ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 2, Number 2, Spring 1972

SOME GENERALIZATIONS OF MEHLER'S FORMULA[1]

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ABSTRACT. A number of earlier results of the authors, involving the classical Hermite polynomials, are applied to prove two generalizations of some interesting extensions of the well-known Mehler formula, given recently by Carlitz.

1. **Introduction.** Let $H_n(z)$ denote the Hermite polynomial defined by

$$\sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!} = \exp(2zt - t^2). \tag{1.1}$$

In an attempt to unify several extensions of the well-known Mehler formula [4, p. 198]

$$\begin{aligned} &\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{t^n}{n!} \\ &\quad = (1 - 4t^2)^{-1/2} \exp\left\{\frac{4xyt - 4(x^2 + y^2)t^2}{1 - 4t^2}\right\}, \end{aligned} \tag{1.2}$$

given recently by Carlitz [2], we proved the following general formulas [5]:

$$\begin{aligned} &\sum_{m,n,p=0}^{\infty} H_{n+p+r}(x)H_{p+m+s}(y)H_{m+n}(z) \frac{u^m}{m!}\frac{v^n}{n!}\frac{w^p}{p!} \\ &= S_1 \sum_{k=0}^{\min(r,s)} 2^{2k}k! \binom{r}{k}\binom{s}{k}\left(\frac{w - 2uv}{\sqrt{\{(1 - 4u^2)(1 - 4v^2)\}}}\right)^k \\ &\quad \cdot H_{r-k}\left(\frac{(x - 2vz)(1 - 4u^2) - 2(y - 2uz)(w - 2uv)}{\sqrt{\{\Delta(1 - 4u^2)\}}}\right) \\ &\quad \cdot H_{s-k}\left(\frac{(y - 2uz)(1 - 4v^2) - 2(x - 2vz)(w - 2uv)}{\sqrt{\{\Delta(1 - 4v^2)\}}}\right), \end{aligned} \tag{1.3}$$

Received by the editors December 16, 1970.

AMS 1970 *subject classifications.* Primary 33A65; Secondary 42A52.

[1]This work was supported in part by the National Research Council of Canada under Grant A7353. See also Abstract 71T-B128, Notices Amer. Math. Soc. 18 (1971), 645.

where, for convenience,

$$\Delta = 1 - 4u^2 - 4v^2 - 4w^2 + 16uvw, \tag{1.4}$$

$$\begin{aligned} S_1 = {} & \Delta^{-(r+s+1)/2}(1 - 4u^2)^{r/2}(1 - 4v^2)^{s/2} \\ & \cdot \exp\left\{\sum x^2 - \frac{1}{\Delta}\left(\sum x^2 - 4\sum u^2x^2 - 4\sum wxy + 8\sum uvxy\right)\right\}, \end{aligned} \tag{1.5}$$

and $\sum x^2$, $\sum u^2x^2$, $\sum wxy$, $\sum uvxy$ are symmetric functions in the indicated variables.

$$\begin{aligned} & \sum_{m,n_1,\cdots,n_k=0}^{\infty} H_{m+n_1+\cdots+n_k+r}(x)H_{m+s}(y)H_{n_1}(z_1)\cdots H_{n_k}(z_k)\frac{u^m}{m!}\frac{v_1^{n_1}}{n_1!}\cdots\frac{v_k^{n_k}}{n_k!} \\ & = S_2 \sum_{k=0}^{\min(r,s)} 2^{2k}k! \binom{r}{k}\binom{s}{k}\left(\frac{u}{\sqrt{(1-4\sum v_i^2)}}\right)^k \\ & \quad \cdot H_{r-k}\left(\frac{x - 2uy - 2\sum v_iz_i}{\sqrt{(1 - 4u^2 - 4\sum v_i^2)}}\right) \\ & \quad \cdot H_{s-k}\left(\frac{y(1 - 4\sum v_i^2) - 2u(x - 2\sum v_iz_i)}{\sqrt{\{(1 - 4u^2 - 4\sum v_i^2)(1 - 4\sum v_i^2)\}}}\right), \end{aligned} \tag{1.6}$$

where

$$\begin{aligned} S_2 = {} & (1 - 4u^2 - 4\sum v_i^2)^{-(r+s+1)/2}(1 - 4\sum v_i^2)^{s/2} \\ & \cdot \exp\left\{x^2 - \frac{(x - 2uy - 2\sum v_iz_i)^2}{1 - 4u^2 - 4\sum v_i^2}\right\}, \end{aligned} \tag{1.7}$$

and the range of each i summation is from $i = 1$ to $i = k$, $k = 1, 2, 3, \cdots$.

The object of the present note is to show how our results (1.3) and (1.6) may be applied to prove a number of extensions of the following elegant formula of Carlitz [**1**, p. 43].

$$\begin{aligned} & \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{\min(r,s)} 2^{2k}k!\binom{r}{k}\binom{s}{k}\left(\frac{-t}{1-4t^2}\right)^k H_{n+r-k}(x)H_{n+s-k}(y) \\ & = (1 - 4t^2)^{-(r+s+1)/2}\exp\left\{\frac{4xyt - 4(x^2 + y^2)t^2}{1 - 4t^2}\right\} \\ & \quad \cdot H_r\left(\frac{x - 2yt}{\sqrt{(1 - 4t^2)}}\right)H_s\left(\frac{y - 2xt}{\sqrt{(1 - 4t^2)}}\right), \end{aligned} \tag{1.8}$$

which, when $r = s$, would yield an earlier result of Chatterjea [3].

2. **The general formulas.** We first prove the formula

$$\sum_{m,n,p=0}^{\infty} H_{m+n}(z) \frac{u^m}{m!} \frac{v^n}{n!} \frac{w^p}{p!} \sum_{k=0}^{\min(r,s)} 2^{2k}k! \binom{r}{k} \binom{s}{k} \left(\frac{2uv-w}{\Delta}\right)^k$$

$$\cdot H_{n+p+r-k}(x) H_{p+m+s-k}(y)$$

$$(2.1) \qquad = S_1 H_r\left(\frac{(x-2vz)(1-4u^2)-2(y-2uz)(w-2uv)}{\sqrt{\{\Delta(1-4u^2)\}}}\right)$$

$$\cdot H_s\left(\frac{(y-2uz)(1-4v^2)-2(x-2vz)(w-2uv)}{\sqrt{\{\Delta(1-4v^2)\}}}\right),$$

where Δ and S_1 are given by (1.4) and (1.5) respectively.

Denoting the left member of (2.1) by Ω, if we make use of the formula (1.3), we get

$$\Omega = \sum_{k=0}^{\min(r,s)} 2^{2k}k! \binom{r}{k} \binom{s}{k} \left(\frac{2uv-w}{\Delta}\right)^k$$

$$\cdot \sum_{m,n,p=0}^{\infty} H_{n+p+r-k}(x) H_{p+m+s-k}(y) H_{m+n}(z) \frac{u^m}{m!} \frac{v^n}{n!} \frac{w^p}{p!}$$

$$= S_1 \sum_{k=0}^{\min(r,s)} 2^{2k}k! \binom{r}{k} \binom{s}{k} \left(\frac{2uv-w}{\sqrt{\{(1-4u^2)(1-4v^2)\}}}\right)^k$$

$$\cdot \sum_{j=0}^{\min(r-k,s-k)} 2^{2j}j! \binom{r-k}{j}\binom{s-k}{j}\left(\frac{w-2uv}{\sqrt{\{(1-4u^2)(1-4v^2)\}}}\right)^j$$

$$\cdot H_{r-k-j}\left(\frac{(x-2vz)(1-4u^2)-2(y-2uz)(w-2uv)}{\sqrt{\{\Delta(1-4u^2)\}}}\right)$$

$$\cdot H_{s-k-j}\left(\frac{(y-2uz)(1-4v^2)-2(x-2vz)(w-2uv)}{\sqrt{\{\Delta(1-4v^2)\}}}\right)$$

$$= S_1 \sum_{i=0}^{\min(r,s)} 2^{2i}i! \binom{r}{i} \binom{s}{i} \left(\frac{2uv-w}{\sqrt{\{(1-4u^2)(1-4v^2)\}}}\right)^i$$

$$\cdot H_{r-i}\left(\frac{(x-2vz)(1-4u^2)-2(y-2uz)(w-2uv)}{\sqrt{\{\Delta(1-4u^2)\}}}\right)$$

$$\cdot H_{s-i}\left(\frac{(y-2uz)(1-4v^2)-2(x-2vz)(w-2uv)}{\sqrt{\{\Delta(1-4v^2)\}}}\right)$$

$$\cdot \sum_{j=0}^{i} (-1)^j \binom{i}{j},$$

whence (2.1) follows immediately.

Next we give the formula

$$\sum_{m,n_1,\cdots,n_k=0}^{\infty} H_{n_1}(z_1)\cdots H_{n_k}(z_k)\,\frac{u^m}{m!}\,\frac{v_1^{\,n_1}}{n_1!}\cdots\frac{v_k^{\,n_k}}{n_k!}$$

$$\cdot\sum_{j=0}^{\min(r,s)} 2^{2j}j!\binom{r}{j}\binom{s}{j}\left(\frac{-u}{1-4u^2-4\sum v_i^2}\right)^j$$

$$\cdot H_{m+n_1+\cdots+n_k+r-j}(x)H_{m+s-j}(y) \tag{2.2}$$

$$= S_2 H_r\left(\frac{x-2uy-2\sum v_i z_i}{\sqrt{(1-4u^2-4\sum v_i^2)}}\right)$$

$$\cdot H_s\left(\frac{y(1-4\sum v_i^2)-2u(x-2\sum v_i z_i)}{\sqrt{\{(1-4u^2-4\sum v_i^2)(1-4\sum v_i^2)\}}}\right),$$

where S_2 is given by (1.7) and, as before, each i summation runs from $i=1$ to $i=k$, $k=1,2,3,\cdots$.

The derivation of (2.2) would make use of our formula (1.6) in a manner already illustrated in the proof of (2.1). The details are, therefore, omitted.

3. **Particular cases.** Some particular cases of (2.1) and (2.2) are worthy of note.

If in (2.1) we set u or $v=0$ and make a slight change of variables, we get

$$\sum_{m,n=0}^{\infty} H_m(z)\frac{u^m}{m!}\frac{v^n}{n!}\sum_{k=0}^{\min(r,s)} 2^{2k}k!\binom{r}{k}\binom{s}{k}\left(\frac{-v}{1-4u^2-4v^2}\right)^k$$

$$\cdot H_{n+r-k}(x)H_{m+n+s-k}(y)$$

$$= (1-4u^2-4v^2)^{-(r+s+1)/2}(1-4u^2)^{r/2}$$

$$\cdot\exp\left\{\frac{-4y^2(u^2+v^2)+4y(vx+uz)-4(vx+uz)^2}{1-4u^2-4v^2}\right\} \tag{3.1}$$

$$\cdot H_r\left(\frac{x(1-4u^2)-2v(y-2uz)}{\sqrt{\{(1-4u^2)(1-4u^2-4v^2)\}}}\right)H_s\left(\frac{y-2uz-2vx}{\sqrt{(1-4u^2-4v^2)}}\right)$$

which, in turn, would reduce to Carlitz's formula (1.8) when $u=0$.

Another interesting special case of (2.1) would occur when $w=0$. Indeed we get

$$\sum_{m,n=0}^{\infty} H_{m+n}(z) \frac{u^m}{m!} \frac{v^n}{n!} \sum_{k=0}^{\min(r,s)} 2^{2k}k! \binom{r}{k} \binom{s}{k} \left(\frac{2uv}{1-4u^2-4v^2}\right)^k \cdot H_{n+r-k}(x)H_{m+s-k}(y)$$

$$(3.2) \quad = (1-4u^2-4v^2)^{-(r+s+1)/2}(1-4u^2)^{r/2}(1-4v^2)^{s/2} \cdot \exp\left\{\frac{-4z^2(u^2+v^2)+4z(uy+vx)-4(uy+vx)^2}{1-4u^2-4v^2}\right\} \cdot H_r\left(\frac{(x-2vz)(1-4u^2)+4uv(y-2uz)}{\sqrt{\{(1-4u^2)(1-4u^2-4v^2)\}}}\right) \cdot H_s\left(\frac{(y-2uz)(1-4v^2)+4uv(x-2vz)}{\sqrt{\{(1-4v^2)(1-4u^2-4v^2)\}}}\right).$$

Formula (3.2) provides an extension of Carlitz's formula (1.2), p. 117 in [2] to which it would reduce when $r = s = 0$.

On the other hand, the most interesting special cases of our formula (2.2) seem to occur when $v_1 = \cdots = v_k = 0$ or when $k = 1$. In the former case we are led at once to Carlitz's formula (1.8), while the latter yields our formula (3.1) which, as we noted above, provides a generalization of Carlitz's result (1.8).

Finally, we remark that in the special case when $w = 2uv$, formulas (1.3) and (2.1) can be shown fairly easily to reduce to the elegant result

$$(3.3) \qquad H_m(z)H_n(z) = \sum_{r=0}^{\min(m,n)} 2^r r! \binom{m}{r} \binom{n}{r} H_{m+n-2r}(z),$$

which is attributed to Nielsen.

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