dz|.$$

This definition of length introduces a metric in D which is conformally invariant. The study of the Gaussian curvature of this "space" leads to simple and surprising results which is our next order of business.

This requires some brief statements about concepts of differential geometry. Take a surface S in R^3 which is given by two parameters u, v in the form

$$(6.24) \quad x_i = f_i(u, v)$$

where the f_i 's have continuous partials at least of order two. This ensures the existence of a normal at every point P on S and the direction cosines of the normal will change continuously with P . Through the normal at P we draw a plane. The intersection of the plane with the surface is a curve having arclength and curvature at P . Let the

plane turn with the normal at P as an axis. Normally the curvature will change (continuously) with the normal section and there will be a maximum value of the curvature and a minimum value. The product of these two curvatures is the total or Gaussian curvature. Its value at P is an intrinsic property of the surface and does not depend upon the embedding of S into the space. Moreover, it is invariant under transformations which preserve arclengths. The total curvature is expressible in terms of the coefficients E, F, G of the first fundamental form and the expression simplifies very much if $F = 0$ and the parametric curves are orthogonal. We have then

$$(6.25) \quad C = - \left[\frac{\partial}{\partial u} \frac{1}{\sqrt{E}} \frac{\partial}{\partial u} \sqrt{G} + \frac{\partial}{\partial v} \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right] (EG)^{-1/2}$$

and this simplifies still further if $G = E$ when we get

$$(6.26) \quad C = - \left\{ \frac{\partial}{\partial u} \left[E^{-1/2} \frac{\partial}{\partial u} E^{1/2} \right] + \frac{\partial}{\partial v} \left[E^{-1/2} \frac{\partial}{\partial v} E^{1/2} \right] \right\} E^{-1}.$$

We now return to the space $L_2H(D)$ and, in particular, to the domain D where we have introduced a metric by setting

$$(6.27) \quad (ds)^2 = K(z, z)[(dx)^2 + (dy)^2].$$

We interpret D as a surface endowed with the metric defined by (6.27). Here we have

$$(6.28) \quad E = G = K(z, z), \quad F = 0.$$

The formulas reduce to

$$(6.29) \quad C(z) = -\frac{1}{2}K^{-1} \Delta \log K = -2K^{-1} \frac{\partial^2}{\partial \bar{z} \partial z} \log K.$$

This is formally negative, actually negative and constant,

$$(6.30) \quad C(z) = -4\pi,$$

provided D is simply-connected. To prove this we go back and note that the curvature is invariant under conformal mapping. Suppose we map D conformally on the unit disk, so that $z = t$ goes into $w = 0$. This we can do since D is simply-connected. Now for the unit disk we have

$$(6.31) \quad K_0(z, z) = \pi^{-1}(1 - z\bar{z})^{-2}$$

so that

$$\log K_0(z, z) = -\log \pi - 2 \log (1 - z\bar{z}).$$

We take the second derivative of this with respect to z and \bar{z} obtaining $2(1 - z\bar{z})^{-2}$ which multiplied by $-2K_0^{-1}$ gives -4π as asserted.

For the Szegő kernel we have already observed that in the case of a simply-connected domain the square of the kernel function yields the derivative of the mapping function on the interior of a circle. This means that for the invariant metric we have to use that Bergman kernel or the square of the Szegő kernel.

Let us return to the Bergman kernel for the unit disk (open) and the corresponding metric

$$(6.32) \quad ds = \pi^{-1/2}(1 - z\bar{z})^{-1} |dz|.$$

Except for the constant factor this agrees with the so-called hyperbolic metric introduced by H. Poincaré in his theory of automorphic functions from 1881 onwards. It is one of the simplest and most elegant realizations of a Lobačevskii noneuclidean geometry in which circular arcs, orthogonal to the unit circle and restricted to the interior, play the role of straight lines. The pencil of circular arcs with a common endpoint on the unit circle corresponds to a family of parallel straight lines. The geodesics in the Bergman metric are circular arcs in the hyperbolic geometry.

If one point is at the origin and the other z_2 somewhere inside the unit circle, then the conformally invariant distance of z_2 from 0 will be the length of the geodesic, in this case a straight line segment from 0 to z_0 . The result is

$$(6.33) \quad \frac{1}{2} \pi^{-1/2} \log \left(\frac{1 + |z_0|}{1 - |z_0|} \right).$$

Note that the distance becomes infinite as $|z_0| \rightarrow 1$. In order to get the length of the geodesic joining two points z_1, z_2 we take a conformal map of the unit disk into itself in such a manner that z_1 goes into 0. The mapping is a Möbius transformation

$$(6.34) \quad w = (z - z_1)/(1 - \bar{z}_1 z)$$

which takes $z = z_2$ into $w = w_2 = (z_2 - z_1)(1 - \bar{z}_1 z_2)^{-1}$. Since the geodesic distance is unchanged we get

$$(6.35) \quad \frac{1}{2} \pi^{-1/2} \log \frac{|1 - \bar{z}_1 z_2| + |z_2 - z_1|}{|1 - \bar{z}_1 z_2| - |z_2 - z_1|}.$$

Our last observation in this setting concerns the principle of hyperbolic measure. This asserts that if the unit disk is mapped conformally onto a proper subset S of itself, then the hyperbolic measure is not increased. This follows from properties of the Bergman kernel. On the one hand we have

$$(6.36) \quad [K_0(z, z)]^{1/2} |dz| = [K_S(w, w)]^{1/2} |dw|$$

by the invariance of the geodesic length under conformal mapping. On the other we have

$$[K_S(w, w)]^{1/2} |dw| \geq [K_0(w, w)]^{1/2} |dw|.$$

For if t is a point common to the unit disk and the subset S and we consider the minimum problem of Lemma 3.7 for U and S , then the minima are

$$[K_0(t, t)]^{-1/2} \text{ for } U \text{ and } [K_S(t, t)]^{-1/2} \text{ for } S.$$

The entries in the competition for U , when restricted to S , are also entries for S , but entries for S may not be continuable analytically to qualify them for U . From this we conclude that

$$(6.37) \quad K_0(t, t) \geq K_S(t, t).$$

Combining (6.35) and (6.36) we get

$$(6.38) \quad \{K_0(z, z)/K_0(w, w)\}^{1/2} \geq |dw/dz|.$$

This gives

$$(6.39) \quad |dw/dz| \leq (1 - |w|^2)/(1 - |z|^2)$$

which implies that the hyperbolic measure is not increased by the conformal mapping. If we have equality in (6.39) for some value of $z = z_0$ with $w = f(z_0)$ then we have equality for all z and $w = f(z)$ is a Möbius transformation (6.34).

It is seen that the principle of hyperbolic measure is merely a special case of a principle of Bergman measure. If a simply-connected domain D is mapped conformally and 1-1 on a subset S , then the geodesic measure is not increased, i.e.

$$(6.40) \quad \int_C [K_D(z, z)]^{1/2} |dz| \geq \int_{f(C)} [K_S(w, w)]^{1/2} |dw|.$$

7. Functions of two complex variables. The theory of reproducing kernels is valid also for functions of several complex variables. This is in the main the creation of Stefan Bergman. In the following we shall restrict ourselves to functions of two variables. The extension to n variables does not introduce any essentially new phenomena. The novelties come in the passage from one to two variables.

There are some fundamental differences between the theories of analytic functions of one and of two complex variables. Two of the most striking differences are the following.

I. *Domains of regularity.* In the theory of functions of one variable, given any domain D in the plane, there exists a function $f(z)$ which is holomorphic in D but in no larger domain so that all points on the boundary are singular points of f . Not so in C^2 . Here there exist domains (connected open sets in C^2 or equivalently in R^4) which cannot be domains of regularity in the following sense. Any function $f(z_1, z_2)$ which is holomorphic in D exists also in a larger domain.

II. *Mapping theorems.* In the one-dimensional case, the Riemann Mapping Theorem asserts that any two bounded, simply-connected domains can be mapped conformally one onto the other and the mapping function is unique once a pair of corresponding points has been selected and corresponding directions have been chosen at these points. In two dimensions one would expect to find a pair of functions

$$(7.1) \quad w_1 = f_1(z_1, z_2), \quad w_2 = f_2(z_1, z_2)$$

both holomorphic for (z_1, z_2) in D_1 and such that (w_1, w_2) lies in D_2 , the correspondence being 1-1 and onto. If such a mapping exists it is said to be *pseudo-conformal*. But already the basic regions of a dicylinder and a sphere

$$(7.2) \quad |z_1| < 1, \quad |z_2| < 1 \quad \text{and} \quad |z_1|^2 + |z_2|^2 < 1$$

admit of no such mapping.

A function $f(z) = f(z_1, z_2)$ is holomorphic at the point $z_1 = t_1$, $z_2 = t_2$, if it can be developed in a power series

$$(7.3) \quad f(z_1, z_2) = \sum_{m,n} a_{mn}(z_1 - t_1)^m (z_2 - t_2)^n$$

convergent in some dicylinder with center at (t_1, t_2) . Here again there is a difference between the one- and the two-dimensional cases. For a function of one variable, there is a definite circle of convergence with a radius determined by the nearest singular point. A function of two complex variables may have isolated singular points but normally there is a singular manifold. As a consequence, the dicylinder of convergence has associated radii

$$(7.4) \quad |z_1 - t_1| = r_1, \quad |z_2 - t_2| = r_2.$$

If we fix r_1 arbitrarily, r_2 is uniquely determined and vice versa.

Equation (7.3) is the Taylor series of f so that

$$(7.5) \quad a_{mn} = \frac{1}{m!n!} \frac{\partial^{m+n}}{\partial z_1^m \partial z_2^n} f(z_1, z_2) \Big|_{z_1=t_1, z_2=t_2} .$$

This raises the question of differentiability. It is clear that a locally holomorphic function must have partial derivatives of all orders. What sort of converse holds? Here the Fundamental Theorem of F. Hartogs (Math. Ann. 62 (1906)) is basic.

Let $f(z_1, z_2)$ be defined for $|z_1| < a, |z_2| < b$ as a single-valued function. For each fixed z_2 with $|z_2| < b$ let $f(z_1, z_2)$ be a holomorphic function of z_1 for $|z_1| < a$ and let $f(z_1, z_2)$ be holomorphic in z_2 for $|z_2| < b$ and each fixed z_1 with $|z_1| < a$. Then $f(z_1, z_2)$ is holomorphic at all points (z_1, z_2) in the given dicylinder.

This means that the existence of continuous first order partials is sufficient provided they satisfy the Cauchy-Riemann equations (note two pairs of equations corresponding to the four real variables $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$.)

The dicylinder and the sphere are special cases of the so-called Reinhardt circular regions. Such a region is characterized by the fact that it is invariant under all transformations

$$(7.6) \quad Z_1 = t_1 + (z_1 - t_1)e^{i\theta_1}, \quad Z_2 = t_2 + (z_2 - t_2)e^{i\theta_2}.$$

Here (t_1, t_2) is fixed, the center of the region.

For a bounded Reinhardt domain R with center at $(0, 0)$ a reproducing kernel may be constructed as follows. Consider the monomes

$$(7.7) \quad z_1^m z_2^n$$

where m and n are arbitrary integers, positive, negative or zero. Set

$$(7.8) \quad a_{mn}^2 = \int_R |z_1|^{2m} |z_2|^{2n} d\omega.$$

If $(0, 0)$ is in R , then the negative values of m and n are severely restricted by convergence considerations. The functions

$$(7.9) \quad (a_{mn})^{-1} z_1^m z_2^n \equiv f_{mn}(z_1, z_2)$$

form a complete orthonormal system for R .

The elements of $L_2H(R)$ are now the series of the form

$$(7.10) \quad f(z_1, z_2) = \sum_{m,n} c_{mn} z_1^m z_2^n$$

with

$$(7.11) \quad \sum_{m,n} |a_{mn} c_{mn}|^2 < \infty.$$

The reproducing kernel is

$$(7.12) \quad K(z_1, z_2; t_1, t_2) = \sum_{m,n} \overline{f_{mn}(z_1, z_2)} f_{mn}(t_1, t_2).$$

It is a simple matter to verify that this is indeed a reproducing kernel.

For the dicylinder $|z_1| < 1$, $|z_2| < 1$, we have

$$(7.13) \quad a_{mn}^2 = \pi^2[(m+1)(n+1)]^{-1}, \quad m \geq 0, n \geq 0,$$

and the reproducing kernel is simply

$$(7.14) \quad \pi^{-2}(1 - \bar{z}_1 t_1)^{-2}(1 - \bar{z}_2 t_2)^{-2}.$$

For a sphere of radius R we obtain instead

$$(7.15) \quad 2\pi^{-2}R^2(R^2 - \bar{z}_1 t_1 - \bar{z}_2 t_2)^{-3}.$$

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