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ALGEBRAS OF INTEGRABLE FUNCTIONS. II

KENNETH O. LELAND[1]

1. **Introduction.** Morera's theorem in complex function theory raises the possibility that this theory can be based on integration rather than differentiation. Heffter [1], Macintyre and Wilbur [9] and this author [7] have given such a development. In this paper a theory of integrable functions will be developed in a more general context of operator valued functions wherein the functions no longer need be analytic.

Let K denote the complex plane. For $a \in K$, set $A_a(z) = az$ for all $z \in K$. Then A_a is a bounded linear transformation of K, thought of as a real Euclidean space E_2 into itself. Set $T_2{}' = \{A_a;\, a \in K\}$. Let f be a continuous function on an open set S in E_2 into the space B_2 of bounded linear transformations of E_2 into itself, and let P be a path (rectifiable arc) in S with endpoints α and β. Then for any subdivision $\alpha = z_1 < \cdots < z_{n+1} = \beta$ of P, a Riemann sum, the vector $R = \sum_{i=1}^{n} f_{z_i}(z_{i+1} - z_i)$ can be formed. If range f lies in $T_2{}'$, then, for $z \in S$, $f_z = f(z) = A_{\phi(z)}$ for some $\phi(z) \in K$, and we may write $R = \sum_{i=1}^{n} \phi(z_i)(z_{i+1} - z_i)$. Taking the limit as the norm of the subdivision defining R approaches zero, we obtain the vector $\theta = \int_{\alpha_P}^{\beta} f(z)\, dz = \int_{\alpha_P}^{\beta} f_z(dz)$. If range $f \subseteq T_2{}'$, we can interpret θ as the complex number $\int_{\alpha_P}^{\beta} \phi(z)\, dz$.

f is said to be integrable if for all closed paths (rectifiable simple closed curves) $C \subseteq S$, we have $\int_C f(z) dz = \int_C f_z(dz) = 0$. If range $f \subseteq T_2{}'$, then $\int_C \phi(z)\, dz = 0$ for all closed paths $C \subseteq S$, and by Morera's theorem ϕ is analytic; consequently, f is itself Fréchet differentiable, where $f_z{}'$ is a linear transformation of E_2 into B_2 for $z \in S$.

The general case studied in this paper is obtained by replacing E_2 by an arbitrary real Euclidean space E of dimension p, $p > 1$. Let T be a *commutative* subalgebra of the Banach algebra B of bounded linear transformations of E into E and let F be the family of continuous integrable functions on open subsets of E into T.

Let E' be a finite-dimensional commutative Banach algebra with identity over the reals and for $a \in E'$, set $A_a(t) = at$ for $t \in E'$. Set $T' = \{A_a;\, a \in E'\}$.

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Let $f \in F$, $S =$ domain f simply connected. Let $z_0 \in S$ and for $z \in S$, set $g(z) = \int_{z_0}^{z} f(z)\, dz$. Then g maps S into E, and for $z \in S$, the Fréchet derivative $g_z{}'$ of g at z is the operator $f(z) = f_z$ of T.

It shall be shown in §2, that the family of continuous integrable functions on a simply connected open set S in E into T forms an algebra. The method of proof is reminiscent of the proof of the Cauchy-Goursat theorem. For the case that $E = E_2 \cong K$, one may refer to [7].

In §3, the relationship between integrability and differentiability is discussed. Examples of integrable but nondifferentiable functions are given.

In §4, T is required to be symmetric, that is, for $x \in T$, the adjoint x^* of x lies in T. In this case questions concerning the analyticity or nonanalyticity of elements of F can be readily answered since in this case an element of F can be expressed locally as a direct sum of functions which can be interpreted as analytic functions from K into K or as continuous functions from R into R. This analysis is based on the fact that E can be expressed as the direct sum of irreducible subspaces invariant under T.

In §5, it is shown that provided that T is presumed semisimple, a necessary and sufficient condition for analyticity of the elements of F is that no element of T have rank one (i.e. range $A = A(E)$ is not one dimensional for all $A \in T$). The possibility that the presumption of semisimplicity may be dispensed with is examined.

Employed in §5 is a definition of analyticity in a real variable context suitable for the purposes of this paper due to the author [2], [5], [6].

In [8] it shall be shown that integrable functions, although not necessarily analytic, satisfy some form of the maximum modulus theorem.

The development of this paper is in no way affected if the only paths of integration permitted are those formed from straight line segments.

Let ω denote the positive integers. If Z is a Banach space, $\delta > 0$, $x \in Z$, set $U_x(\delta) = \{t \in Z;\ \|t - x\| < \delta\}$, $U(\delta) = U_0(\delta)$ and $U = U_0(1)$, and set $V_x(\delta) = \{t \in Z;\ \|t - x\| = \delta\}$, $V(\delta) = V_0(\delta)$, and $V = V_0(1)$. If f is a function with domain S and $H \subseteq E$, then the restriction $f \mid H$ of f to H is the function g with domain $H \cap S$ such that $g(x) = f(x)$ for all $x \in H \cap S$. For H a subspace of E and G a family of functions defined on subsets of E, $G \mid H$ denotes the family $\{g \mid H;\ g \in G\}$.

A subspace H of E is said to be invariant if $A(x) \in H$ for all $A \in T$, $x \in H$.

2. **The algebras.** The principal result of this section, Theorem 2.1, is the proof that the product of two elements of F with common simply

connected domains also lies in F. Our basic tool is the "smoothing" lemma, Lemma 2.1, which allows us to approximate elements of F by "smoothed" elements of F satisfying a Lipschitz type condition, enabling us to use arguments reminiscent of the proof of the Cauchy-Goursat theorem to achieve our conclusion.

Trivially F is closed under the operations of addition (defined on the intersection of the domains of the functions being added), multiplication by scalars, and translations of the form $x \to x - x_0$, $x, x_0 \in E$. It may be readily shown that if $f_1, f_2, \cdots$ is a sequence of elements of F with common domain S which converges uniformly on S to a limit function f, then f also lies in F.

LEMMA 2.1. *Let $f \in F$ and let H be a compact subset of domain f with interior S. Then there exists a sequence $g_1, g_2, \cdots$ of elements of F with common domain S such that*:

(1) *The sequence $g_1, g_2, \cdots$ converges uniformly on S to $f \mid S$.*

(2) *For $i \in \omega$, there exists $N_i > 0$ such that for $x, y \in S$,*

$$\|g_i(y) - g_i(x)\| \leqq N_i \|y - x\|.$$

Our proof is based upon the discussion of "smoothing" operators as introduced by the author in [5].

PROOF. Let $e_1, \cdots, e_p$ be an orthonormal basis of E and let Q be the cube $\{-\frac{1}{2} < (x, e_i) \leqq \frac{1}{2};\ i = 1, \cdots, p\}$. For $x \in E$, $s > 0$ such that $x + s\bar{Q} = \{x + sy;\ y \in \bar{Q}\} \subseteq$ domain f, set

$$f_s(x) = s^{-p} \int_{x+sQ} f(t)\, dm(t),$$

where m is Lebesgue measure on E. Let $\alpha > 0$,

$$\alpha < \inf\{\|x - y\|;\ x \in H, y \in E - \text{domain } f\}$$

and set $H_0 = H + a\bar{Q}$, where $a = \alpha/\sqrt{p}$. Then H_0 is compact and $H \subseteq H_0 \subseteq$ domain f. Let $0 < s \leqq a$. Then $H \subseteq$ domain f_s.

We now show that $f_s \mid S$ is the uniform limit of a sequence of elements of F with common domain S and hence $f_s \mid S$ lies in F. Let $\epsilon > 0$. Then there exits $\delta > 0$, such that $x, y \in H_0$, $\|x - y\| < \delta$ implies $\|f(y) - f(x)\| \leqq \epsilon$. Let $0 < r < \delta/\sqrt{p}$ and $t_1, \cdots, t_n \in sQ$, $n \in \omega$, such that $\{t_i + rQ;\ i = 1, \cdots, n\}$ is a partition of sQ. Then for $x \in H \subseteq H_0$, $i = 1, \cdots, n$, $t \in x + t_i + rQ \subseteq x + sQ \subseteq H + sQ \subseteq H_0$, we have $\|t - (x + t_i)\| \leqq r\sqrt{p} < \delta$ and hence $\|f(t) - f(x + t_i)\| \leqq \epsilon$. Thus for $x \in H$,

$$\left\| \left[\sum_1^n (r/s)^p f(x+t_i)\right] - f_s(x) \right\|$$

$$= s^{-p} \left\| \left[\sum_1^n r^p f(x+t_i)\right] - \left[\sum_1^n \int_{x+t_i+rQ} f(t)\,dm(t)\right] \right\|$$

$$= s^{-p} \left\| \sum_1^n \int_{x+t_i+rQ} [f(x+t_i) - f(t)]\,dm(t) \right\|$$

$$\leqq s^{-p} \sum_1^n \int_{x+t_i+rQ} \|f(x+t_i) - f(t)\|\,dm(t)$$

$$\leqq s^{-p} \sum_1^n r^p \epsilon = s^{-p}(nr^p)\epsilon = s^{-p}(s^p)\epsilon = \epsilon,$$

where nr^p is clearly the volume of sQ. Thus $f_s \mid H$ is the uniform limit on H of elements of F of the form $g(x) \equiv \sum_1^n (r/s)^p f(t_i + x)$, and thus $f_s \mid S$ lies in F.

Now for $x \in H$, $s < \min\{a, \delta/\sqrt{p}\}$, $s > 0$,

$$\|f(x) - f_s(x)\| = s^{-p} \left\| \int_{x+sQ} [f(x) - f(t)]\,dm(t) \right\| \leqq s^{-p}(\epsilon s^p) = \epsilon.$$

Let $0 < s \leqq a$. We now verify (2) for $f_s \mid S$. Now there exists $N_s > 0$, such that $m[(x+sQ) \dotplus (y+sQ)] \leqq N_s\|y - x\|$, for $x, y \in E$, where $A \dotplus B = (A - B) \cup (B - A)$ for arbitrary sets A, B. Thus for $x, y \in H$,

$$\begin{aligned} \|f_s(y) - f_s(x)\| &= s^{-p} \left\| \int_{y+sQ} f(t)\,dm(t) - \int_{x+sQ} f(t)\,dm(t) \right\| \\ &\leqq s^{-p} \int_{(x+sQ)\dotplus(y+sQ)} \|f(t)\|\,dm(t) \\ &\leqq s^{-p} m[(x+sQ) \dotplus (y+sQ)] M \\ &\leqq s^{-p}[N_s\|y - x\|] M \leqq N_s'\|y - x\|, \end{aligned}$$

where $M = \sup\{\|f(t)\|;\, t \in H_0\}$, and $N_s' = N_s M s^{-p}$.

For $n \in \omega$, $x \in S$, set $g_n(x) = f_{a/n}(x)$. Then $g_1, g_2, \cdots$ is the desired sequence of functions.

THEOREM 2.1. *Let S be a simply connected open set in E and f, g elements of F with domain S. Then $fg \in F$.*

PROOF. It suffices to show that $\int_\Delta f(z)g(z)\,dz = 0$ for a triangle Δ such that Δ and its interior J lie in S. Now there exists an open set

$D \subseteq S$, with $\bar{D}$ compact, such that $\bar{J} \subseteq D$ and $\bar{D} \subseteq S$. Let $f_1, f_2, \cdots$ and $g_1, g_2, \cdots$ be the sequences of functions given by Lemma 2.1 for f and g and $H = \bar{D}$. If we show for all $n \in \omega$, that $\int_\Delta f_n(z)g_n(z)\,dz = 0$, then

$$\int_\Delta f(z)g(z)\,dz = \lim_{n\to\infty} \int_\Delta f_n(z)g_n(z)\,dz = \lim_{n\to\infty} 0 = 0.$$

Fix $n \in \omega$. Then from Lemma 2.1, there exist $N_n, M_n > 0$, such that for $x, y \in$ interior $\bar{D} \supseteq D \supseteq \bar{J}$,

$$\|f_n(y) - f_n(x)\| \leqq N_n\|y - x\| \quad \text{and} \quad \|g_n(y) - g_n(x)\| \leqq M_n\|y - x\|.$$

Let $k \in \omega$, and divide Δ into 4^k congruent triangles of perimeter $s/2^k$, $\Delta_1, \cdots, \Delta_{4^k}$, where s is the perimeter of Δ, and let $z_i \in \Delta_i$, for $i = 1, \cdots, 4^k$. Now for $i = 1, \cdots, 4^k$, $f_n(z_i)g_n$, $f_ng_n(z_i) = g_n(z_i)f_n$, and $f_n(z_i)g_n(z_i)$ are integrable and hence

$$\int_{\Delta_i} f_n(z)g_n(z)\,dz = \int_{\Delta_i} [f_n(z) - f_n(z_i)]\,[g_n(z) - g_n(z_i)]\,dz,$$

and thus

$$\begin{aligned}\left\|\int_{\Delta_i} f_n(z)g_n(z)\,dz\right\| &\leqq \int_{\Delta_i} \|f_n(z) - f_n(z_i)\| \cdot \|g_n(z) - g_n(z_i)\|\,ds \\ &\leqq \int_{\Delta_i} N_n\|z - z_i\| \cdot M_n\|z - z_i\|\,ds \leqq \int_{\Delta_i} N_n(s/2^k) \cdot M_n(s/2^k)\,ds \\ &\leqq [N_nM_ns^2/4^k]\,(s/2^k) = N_nM_ns^3/8^k.\end{aligned}$$

Thus

$$\begin{aligned}\left\|\int_\Delta f_n(z)g_n(z)\,dz\right\| &= \left\|\sum_{i=1}^{4^k} \int_{\Delta_i} f_n(z)g_n(z)\,dz\right\| \\ &\leqq \sum_{i=1}^{4^k} \left\|\int_{\Delta_i} f_n(z)g_n(z)\,dz\right\| \\ &\leqq 4^k[N_nM_ns^3/8^k] = N_nM_ns^3/2^k.\end{aligned}$$

Letting $k \to \infty$, we obtain $\int_\Delta f_n(z)g_n(z)\,dz = 0$.

REMARK. In his argument for the complex function case Heffter [1] requires that one of the functions f, g be differentiable. Macintyre and Wilbur [9] weaken this, requiring only that one of the functions satisfy a Lipschitz condition. The need for such additional assumptions has been obviated in our argument by the use of the "smoothing" lemma, Lemma 2.1. Commutativity comes into play only once, appearing in the argument that functions of the form $f(z)c$, where f is integrable and $c \in T$, are also of the form $cf(z)$ and hence integrable.

COROLLARY 2.1. *Let f be an element of F with simply connected domain S such that $g(z) = f(z)^{-1}$ exists for all $z \in S$. Then $g \in S$.*

PROOF. Let $\Delta, J, D, f_1, f_2, \cdots$ be as given in the proof of Theorem 2.1. Now $f(\bar{J})$ is a compact subset of the open set Z of invertible elements of T. Hence there exists $\delta > 0$, such that $x \in f(\bar{J}), z \in T$, $\|z - x\| < \delta$ implies $z \in Z$. Since the sequence $f_1, f_2, \cdots$ converges uniformly on $\bar{J}$, there must exist $q \in \omega$, such that $f_n(x) \in Z$ for $x \in \bar{J}$, $n \in \omega_q = \{n \in \omega; n \geqq q\}$. For $n \in \omega_q$, $x \in \bar{J}$, set $g_n(x) = f_n(x)^{-1}$. Now for $n \in \omega_q$, there exists $N_n > 0$ such that for $x, y \in \bar{J}$, $\|f_n(y) - f_n(x)\| \leqq N_n\|y - x\|$, and hence

$$\begin{aligned}\|g_n(y) - g_n(x)\| &= \|g_n(y)g_n(x)[f_n(x) - f_n(y)]\| \\ &\leqq (N_n')^2[N_n\|y - x\|] = N_n''\|y - x\|,\end{aligned}$$

where $N_n' = \sup\{\|g(t)\|; t \in \bar{J}\}$ and $N_n'' = (N_n')^2 N_n$.

Let $k \in \omega$ and divide Δ into 4^k congruent triangles of perimeter $s/2^k$, $\Delta_1, \cdots, \Delta_{4^k}$, where s is the perimeter of Δ, and let $z_i \in \Delta_i$ for $i = 1, \cdots, 4^k$. Now for $i = 1, \cdots, 4^k$, $g_n(z_i)f_n$ is integrable and hence

$$\begin{aligned}-g_n(z_i)\int_{\Delta_i} &[f_n(z) - f_n(z_i)]\,[g_n(z) - g_n(z_i)]\,dz \\ &= -g_n(z_i)\int_{\Delta_i} -f_n(z_i)g_n(z)\,dz = \int_{\Delta_i} g_n(z)\,dz.\end{aligned}$$

Then $\|\int_{\Delta_i} g_n(z)\,dz\| \leqq N_n'[N_nN_n'']\,s^3/8^k$ and $\|\int_{\Delta} g_n(z)\,dz\| \leqq N_nN_n'N_n''s^3/2^k$. Letting $k \to \infty$ and then letting $n \to \infty$, the theorem follows.

Let A be a path in E' with endpoints α and β and let f and g be continuous functions on A and $A \times A$ respectively into E'. Then clearly $\theta = \int_{\alpha_A}^{\beta} f(z)\,dz$ is defined, where $A_\theta = \int_{\alpha_A}^{\beta} A_{f(z)}(dz)$. Let $P: \alpha = x_0 < \cdots < x_{n+1} = \beta$, $n \in \omega$, be a subdivision of A and set

$$\begin{aligned}R &= \sum_{i=0}^{n}\sum_{j=0}^{n} g(x_i, x_j)(x_{i+1} - x_i)(x_{j+1} - x_j) \\ &= \sum_{i=0}^{n}\left[\sum_{j=0}^{n} g(x_i, x_j)(x_{j+1} - x_j)\right](x_{i+1} - x_i).\end{aligned}$$

Then clearly as the norm of P converges to zero, R converges to a limit $\mu = \int_\alpha^\beta \int_{\alpha_{A\times A}}^{\beta} g(x, y)\,dx\,dy$. Clearly $\mu = \int_\alpha^\beta [\int_{\alpha_A}^{\beta} g(x, y)\,dx]\,dy = \int_{\alpha_A}^{\beta}[\int_{\alpha_A}^{\beta} g(x, y)\,dy]\,dx$.

A continuous function f on an open set $S \subseteq E'$ into E' is said to be integrable if $\int_C f(z)\,dz = 0$ for all paths $C \subseteq S$.

Theorem 2.2. *Let S be a simply connected open set in E', $0 \in S$, f a continuous integrable function on S into E', and set $P(z) = \int_0^z f(x)\,dx$ for $z \in S$. Then P is an integrable function on S into E'.*

Proof. Our argument is based on that of [9]. Let $z \in S$ and let A be a path in S with endpoints 0 and z. Then

$$\begin{aligned}\int_0^z P(x)\,dx &= \int_{0_A}^z \left[\int_{0_A}^x f(t)\,dt\right]dx = \int_{0_A}^z \left[\int_{t_A}^z f(t)\,dx\right]dt \\ &= \int_{0_A}^z f(t)\left[\int_{t_A}^z e'\,dx\right]dt = \int_{0_A}^z f(t)(t-z)\,dt,\end{aligned} \tag{1}$$

where e' is the identity element of E'. Set $g(t) = t - z$ for $t \in E'$. Then trivially g is integrable, and thus A_f and A_g are integrable. From Theorem 2.1, $A_{fg} = A_f A_g$ is integrable and thus fg is integrable and thus the last integral of (1) is dependent of the choice of path A linking 0 and z, and consequently P is integrable.

Remark. The requirement that E be finite dimensional may be removed in this section. Suppose S is a simply connected open set in E and f, g elements of F with domain S, and let C be a closed path in S. Then there exists a sequence of closed paths $C_1, C_2, \cdots$, formed from straight line segments which approximate C such that $\int_{C_n} f(z)\,dz \to \int_C f(z)\,dz$ as $n \to \infty$, etc. Now for $n \in \omega$, there exists a finite-dimensional subspace E_n of E containing C_n. Lemma 2.1 readily generalizes to the case of f/E_n and g/E_n, and thus Theorem 2.1 can be reworked to yield $\int_{C_n} f(z)g(z)\,dz = 0$. Taking the limit as $n \to \infty$, we obtain $\int_C f(z)g(z)\,dz = 0$. Thus Theorem 2.1 generalizes to the infinite-dimensional case. Corollary 2.1 and Theorem 2.2 similarly generalize.

3. **Nonanalytic examples.** So far in our discussion no use of the notion of differentiation has been made. Indeed it is quite easy to construct integrable functions which are not differentiable. For $z = (x, y) \in E_2 = R \oplus R$, set $P(z) = (x, 0)$, and set $T = \{re + sP;\ r, s \in R\}$. Since $P^2 = P$, T is a commutative Banach algebra with identity. Let h be an arbitrary continuous function with domain R into R and set $f(z) = h(x)P$ for $z = (x, y) \in E_2$. We show that f is integrable. For $x \in R$, set $u(x) = \int_0^x h(t)\,dt$ and for $z = (x, y) \in E_2$, set $\phi(z) = [u(x), 0]$. Then for $z = (x, y)$, $\rho = (s, t)$ lying in E_2, we have $\phi_z'(\rho) = [u'(x)s, 0] = [h(x)s, 0] = h(x)P(\rho) = f_z(\rho)$, and thus $\phi_z' = f(z)$. Then clearly for any path $A \subseteq E_2$ with endpoints α and β, $\int_{\alpha_A}^{\beta} f(z)\,dz = \phi(\beta) - \phi(\alpha)$ and thus for any closed path C, $\int_C f(z)\,dz = 0$.

Alternately suppose C is a closed path composed of line segments, none parallel to $\{0\} \times R$. Then $\int_C f(z)\,dz$ can be expressed as $\mu \equiv (1,0)\sum_0^n \int_{x_i}^{x_{i+1}} h(x)\,dx$, where $x_0, x_1, \cdots, x_{n+1} = x_0$, $n \in \omega$, are the projections of the endpoints of segments of C onto R. Since $x_{n+1} = x_0$, $\mu = 0$.

A continuous function f on an open set $S \subseteq E'$ is said to be E'-differentiable if f is (Fréchet) differentiable, and if for all $x \in S$, there exists $c \in E'$, such that $f_x'(t) = ct$ for all $t \in E$. An E'-differentiable function is integrable. The standard Cauchy-Goursat argument for the complex case [1], [9] involving nesting of triangles readily transfers over to the context of E'. In this more general context, however, E'-differentiability does not imply analyticity. Let E' be the algebra generated by e and μ, where $\mu^2 = 0$ and $e\mu = \mu e = \mu$. Let h be an arbitrary continuous function with domain R into R, and set $u(x) = \int_0^x h(t)\,dt$ for $x \in R$. For $z = xe + y\mu \in E'$, set $\phi(z) = u(x)\mu$. Then for $z = xe + y\mu$, $\rho = re + s\mu \in E'$, $\phi_z'(\rho) = [u'(x)r]\,\mu = h(x)r\mu = [h(x)\mu]\rho$, where $h(x)\mu \in E'$. Thus ϕ is E'-differentiable and hence integrable; however ϕ'' need not exist, since h' need not exist.

If one restricts himself to complex Euclidean spaces K_n, $n \in \omega$, (which may be interpreted as real Euclidean spaces E_{2n}) and complex homogeneous operators, then the differentiability of the elements of F follows from the fact that the integrals of the elements of F are *complex* (Fréchet) differentiable and hence infinitely differentiable and analytic. In §5 we will show how to insure analyticity by imposing much less drastic conditions on E and T working in a real variable context.

4. **The symmetric case.** If E can be represented as a direct sum of invariant subspaces, then for $f \in F$, f can be expressed as a direct sum of integrable functions defined on the subspaces. If T is symmetric, then E can be expressed as the direct sum of mutually orthogonal one- and two-dimensional invariant subspaces, and for $f \in F$, f can be expressed as the direct sum of functions which can be interpreted as continuous functions on R into R or as complex analytic functions on K into K.

We note that if, for E, we take $E_2 \cong K$, then for a complex number A, we associate A^* with the conjugate $\bar{A}$ of A.

LEMMA 4.1. *Let f be an integrable function on U into B, and S and W subspaces of E such that for $x \in U$, $t \in S$, $f(x)(t) \in W$. Then for $x, y \in U$ such that $y - x \in S$, we have $[f(x) - f(y)](t) \in W$ for all $t \in E$.*

PROOF. For $x \in U$, set $g(x) = \int_0^x f(z)\,dz$. Let $x, y \in U$ such that $x \neq y$ and $y - x \in S$, and let $x = x_0 < \cdots < x_{n+1} = y$, $n \in \omega$, be a subdivision of the interval $[x, y]$ of E. Then for $i = 0, \cdots, n$, $\Delta x_i = x_{i+1} - x_i \in S$, and hence $f(x_i)(\Delta x_i) \in W$. Thus the Riemann sum $\sum_0^n f(x_i)(\Delta x_i) \in W$. Hence $\int_x^y f(z)\,dz \in W$ and $g(y) - g(x) \in W$.

Let $t \in E$ and $r \in R$ be such that $x + rt$, $y + rt \in U$. Then $(y + rt) - (x + rt) = y - x \in S$, and hence $g(y + rt) - g(x + rt) \in W$. Thus for all $t \in E$,

$$\begin{aligned} f(y)(t) - f(x)(t) = {} & \lim_{r\to 0} [g(y + rt) - g(y)]\, r^{-1} \\ & - \lim_{r\to 0} [g(x + rt) - g(x)]\, r^{-1} \\ = {} & \lim_{r\to 0} \{[g(y + rt) - g(x + rt)] + [g(y) - g(x)]\} r^{-1} \\ \in {} & W. \end{aligned}$$

THEOREM 4.1. *Let $H_1, \cdots, H_n$, $n \in \omega$, be invariant subspaces of E distinct from $\{0\}$ such that $E = H_1 \oplus \cdots \oplus H_n$ and let $f \in F$ be such that $S = (U \cap H_1) \oplus \cdots \oplus (U \cap H_n) \subseteq$ domain f. Then there exist integrable functions $f_1, \cdots, f_n$ such that for $i = 1, \cdots, n$, $U \cap H_i \subseteq$ domain $f_i \subseteq H_i$ and range $f_i \subseteq T \mid H_i$, and such that for $x \in S$, $t \in E$,*

$$f(x)(t) = \sum_{i=1}^{n} f_i(x_i)(t_i),$$

where $x = x_1 + \cdots + x_n$ and $t = t_1 + \cdots + t_n$ and $x_i, t_i \in H_i$ for $i = 1, \cdots, n$.

PROOF. For $i = 1, \cdots, n$, set $g_i(x)(t) = f(x)(t_i)$ for $x \in S$, $t \in E$, and set $f_i = g_i \mid H_i$. Let C be a closed path in U. Now for $i = 1, \cdots, n$, $x \in S$, $t \in E$, we have $t_i \in H_i$ and hence $g_i(x)(t) = f(x)(t_i) \in H_i$ and $\int_C g_i(z)\,dz \in H_i$. Thus $\{\int_C g_i(z)\,dz;\ i = 1, \cdots, n\}$ is a linearly independent set. Now for $x \in S$, $t \in E$,

$$f(x)(t) = f(x)\left(\sum_{i=1}^{n} t_i\right) \sum_{i=1}^{n} f(x)(t_i) = \sum_{i=1}^{n} g_i(x)(t),$$

and hence $0 = \int_C f(z)\,dz = \int_C \left[\sum_{i=1}^n g_i(z)\right] dz = \sum_{i=1}^n \int_C g_i(z)\,dz$. Thus for $i = 1, \cdots, n$, $\int_C g_i(z)\,dz = 0$ and g_i is integrable.

Let $i = 1, \cdots, n$. Then for all $x \in S$, $t \in W_i = H_1 \oplus \cdots \oplus H_{i-1} \oplus H_{i+1} \oplus \cdots \oplus H_n$, we have $g_i(x)(t) = 0$ and $x - x_i \in W_i$. Hence from Lemma 4.1, for all $x \in S$, $g_i(x) = g_i(x_i)$. Thus for $x \in S$, $t \in E$,

$$f(x)(t) = \sum_{k=1}^{n} g_i(x)(t) = \sum_{k=1}^{n} g_i(x)(t_i)$$

$$= \sum_{k=1}^{n} g_i(x_i)(t_i) = \sum_{k=1}^{n} f_i(x_i)(t_i).$$

A subspace H of E is said to be an irreducible invariant subspace of E, if H is an invariant subspace of E distinct from $\{0\}$, and H contains no proper invariant subspace distinct from $\{0\}$.

LEMMA 4.2. *Suppose H is an irreducible invariant subspace of E. Then $T_0 = T \mid H$ is a field with the same dimension as H. Moreover a multiplication can be defined on H such that H becomes a field isomorphic to T_0, and such that for $A \in T_0$ there exists $a \in H$ such that $A(t) = at$ for all $t \in H$. Furthermore if the dimension of H is two and T is symmetric then T_0 and H are isomorphic and isometric to K.*

We observe that it follows from the fundamental theorem of algebra that T_0 is isomorphic (but not necessarily isometric) to R or K, and thus that H has dimension one or two. Suppose H had dimension two. Let f be an integrable function on an open set in H into T_0. If we renorm H and T_0 so that H and T_0 are isometric to K, then f becomes an integrable function from K to K. Whence from Morera's theorem f is analytic and can be expanded in power series. If H has dimension one, then under suitable renorming H and T_0 are isometric to R, and f becomes an arbitrary continuous function from R to R.

PROOF OF LEMMA 4.2. Let $A \in T_0$, $A \neq 0$. Then A is one-to-one on H. Indeed, for $D \in T$, setting $D_0 = D \mid H$, we have $D[A(H)] = D_0[A(H)] = A[D_0(H)] \subseteq A(H)$, and hence $A(H)$ is an invariant subspace of H; and consequently, since H is irreducible, $A(H) = H$.

Let $x \in H \cap V$, and set $\theta(A) = A(x)$ for all $A \in T_0$. Now range θ $= \{A(x);\ A \in T\}$ is clearly an invariant subspace of H and hence range $\theta = H$. Suppose for $A, B \in T_0$, $\theta(A) = \theta(B)$. Then $A(x) = B(x)$ and $(A - B)(x) = 0$. Thus $A - B$ is not one-to-one on H, and hence $A - B = 0$, and thus θ is an isomorphism of T_0 onto H.

Let $A \in T_0$, $A \neq 0$. Since $A(H) = H$, there exists $y \in H$, such that $A(y) = x$. Now there exists $\alpha \in T_0$ such that $\theta(\alpha) = y$. Then $\alpha(x) = y$ and $\theta(A\alpha) = (A\alpha)(x) = A[\alpha(x)] = A(y) = x$. Since $\theta(e) = x$, where e is the identity element of T_0, we have $A\alpha = e$ and $\alpha = A^{-1}$, and thus T_0 is a field.

For $s, t \in H$, set $st = \theta[\theta^{-1}(s)\theta^{-1}(t)]$. Let $A \in T_0$ and set $a = A(x)$. Let $t \in H$ and set $B = \theta^{-1}(t)$. Then $B(x) = t$. Now $at = \theta[\theta^{-1}(a)\theta^{-1}(t)] = \theta[AB] = (AB)(x) = A[B(x)] = A(t)$.

Now suppose T is symmetric and H had dimension two. Then T_0 is symmetric. Let I be the element of T_0 such that $I^2 = -e$. Now $(I^*)^2 = (I^2)^* = (-e)^* = -e$ and thus $I^* = \pm I$. Now $I^* = -I$ since otherwise we would have $0 < \|Ix\|^2 = [Ix, Ix] = [II^*x, x] = [I^2x, x] = -[x, x] = -1$. Let $r, s \in R$. Then for all $y \in V \cap H$, $\|(r + sI)(y)\|^2 = [(r + sI)(r + sI)^*y, y] = [(r + sI)(r - sI)y, y] = [(r^2 + s^2)y, y] = (r^2 + s^2)\|y\|^2 = r^2 + s^2$ and thus $\|re + sI\| = (r^2 + s^2)^{1/2}$. Set $i = I(x)$. Then $[x, i] = [x, Ix] = [I^*x, x] = -[Ix, x] = -[x, i]$ and thus $[x, i] = 0$. Also $[i, i] = [Ix, Ix] = [II^*x, x] = [-I^2x, x] = [x, x] = 1$. Thus $\|rx + si\|^2 = [rx + si, rx + si] = r^2[x, x] + 2rs[x, i] + s^2[i, i] = r^2 + 0 + s^2$.

THEOREM 4.2. *Suppose T is symmetric and $f \in F$ with* domain U. *Then there exists a sequence $Z_1, \cdots, Z_n$, $n \in \omega$, in $\{R, K\}$ and a sequence of functions $f_1, \cdots, f_n$ such that:*

(1) *For $i = 1, \cdots, n$, if $Z_i = R$, then f_i is an arbitrary continuous function on $(-1, 1)$ into R.*

(2) *For $i = 1, \cdots, n$, if $Z_i = K$, f_i is a complex analytic function on $U \subseteq K$ into K.*

(3) *Setting $\phi(x) = [f_1(x_1), f_2(x_2), \cdots, f_n(x_n)]$ for $x = (x_1, \cdots, x_n) \in W = Z_1 \oplus \cdots \oplus Z_n$, there exists an isometry θ of E onto W and an isometry μ of T onto W, such that $\mu^{-1}\{\phi[\theta(x)]\} = f(x)$ for all $x \in U$.*

PROOF. We decompose E. Since E is finite dimensional, there exists an irreducible invariant subspace E_1 of E. Let W_1 be the orthogonal complement $\{y \in E;\ (x, y) = 0$ for all $x \in E_1\}$ of E_1. Then $E = E_1 \oplus W_1$. Now for $y \in W_1$, $A \in T$, we have $A^* \in T$, and for all $x \in E_1$, $A^*(x) \in E_1$ and $[A(y), x] = [A^*(x), y] = 0$; and consequently $A(y) \in W_1$. Thus W_1 is an invariant subspace of E. We next extract an irreducible invariant subspace E_2 of E from W_1 and form the orthogonal complement W_2 of E_2 in W_1. Continuing in this manner until we exhaust E, we obtain a sequence $E_1, \cdots, E_n$, $n \in \omega$, of mutually orthogonal irreducible invariant subspaces of E such that $E = E_1 \oplus \cdots \oplus E_n$.

From Theorem 4.1, there exist integrable functions $g_1, \cdots, g_n$ such that for $i = 1, \cdots, n$, domain $g_i = U \cap E_i$ and range $g_i \subseteq T \mid E_i$, and such that for $x \in U$, $t \in E$, $f(x) = \sum_{i=1}^{n} g_i(x_i)(t_i)$, where x_i and t_i are the projections of x and t respectively into E_i for $i = 1, \cdots, n$.

Let $i = 1, \cdots, n$. Then from Lemma 4.2, there is an isometry θ_i of E_i onto an element Z_i of $\{R, K\}$ and an isometry μ_i of $T_i = T \mid E_i$ onto Z_i, such that for $A \in T_i$, $x \in E_i$, we have $[\mu_i(A)] \cdot [\theta_i(x)] = \theta_i[A(x)]$. For $x \in U \subseteq Z_i$, set $f_i(x) = \mu_i\{g_i[\theta_i^{-1}(x)]\}$. Let P be a path in $U \subseteq Z_i$ with endpoints α and β and let $N : \alpha = x_0 < \cdots < x_{n+1} = \beta$, $n \in \omega$, be a subdivision of P, and let R be the Riemann sum $\sum_0^n f_i(x_i)(x_{i+1} - x_i)$. Then

$$\bar{R} = \theta_i^{-1}(R) = \sum_0^n \{\mu_i^{-1}[f_i(x_i)]\}[\theta_i^{-1}(x_{i+1} - x_i)]$$

$$= \sum_0^n \{\mu_i^{-1}[\mu(g_i(\theta_i^{-1}x_i))]\}(y_{i+1} - y_i) = \sum_0^n g_i(y_i)(y_{i+1} - y_i)$$

is a Riemann sum for the subdivision $\bar{N} : \bar{\alpha} = y_0 < \cdots < y_{n+1} = \bar{\beta}$ of the path $\bar{P} = \theta_i^{-1}(P)$ of E_i, where $y_i = \theta_i^{-1}(x_i)$ for $i = 0, 1, \cdots, n+1$. Now norm N = norm $\bar{N}$. Hence taking the limit as norm N converges to zero we obtain $\theta_i^{-1}[\int_{\alpha_P}^{\beta} f(x)\,dx] = \int_{\bar{\alpha}_{\bar{P}}}^{\bar{\beta}} g(y)\,dy$. Thus f_i is an integrable function from Z_i into Z_i.

We observe that if θ_i and μ_i are merely required to be isomorphisms rather than isometries then there exists a number $C = \|\theta_i^{-1}\| < \infty$ such that norm $\bar{N} \leqq C \cdot$ (norm N). Hence as norm $N \to 0$, norm $\bar{N} \to 0$, and the desired equality of integrals exists.

For $z \in W$, set $\phi(x) = [f_1(x_1), \cdots, f_n(x_n)]$. For $x \in E$, set $\theta(x) = [\theta_1(x_1), \cdots, \theta_n(x_n)]$ and for $A \in T$, writing A_i for $A \mid E_i$ for $i = 1, \cdots, n$, set $\mu(A) = [\mu_1(A_1), \cdots, \mu_n(A_n)]$. Then $\mu^{-1}\{\phi[\theta(x)]\} = f(x)$ for all $x \in U$.

5. **Analyticity.** The principal result of this section, Theorem 5.3, is that if T is presumed to be semisimple then a necessary and sufficient condition for analyticity is that no element of T is of rank one. It is conjectured that the requirement of semisimplicity can be dispensed with. Indeed regardless of whether or not T is semisimple we have from Theorem 5.1 that if T contains an element of rank one, then F contains a nondifferentiable element.

We observe that a necessary and sufficient condition for T to be semisimple is that T contain no nilpotent elements, i.e. elements x, such that $x \neq 0$, but $x^k = 0$ for some $k \in \omega$.

A definition of analyticity suitable for the real variable context of this paper has been given by this author [2], [5], [6]. If the elements of F satisfy a uniform Lipschitz condition descended from Schwarz's lemma in complex function theory, then the elements of F are (Fréchet) differentiable, indeed infinitely (Fréchet) differentiable, and expandable in power series.

Let Z be a family of continuous functions on open subsets of E into a Banach space C. Z is called a TR family if:

(1) $f + g \in Z$ for $f, g \in Z$.

(2) $rf \in Z$ for $f \in Z$, $r \in R$.

(3) For $f \in Z$, S an open set in E, setting $g = f \mid S$, $g \in Z$.

(4) For $f \in Z$, $y \in E$, the translate f_y of f lies in Z, where $f_y(x) = f(x - y)$ for $x \in y +$ (domain f).

(5) For $f \in Z$, $r > 0$, setting $g(x) = f(rx)$ for $x \in r^{-1}$(domain f), we have $g \in Z$.

Trivially F is a TR family.

Z is called a TRL family if Z is a TR family and

(6) there exists $N > 0$, such that for $f \in Z$, $\bar{U} \subseteq$ domain f, $x \in$ domain f, we have

$$\|f(x) - f(0)\| \leqq N \sup \{\|f(t)\|;\, t \in U\}\|x\|.$$

We denote the least upper bound of the family of all such numbers N by $N(Z)$.

If Z is a TRL family, $f \in Z$, and $z_0 \in$ domain f, then for some $\delta > 0$, the series $\sum_0^\infty f_{z_0}^{(n)}(z - z_0)/n!$ converges uniformly on $U_{z_0}(\delta)$ to $f(z)$, where $f_{z_0}^{(n)}$ is a homogeneous function on E into T of degree n, derived from the nth Fréchet derivative of f at z_0, for $n \in \omega$.

THEOREM 5.1. *A necessary condition for F to be a TRL family of analytic functions is that no element of T have a one-dimensional range.*

PROOF. Let $A \in T$, such that $A(E)$ is one dimensional, and set $H = \{x \in E;\, A(x) = 0\}$. Then the dimension of H is $p - 1$. Let w be an element of E such that $w_0 = A(w) \neq 0$. Then $E = \{rw;\, r \in R\} \oplus H$. Let h be an arbitrary continuous function with domain R into R. For $r \in R$, set $u(r) = \int_0^r h(t)\,dt$. For $z = rw + x$, $r \in R$, $x \in H$, set $f(z) = h(r)A$ and set $\phi(z) = u(r)w_0$. Then for $z = rw + x$, $\rho = sw + y, r, s \in R$, $x, y \in H$, we have $A(\rho) = A(sw) + A(y) = sw_0 + 0$, and $\phi_z{}'(\rho) = [u'(r)s]\,w_0 = [h(r)]\,sw_0 = h(r)A(\rho) = [f(z)]\,(\rho)$ and $\phi'(z) = f(z)$. Then as in the examples of §3 for any path $P \subseteq E$ with endpoints α and β, $\int_{\alpha_P}^{\beta} f(z)\,dz = \phi(\beta) - \phi(\alpha)$, and thus for any closed path $C \subseteq E$, $\int_C f(z)\,dz = 0$, and f is integrable. Clearly since h need not satisfy any Lipschitz conditions, f need not satisfy condition (6) of the definition of a TRL family for any $N > 0$.

Let G be the family of all functions g on open sets $S \subseteq E$ into E, such that there exists $f \in F$ with domain S, such that if x and y are points of the same component of S, then $g(y) - g(x) = \int_x^y f(z)\,dz$.

For $f \in F$, H a subspace of E, set $f_H(z) = f(z)|H \in T|H$, and set $f|'H = f_H|H$. Set $F|'H = \{f|'H;\, f \in F\}$.

THEOREM 5.2. *Let H be a subspace of E, such that $F\,|'H$ and $G\mid H$ are TRL families of analytic functions, and such that for $A \in T$, $A(x)$*

$= 0$ *for all* $x \in H$ *implies* $A(x) = 0$ *for all* $x \in E$. *Then* F *and* G *are TRL families of analytic functions.*

PROOF. For $A \in T$, set $\theta(A) = A \mid H$, and set $T_0 = T \mid H$. If for $A, B \in T$, $\theta(A) = \theta(B)$, then for $x \in H$, $A(x) = B(x)$ and $(A - B)(x) = 0$, and hence by hypothesis, $A - B = 0$ and $A \equiv B$. Thus θ is an isomorphism and there exists $N_1 > 0$ such that for $A \in T_0$, $\| \theta^{-1}(A)\| \leqq N_1\|A\|$.

Clearly G is a TR family. We now show that G is a TRL family. Let $f \in F$, $g \in G$, such that $\bar{U} \subseteq$ domain $f =$ domain g, and $g(y) - g(x) = \int_x^y f(z)\, dz$ for $x, y \in U$. Let $x_0 \in U$ and set $r_0 = 1 - \|x_0\|$. Then $r_0U + x_0 \subseteq U$. For $x \in U$, set $h(x) = g(r_0x + x_0)$. Then $h \in G$. Set $M = \{\|g(t)\|;\ t \in U\}$, set $G_0 = G \mid H$, and set $N_0 = N(G_0)$. Then for $x \in U \cap H$,

$$\|h(x) - h(0)\| \leqq N_0 \sup \{\|h(t)\|;\ t \in U \cap H\}\|x\| \leqq N_0M\|x\|.$$

Now for $\alpha \in V \cap H$, $0 < r < r_0$,

$$\begin{aligned} f(x_0)(\alpha) &= \lim_{s\to 0} [g(x_0 + s\alpha) - g(x_0)]\, s^{-1} \\ &= \lim_{s\to 0} [h(sr_0^{-1}\alpha) - h(0)]\, s^{-1}, \end{aligned}$$

and

$$\| [h(sr_0^{-1}\alpha) - h(0)]\, s^{-1}\| \leqq MN_0s^{-1}\|sr_0^{-1}\alpha\| = MN_0r_0^{-1}$$

and thus $\|f(x_0)(\alpha)\| \leqq MN_0r_0^{-1}$, and hence $\| \theta f(x_0)\| \leqq MN_0r_0^{-1}$ and $\|f(x_0)\| \leqq MN_0N_1r_0^{-1}$. Then for $x \in U(\frac{1}{2})$,

$$\begin{aligned} \|g(x) - g(0)\| &= \left\| \int_0^x f(z)\, dz \right\| \leqq \int_0^x \|f(z)\|\, ds \\ &\leqq \int_0^x MN_0N_1(\tfrac{1}{2})^{-1}\, ds = 2MN_0N_1\|x\| \end{aligned}$$

and thus for $x \in U$,

$$\|g(x) - g(0)\| \leqq NM\|x\|,$$

where $N = \max \{2N_0N_1, 4\}$. Thus G is a TRL family and hence by [2], F is a TRL family.

LEMMA 5.1. *Let* f *be an integrable function on* U *such that for* $x \in U$, $f(x) = r(x)e$, *where* r *is a continuous function on* U *into* R. *Then* f *is constant.*

PROOF. Let $x, y \in U$, $x \neq y$ and set $S = W = \{s(y - x);\ s \in R\}$.

Then for $z \in U$, $t \in S$, $f(z)(t) = r(z)t \in W$, and hence from Lemma 5.2, $[f(y) - f(x)](t) \in S$ for all $t \in E$. Thus $[r(y) - r(x)]t$ is a multiple of $y - x$ for all $t \in E$, and hence $r(y) - r(x) = 0$ and $r(y) = r(x)$.

THEOREM 5.3. *If T is semisimple, then a necessary and sufficient condition that F be a TRL family of analytic functions is that no element of T have rank one.*

PROOF. Necessity follows from Theorem 5.1. Let $\mathcal{H}_0 = \{A(E); A \in T, A \neq 0\}$ and let $\mathcal{H}$ be the family of all $H \in \mathcal{H}_0$ such that $\sigma \in \mathcal{H}_0$, $\sigma \subseteq H$, implies $\sigma = H$. Clearly $\mathcal{H}$ is nonempty. Let W be the subspace of E generated by the union of the elements of $\mathcal{H}$. To show sufficiency we shall first show that $F\,|'W$ and $G \mid W$ are TRL families of analytic functions. We then exploit the semisimplicity of T to show that the elements of T are determined by their behavior on W allowing us to employ Theorem 5.2.

Let $H \in \mathcal{H}$ and set $T_0 = T \mid H$. Suppose T_0 is one dimensional. Then all elements of T_0 are of the form re_0, where $r \in R$, and $e_0 = e \mid H$. Since H is not one dimensional, from Lemma 5.1, the elements of $F|'H$ are locally constant functions and thus trivially $F|'H$ and $G \mid H$ are TRL families.

Suppose T_0 is not one dimensional. Employing Theorem 4.2, we shall show that any element $f \in F|'H$ can be expressed locally as a direct sum of complex analytic functions from K into K, thus yielding that $F\,|'\,H$ and $G \mid H$ are TRL families.

Let $x \in H_0 = H - \{0\}$. Then $\bar{x} = \{P(x);\ P \in T\}$ is an invariant subspace of H. Let σ be an irreducible invariant subspace of $\bar{x}$. For $P \in T_0 = T \mid H$, set $\theta(P) = P\,|\sigma$. Suppose for some $P \in T$, $P_0 = P \mid H$, that $\theta(P_0) = 0$. Then $P(\sigma) = \{0\}$ and P is not one-to-one on H. Then $H \neq P(H) \subseteq H$, and $H = P(A)$ and $P(H) = P[A(E)] = (PA)(E) = \text{range}\, PA$ for some $A \in T$. From the minimality of H, $P(H) = \{0\}$, and $P_0 = 0$ and thus θ is an isomorphism. Thus $T_1 = T_0|\sigma = T|\sigma$ is isomorphic to T_0 and hence T_1 does not have dimension one. From Lemma 4.2, T_1, and hence T_0 is a field isomorphic to K. Let $y \in \sigma$, $y \neq 0$, $z \in \bar{x}$. Then there exist $P, Q \in T_0$ such that $y = P(x)$ and $z = Q(x)$. Whence $z = (QP^{-1})(y)$. Then $\bar{x} \subseteq \sigma$ and $\bar{x} = \sigma$ and $\bar{x}$ is irreducible. Clearly $T \mid \bar{x}$ is isomorphic to K. Let $f \in F\,|'\,H$.

Now $\{\bar{x};\ x \in H_0\}$ generates H. Hence there exist $x_1, \cdots, x_k \in H_0$, $k \in \omega$, such that $H = \bar{x}_1 \oplus \cdots \oplus \bar{x}_k$. Then following the proof of Theorem 4.2, there exists an isomorphism π of H onto a complex Hilbert space H' of complex dimension k, and an isomorphism μ of

T_0 onto K, such that $g = \mu f \pi^{-1}$ is a complex analytic function on $\pi(\text{domain} f)$ into K.

There exist ρ, $\delta > 0$, such that $\pi[U(\rho) \cap H] \subseteq U(\delta) \subseteq \pi(U \cap H)$. Then $\pi[U(\rho) \cap H] \subseteq \text{domain}\, g$. Let $x \in U(\rho) \cap H$, $x' = \pi(x)$. Then from complex function theory,

$$\begin{aligned}\|f(x) - f(0)\| &\leqq \|\mu^{-1}\| \cdot \|g(x') - g(0)\| \\ &\leqq 2\|\mu^{-1}\| \sup \{\|g(t)\|; t \in U(\delta)\}\delta^{-1}\|x'\| \\ &\leqq 2\|\mu^{-1}\|[\|\mu\| \sup \{\|f(t)\|; t \in \pi^{-1}[U(\delta)]\}]\delta^{-1}[\|\pi\| \cdot \|x\|] \\ &\leqq NM\|x\|,\end{aligned}$$

where $M = \sup \{\|f(t)\|; t \in U \cap H\}$ and

$$N = \max \{2\|\mu^{-1}\| \cdot \|\mu\| \cdot \|\pi\|\delta^{-1}, 2/\rho\}.$$

For $x \in U(\rho) \cap H$, $\|x\| \geqq \rho$, we have

$$\|f(x) - f(0)\| \leqq 2M = [2M/\rho]\rho \leqq [2M/\rho]\, \|x\| \leqq NM\|x\|.$$

Thus $F_H = F \,|'\, H$ is a TRL family and $N(F_H) \leqq N$.

W can be expressed as the direct sum of a finite subcollection $\{H_1, \cdots, H_n\}$ of $\mathcal{H}$, $n \in \omega$. Then for $x \in W$, there exist unique elements $x_1, \cdots, x_n$ of W, such that $x_i \in H_i$ for $i = 1, \cdots, n$, and $x = x_1 + \cdots + x_n$. Moreover there exists $\rho > 1$ such that for $x \in W$, $i = 1, \cdots, n$, we have $\|x_i\| \leqq \rho\|x\|$. For $i = 1, \cdots, n$, set $N_i = N[F \,|'\, H_i]$. Let $f \in F \,|'\, W$, $\bar{U} \cap W \subseteq \text{domain} f$.

For $i = 1, \cdots, n$, set $f_i = f \,|'\, H_i$. Then from Theorem 4.1, for $z \in U(\rho^{-1}) \cap W$, $t \in W$, $f(z)(t) = \sum_{i=1}^n f_i(z_i)(t_i)$. Whence for $x \in U(\rho^{-1}) \cap W$, we have $x_1, \cdots, x_n \in U$ and

$$\begin{aligned}\|f(x) - f(0)\| &\leqq \sum_{i=1}^{n} \|f_i(x_i) - f_i(0)\| \\ &\leqq \sum_{i=1}^{n} N_i \sup \{\|f_i(s)\|; s \in U \cap H_i\}\|x_i\| \qquad (1) \\ &\leqq \sum_{i=1}^{n} N_i M[\rho\|x\|] \leqq NM\|x\|,\end{aligned}$$

where $N = \max \{\rho \sum_{i=1}^n N_i, 2\rho\}$ and

$$M = \sup \{\|f(s)\|; s \in U \cap W\}.$$

Trivially (1) holds for $x \in U \cap W$, $\|x\| \geqq \rho^{-1}$, and thus $F \,|'\, W$ is a

TRL family. By a similar argument we show that $G \mid W$ is a TRL family.

Let A be an arbitrary element of T such that $A(x) = 0$ for all $x \in W$. If we show that $A \equiv 0$, then the theorem follows from Theorem 5.2. Suppose $A \not\equiv 0$ and consider the sequence $E \supseteq A(E) \supseteq A^2(E) \supseteq \cdots$. Since T is semisimple, there exists $k \in \omega$ such that $A^k(E) = A^{k+1}(E) = A[A^k(E)] \neq \{0\}$ and A is one-to-one on $A^k(E)$. Now there exists $H \in \mathscr{H}$ such that $H \subseteq A^k(E)$. Whence $A(H) \neq \{0\}$. But $H \subseteq W$ and $A(H) \subseteq A(W) = \{0\}$.

REMARK. The method of proof of Theorem 5.3 suggests that the requirement

(1) No element of T has rank one;

be replaced by

(2) $F|' H$ is a TRL family for all one-dimensional elements H of $\mathscr{H}_0$.

We observe from [6] that if (2) holds, there exists $n \in \omega$, such that $F|' H$ is isomorphic to the family of polynomials and of restrictions of polynomials to open subsets of R into R of degree less than n.

It can be readily shown that the requirement of semisimplicity combined with (2) yields the conclusion that (1) must hold and thus that F is a TRL family. Indeed suppose there exists $A \in T$, such that $H = A(E)$ has dimension one. Now if $A(H) = \{0\}$, then $A^2(E) = A(H) = \{0\}$, and hence, since T is presumed semisimple, $A = 0$. Thus $A(H) = H$. Let $w \in H$, $w \neq 0$, let h be a continuous nondifferentiable function on R into R, and for $x \in E$, set $f(x) = h(x')A$, where $x' \in R$, $x'w = A(x)$. Then as in the proof of Theorem 5.1, f is integrable and nondifferentiable, contradicting (2).

Neither semisimplicity nor condition (2) is sufficient separately to give the conclusion that F is a TRL family as the following examples show. For E we take $R \oplus R$. For $(x, y) \in E$, set $P_1(x, y) = (x, 0)$ and $P_2(x, y) = (y, 0)$. For $i = 1, 2$, set $T_i = \{re + sP_i;\ r, s \in R\}$, and let F_i be the family of integrable functions on open subsets of E into T_i. Both P_1 and P_2 have one-dimensional ranges, and hence from Theorem 5.1, F_1 and F_2 are not TRL families of analytic functions. We show that T_1 is semisimple and that $F_2|' H$ is a family of constant functions and hence trivially a TRL family, where $H = \{(x, 0);\ x \in R\}$ is, as we shall show, the only one-dimensional element of $\{A(E);\ A \in T_2\}$.

Let $r, s \in R$, $k \in \omega$, $k > 1$, and set $A = re + sP_1$. Suppose $A^k = 0$. Then for some $c \in R$, $0 = A^k = (re + sP_1)^k = r^k e + (rc + s^k)P_1$ and hence $r^k = 0$ and $rc + s^k = 0$. Then $r = s = 0$ and $A = 0$, and T_1 is semisimple.

Let $A' \in T_2$ and set $H' = A'(E)$. Suppose $H' \neq H$ and H' is one dimensional. Then $P_2(H') \neq \{0\}$ and $H \cap H' = \{0\}$. Since H' is an invariant subspace of E, $P_2(H') \subseteq H'$. Now $P_2(H') \subseteq P_2(E) = H$ and thus $P_2(H') \subseteq H \cap H' = \{0\}$. But then $P_2(H') = \{0\}$. Thus $A'(E) = H$.

Let α and β be continuous functions on U into R such that $\alpha e + \beta P_2$ is integrable. Now $f = \alpha P_2 = (\alpha e + \beta P_2)P_2$ is integrable and $f(z)(H) = \{0\}$ for $z \in U$, and hence applying Lemma 4.1, $f = \alpha P_2$ is constant on H. Now for $z_1, z_2 \in H$,

$$\alpha(z_1)P_2(0,1) = \alpha(z_2)P_2(0,1) \quad \text{and} \quad \alpha(z_1)(1,0) = \alpha(z_2)(1,0),$$

and thus $\alpha(z_1) = \alpha(z_2)$ and $\alpha e|' H$ is constant. Trivially $\beta P_2|' H$ is a null operator valued constant function. Thus $F_2|' H$ is a TRL family.

It is conjectured that the requirement that there exist no one-dimensional range spaces is a necessary and sufficient condition of analyticity. If this conjecture fails we propose the condition that there exist no one-dimensional invariant subspaces. This stronger condition forces E and all invariant subspaces of E to be of even dimension, causing one to suspect that some sort of complexification of E and T is possible.

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Illinois Institute of Technology, Chicago, Illinois 60616