ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 2, Number 1, Winter 1972

# SPECTRAL REPRESENTATION OF SELFADJOINT EXTENSIONS OF A SYMMETRIC OPERATOR

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ABSTRACT. It is shown that the spectral multiplicity of a minimal selfadjoint extension $A$ of a simple closed symmetric operator $A_1$ with deficiency indices $m$, $n$ cannot exceed $m + n$. In the case that $A_1$ has deficiency indices 1, 1, it is shown that any minimal selfadjoint extension $A$ can be represented as a multiplication operator in a space $L_{P}^2(-\infty, \infty)$, where $P(t)$ is a 2 by 2 nondecreasing Hermitian matrix function of $t$. In this case the spectrum and spectral multiplicity of $A$ are studied by use of $P(t)$ and its relation to the matrix $\Phi(\lambda) = \int_{-\infty}^{\infty}[(t-\lambda)^{-1} - t(1+t^2)^{-1}]\,dP(t)$, where $\lambda$ is a complex variable. A criterion is given for when the spectral multiplicity of $A$ is two and for when it is one. It follows from this criterion that if $A_1$ has a selfadjoint extension $A_0$ in the original space with a singular spectral function, then the spectral multiplicity of any minimal selfadjoint extension $A$ is one.

1. **Introduction.** Let $A_1$ be a closed symmetric operator with deficiency indices $m$, $n$ in a Hilbert space $\mathfrak{H}_1$. We suppose that $A_1$ is *simple*, i.e., that $A_1$ does not have a reducing subspace in which it is selfadjoint. A selfadjoint operator $A$ in a Hilbert space $\mathfrak{H}$ is called an *extension* of $A_1$ if $\mathfrak{H}_1 \subseteq \mathfrak{H}$ and $A_1 \subset A$. A selfadjoint extension $A$ is said to be *minimal* if the only subspace of $\mathfrak{H} \ominus \mathfrak{H}_1$ which reduces $A$ is $\{0\}$. In this article it is shown that the spectral multiplicity of a minimal selfadjoint extension of $A_1$ cannot exceed $m + n$. In the case that $A_1$ has deficiency indices 1, 1 a spectral representation is given for any minimal selfadjoint extension $A$ of $A_1$ in the form of a multiplication

Received by the editors July 16, 1970.

*AMS* 1970 *subject classifications.* Primary 47A20, 47B25, 47A10; Secondary 47E05.

*Key words and phrases.* Symmetric operator, deficiency indices, selfadjoint operator, selfadjoint extension, spectrum, spectral multiplicity, spectral function, generalized spectral function, spectral matrix, spectral representation, multiplication operator, singular Sturm-Liouville operator, Cayley transform, isometric operator, unitary operator, unitary extension, generating subspace, analytic function with nonnegative imaginary part, resolution of the identity, resolvent, generalized resolvent, generalized resolution of the identity.

[1]This research was supported by the National Science Foundation Grant No. GP-12886.

operator in a space $L_P^2(-\infty, \infty)$ where $P(t)$ is a 2 by 2 nondecreasing Hermitian matrix function of $t$, the "spectral matrix". The spectrum and spectral multiplicity of $A$ are studied by use of the spectral matrix and its relation to the matrix

$$\Phi(\lambda) = \int_{-\infty}^{\infty} [(t-\lambda)^{-1} - t(1+t^2)^{-1}]\, dP(t).$$

The work complements that of Donoghue [4], who restricted himself to the case that $\mathfrak{H} = \mathfrak{H}_1$, i.e., to the case that $A$ is a selfadjoint extension of $A_1$ in the original space $\mathfrak{H}_1$. A different approach to the spectral representation of $A$ using an abstract analog of the expansion theorems connected with ordinary differential operators was provided by the author in [8] and in previous articles. Some of the methods of the present article are close to those of Kac [9], who carried out a similar project for singular Sturm-Liouville operators.

2. **Generating subspaces.** Let $A_1$ be a closed symmetric operator with domain $\mathfrak{D}(A_1)$ in a Hilbert space $\mathfrak{H}_1$. Let $\lambda_0$ be a fixed nonreal number. Let

$$\mathfrak{L}_1(\lambda_0) = (A_1 - \bar{\lambda}_0 E)\mathfrak{D}(A_1), \qquad \mathfrak{M}_1(\lambda_0) = \mathfrak{H}_1 \ominus \mathfrak{L}_1(\lambda_0),$$
$$m_1 = \dim \mathfrak{M}_1(\bar{\lambda}_0), \qquad n_1 = \dim \mathfrak{M}_1(\lambda_0),$$
$$U_1(\lambda_0) = (A_1 - \bar{\lambda}_0 E)(A_1 - \lambda_0 E)^{-1}.$$

Here $E$ stands for the identity operator. $U_1(\lambda_0)$ is called the Cayley transform of $A_1$; it is an isometry mapping $\mathfrak{L}_1(\bar{\lambda}_0)$ onto $\mathfrak{L}_1(\lambda_0)$. It is true that $A_1 = (\lambda_0 U_1(\lambda_0) - \bar{\lambda}_0 E)(U(\lambda_0) - E)^{-1}$. $m_1, n_1$ are called the deficiency indices of $A_1$ with respect to $\lambda_0$ and depend only on the half-plane (upper or lower) in which $\lambda_0$ lies.

According to Naĭmark [11], all selfadjoint extensions of $A_1$ may be constructed as follows: Let $A_2$ be a closed Hermitian operator in a Hilbert space $\mathfrak{H}_2$. (By Hermitian it is meant that $(A_2 f, g) = (f, A_2 g)$ for all $f, g \in \mathfrak{D}(A_2)$, but it is not necessarily true that $\mathfrak{D}(A_2)$ is dense in $\mathfrak{H}_2$.) Let $\mathfrak{L}_2(\lambda_0)$, $\mathfrak{M}_2(\lambda_0)$, $m_2$, $n_2$, $U_2(\lambda_0)$ be defined as for $A_1$. Suppose that $m_1 + m_2 = n_1 + n_2$ and that $m_2 \leqq n_1$. Then the operator $A' = A_1 \oplus A_2$ in the Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ is a closed Hermitian operator with $\mathfrak{L}'(\lambda_0) = \mathfrak{L}_1(\lambda_0) \oplus \mathfrak{L}_2(\lambda_0)$, $\mathfrak{M}'(\lambda_0) = \mathfrak{M}_1(\lambda_0) \oplus \mathfrak{M}_2(\lambda_0)$, $U'(\lambda_0) = U_1(\lambda_0) \oplus U_2(\lambda_0)$, and with equal deficiency indices $m_1 + m_2$, $n_1 + n_2$. Now let $V$ be an isometric operator mapping $\mathfrak{M}'(\bar{\lambda}_0)$ onto $\mathfrak{M}'(\lambda_0)$ and satisfying the condition that if $f \in \mathfrak{M}_2(\bar{\lambda}_0)$ and $Vf \in \mathfrak{M}_2(\lambda_0)$, then $f = 0$. Then, $U(\lambda_0) = U'(\lambda_0) \oplus V$ is a unitary operator in $\mathfrak{H}$, and it is the Cayley transform of a selfadjoint operator $A$ in $\mathfrak{H}$ which is an extension of $A_1$.

What has been done above, briefly, is to tack a Hermitian operator $A_2$ onto $A_1$ so as to obtain a Hermitian operator $A'$ with equal deficiency indices. A selfadjoint extension $A$ of $A'$ is then obtained by a method similar to the one used by von Neumann to extend a symmetric operator $A'$ with equal deficiency indices to a selfadjoint operator in the original space. If $\mathfrak{H}_2 = \{0\}$, then, of course, Naĭmark's method reduces to von Neumann's. Naĭmark's method, and, in particular, the operator $V$ which is tacked onto $U'(\lambda_0)$ is studied in more detail in Gilbert [**6**].

In the following, $E(t)$ will stand for the spectral function of a selfadjoint operator $A$ in a Hilbert space $\mathfrak{H}$. It is assumed that $E(t)$ is normalized by the conditions $E(-\infty) = 0$, $E(\infty) = E$, $E(t-0) = E(t)$. l.h. $\{\cdots\}$ and c.l.h. $\{\cdots\}$ will stand for the linear hull and closed linear hull, respectively, of the set $\{\cdots\}$. $\Delta$ will stand for an interval $[a, b)$, and $E(\Delta) = E(b) - E(a)$. A subspace $\mathfrak{G}$ is said to be a *generating subspace* for a selfadjoint operator $A$ if l.h. $\{E(\Delta)f : f \in \mathfrak{G}$, $\Delta$ is arbitrary$\}$ is dense in $\mathfrak{H}$.

THEOREM 1. *Let $A$ be a selfadjoint operator in a Hilbert space $\mathfrak{H}$. Let $U = U(\lambda_0)$ be the Cayley transform of $A$ for a nonreal number $\lambda_0$. Let $\mathfrak{G}$ be a subspace of $\mathfrak{H}$. Suppose that l.h. $\{U^k f : f \in \mathfrak{G}, k = 0, \pm 1, \pm 2, \cdots\}$ is dense in $\mathfrak{H}$. Then, $\mathfrak{G}$ is a generating subspace for $A$.*

PROOF. Suppose $h$ is perpendicular to l.h. $\{E(\Delta)f : f \in \mathfrak{G}$, $\Delta$ is arbitrary$\}$. Then, $(E(\Delta)f, h) = 0$ for all $\Delta$ and for all $f \in \mathfrak{G}$. Hence, $(F(s)f, h) = 0$ for all $s$, $0 \leqq s \leqq 2\pi$, and all $f \in \mathfrak{G}$, where $F(s)$ is the spectral function of $U$. ($F(s)$ and $E(t)$ are related by the equation $E(t) = F(s)$, where $t = \operatorname{Re}\lambda_0 + (\operatorname{Im}\lambda_0)\cot(s/2)$ if $\operatorname{Im}\lambda_0 < 0$, and $t = -\operatorname{Re}\lambda_0 - (\operatorname{Im}\lambda_0)\cot(s/2)$ if $\operatorname{Im}\lambda_0 > 0$, $0 < s < 2\pi$.) Thus, $(U^k f, h) = \int_0^{2\pi} e^{iks}\, d(F(s)f, h) = 0$ for all $f \in \mathfrak{G}$ and all $k = 0, \pm 1, \cdots$, and therefore $h$ is perpendicular to l.h. $\{U^k f : f \in \mathfrak{G}, k = 0, \pm 1, \cdots\}$. Hence, $h = 0$. This proves the theorem.

REMARK. Actually, it can be shown that for any subspace $\mathfrak{G}$, c.l.h. $\{E(\Delta)f : f \in \mathfrak{G}$, $\Delta$ is arbitrary$\}$ = c.l.h. $\{U^k f : f \in \mathfrak{G}, k = 0, \pm 1, \pm 2, \cdots\}$ = c.l.h. $\{R(\lambda)f : f \in \mathfrak{G}, \operatorname{Im}\lambda \neq 0\}$, where $R(\lambda)$ is the resolvent of $A$.

THEOREM 2. *Let $A$ be a minimal selfadjoint extension of the simple closed symmetric operator $A_1$. Let $\lambda_0$ be a nonreal number. Let $U = U(\lambda_0)$ be the Cayley transform of $A$ corresponding to $\lambda_0$. Then, l.h. $\{U^k f : f \in \mathfrak{M}_1(\bar{\lambda}_0) \oplus \mathfrak{M}_2(\bar{\lambda}_0), k = 0, \pm 1, \cdots\}$ is dense in $\mathfrak{H}$, and l.h. $\{U^k f : f \in \mathfrak{M}_1(\lambda_0) \oplus \mathfrak{M}_2(\lambda_0), k = 0, \pm 1, \cdots\}$ is dense in $\mathfrak{H}$.*

PROOF. Let $\mathfrak{A}$ = c.l.h. $\{U^k f : f \in \mathfrak{M}_1(\bar{\lambda}_0) \oplus \mathfrak{M}_2(\bar{\lambda}_0), k = 0, \pm 1, \cdots\}$.

$U$ maps $\mathfrak{A}$ onto $\mathfrak{A}$, because $\mathfrak{A}$ is invariant under $U$ and $U^{-1}$. Since $U$ is a unitary map, it follows that $U$ maps $\mathfrak{A}^\perp$ onto $\mathfrak{A}^\perp$. We shall show that $\mathfrak{A}^\perp = \{0\}$, and therefore $\mathfrak{A} = \mathfrak{H}$.

We note that since $\mathfrak{M}_1(\bar{\lambda}_0) \oplus \mathfrak{M}_2(\bar{\lambda}_0) \subset \mathfrak{A}$, it follows that $\mathfrak{A}^\perp \subset \mathfrak{L}_1(\bar{\lambda}_0) \oplus \mathfrak{L}_2(\bar{\lambda}_0)$. Let $f \in \mathfrak{A}^\perp$. Then, $f = f_1 + f_2$, where $f_1 \in \mathfrak{L}_1(\bar{\lambda}_0)$, and $f_2 \in \mathfrak{L}_2(\bar{\lambda}_0)$; and $Uf = Uf_1 + Uf_2$, where $Uf \in \mathfrak{A}^\perp \subset \mathfrak{L}_1(\bar{\lambda}_0) \oplus \mathfrak{L}_2(\bar{\lambda}_0)$, $Uf_1 \in \mathfrak{L}_1(\lambda_0) \subset \mathfrak{H}_1$, and $Uf_2 \in \mathfrak{L}_2(\lambda_0) \subset \mathfrak{H}_2$. It follows that $Uf_1 \in \mathfrak{L}_1(\bar{\lambda}_0)$, and $Uf_2 \in \mathfrak{L}_2(\bar{\lambda}_0)$. We have thus shown that if $P$ is the operator of orthogonal projection on $\mathfrak{L}_1(\bar{\lambda}_0)$, then $UPf = PUf$ for all $f \in \mathfrak{A}^\perp$. From this it follows that $U$ maps $P\mathfrak{A}^\perp$ onto $P\mathfrak{A}^\perp$. Since $U$ restricted to $\mathfrak{L}_1(\bar{\lambda}_0)$ is the Cayley transform $U_1$ of $A_1$, we have that $U_1$ maps $P\mathfrak{A}^\perp$ onto $P\mathfrak{A}^\perp$. This means that $A_1$ is reduced by $P\mathfrak{A}^\perp$, and the part of $A_1$ in $P\mathfrak{A}^\perp$ is a selfadjoint operator. Since $A_1$ is simple, $P\mathfrak{A}^\perp = \{0\}$. Thus $\mathfrak{A}^\perp \subset \mathfrak{H}_2$. Since $\mathfrak{A}^\perp$ reduces $U$ and therefore $A$, and since $A$ is minimal, $\mathfrak{A}^\perp = \{0\}$, which is what we set out to prove.

Since $U$ maps $\mathfrak{M}_1(\bar{\lambda}_0) \oplus \mathfrak{M}_2(\bar{\lambda}_0)$ onto $\mathfrak{M}_1(\lambda_0) \oplus \mathfrak{M}_2(\lambda_0)$, the fact that l.h. $\{U^k f : f \in \mathfrak{M}_1(\bar{\lambda}_0) \oplus \mathfrak{M}_2(\bar{\lambda}_0),\ k = 0, \pm 1, \cdots\}$ is dense in $\mathfrak{H}$ immediately implies that l.h. $\{U^k f : f \in \mathfrak{M}_1(\lambda_0) \oplus \mathfrak{M}_2(\lambda_0),\ k = 0, \pm 1, \cdots\}$ is dense in $\mathfrak{H}$. This completely proves the theorem.

Theorem 3. *Let $A$ be a minimal selfadjoint extension of the simple closed symmetric operator $A_1$. Let $\lambda_0$ be a nonreal number. Then, $\mathfrak{M}_1(\bar{\lambda}_0) \oplus \mathfrak{M}_2(\bar{\lambda}_0)$, $\mathfrak{M}_1(\lambda_0) \oplus \mathfrak{M}_2(\lambda_0)$, and $[\mathfrak{M}_1(\lambda_0) \dotplus \mathfrak{M}_1(\bar{\lambda}_0)]^c$ are all generating subspaces for $A$. (Here, $[\quad]^c$ stands for closure.)*

Proof. That $\mathfrak{M}_1(\bar{\lambda}_0) \oplus \mathfrak{M}_2(\bar{\lambda}_0)$ and $\mathfrak{M}_1(\lambda_0) \oplus \mathfrak{M}_2(\lambda_0)$ are generating subspaces for $A$ follows immediately from Theorems 1 and 2. In order to prove that $\mathfrak{M}_1(\lambda_0) \dotplus \mathfrak{M}_1(\bar{\lambda}_0)$ is a generating subspace, we observe that $[P_2 U \mathfrak{M}_1(\bar{\lambda}_0)]^c = \mathfrak{M}_2(\lambda_0)$, where $P_2$ is the operator of orthogonal projection onto $\mathfrak{M}_2(\lambda_0)$. (This follows from Gilbert [6, Theorem 2, item (4)].) Also, $U$ maps $\mathfrak{M}_1(\bar{\lambda}_0)$ into $\mathfrak{M}_1(\lambda_0) \oplus \mathfrak{M}_2(\lambda_0)$. Hence,

$$[\mathfrak{M}_1(\lambda_0) + U\mathfrak{M}_1(\bar{\lambda}_0)]^c = \mathfrak{M}_1(\lambda_0) \oplus \mathfrak{M}_2(\lambda_0).$$

Therefore,

$$\begin{aligned} &\text{c.l.h. } \{U^k f : f \in [\mathfrak{M}_1(\lambda_0) \dotplus \mathfrak{M}_1(\bar{\lambda}_0)]^c, k = 0, \pm 1, \cdots\} \\ &\qquad = \text{c.l.h. } \{U^k f : f \in \mathfrak{M}_1(\lambda_0) \oplus \mathfrak{M}_2(\lambda_0), k = 0, \pm 1, \cdots\}. \end{aligned}$$

Since the latter set equals $\mathfrak{H}$ by Theorem 2, $[\mathfrak{M}_1(\lambda_0) \dotplus \mathfrak{M}_1(\bar{\lambda}_0)]^c$ is a generating subspace for $A$ by Theorem 1. This proves Theorem 3.

(The author wishes to express his appreciation to Professor Robert McKelvey for pointing out that $\mathfrak{M}_1(\lambda_0) \dotplus \mathfrak{M}_1(\bar{\lambda}_0)$ generates the space.

The proof given here is different from McKelvey's.)

REMARK 1. We have used $[\mathfrak{M}_1(\lambda_0) \dot{+} \mathfrak{M}_1(\bar{\lambda}_0)]^c$ in Theorem 3 rather than $\mathfrak{M}_1(\lambda_0) \dot{+} \mathfrak{M}_1(\bar{\lambda}_0)$ for the purely technical reason that we have been discussing generating *subspaces* rather than generating linear manifolds or generating sets.

REMARK 2. Let $A_1$ be a simple closed symmetric operator with deficiency indices $m_1$, $n_1$ with respect to the nonreal number $\lambda_0$. Let $A$ be a minimal selfadjoint extension of $A_1$. Since $\mathfrak{M}_1(\bar{\lambda}_0) \oplus \mathfrak{M}_2(\bar{\lambda}_0)$ is a generating subspace for $A$ and has dimension $m_1 + m_2$, it follows that the spectral multiplicity of $A$ cannot exceed $m_1 + m_2$. Since $m_2 \leqq n_1$, it also follows that the spectral multiplicity cannot exceed $m_1 + n_1$.

## 3. Boundary values of analytic functions with nonnegative imaginary parts.

In the remainder of the paper certain facts will be needed about boundary values of analytic functions with nonnegative imaginary parts. These facts are collected here. Following Kac [9], we define an *R-function* $\theta(\lambda)$ to be a function which is analytic in the upper and lower half-planes and satisfies the conditions $\theta(\bar{\lambda}) = \theta(\lambda)^-$, $\operatorname{Im}\lambda \operatorname{Im}\theta(\lambda) \geqq 0$ $(\operatorname{Im}\lambda \neq 0)$. Here $\theta(\lambda)^-$ stands for the complex conjugate of $\theta(\lambda)$, and Im stands for imaginary part. Each $R$-function admits a representation of the form

$$(1) \qquad \theta(\lambda) = \alpha + \beta\lambda + \int_{-\infty}^{\infty} [(t-\lambda)^{-1} - t(1+t^2)^{-1}]\, d\mu(t),$$

where $\beta \geqq 0$ and $\alpha$ are real constants, and $\mu$ is a nondecreasing function for which $\int_{-\infty}^{\infty} (1+t^2)^{-1}\, d\mu(t) < \infty$. Conversely, if $\theta(\lambda)$ has a representation of the form (1), then $\theta(\lambda)$ is an $R$-function. $\alpha$ and $\beta$ in the representation (1) are unique, and $\mu$ is unique if it is normalized in some way, say, by the conditions

$$\mu(0) = 0, \qquad \mu_{(t)} = (1/2)[\mu(t-0) + \mu(t+0)].$$

The particular normalization used for $\mu$ will not be important to us. $\mu$ is called the *spectral function* of $\theta$.

$\theta(\lambda)$ is called an $R_1$-*function* if it is the difference of two $R$-functions. Each $R_1$-function $\theta(\lambda)$ admits a representation (1) in which $\alpha, \beta$ are real numbers and $\mu$ is a function of bounded variation on each finite segment such that $\int_{-\infty}^{\infty} (1+t^2)^{-1} |d\mu(t)| < \infty$.

The following properties are true for $R_1$-functions. (See Kac [9] or Aronszajn and Donoghue [3].)

(I) If $t_1$, $t_2$ are continuity points of $\mu$,

$$\mu(t_2) - \mu(t_1) = (1/\pi) \lim_{\eta \to 0+} \int_{t_1}^{t_2} \operatorname{Im} \theta(s + i\eta)\, ds.$$

This is called the Stieltjes inversion formula.

(II) If at the point $t$ there exists a finite or infinite symmetric derivative $\mu^{(')}(t) = \lim_{h\to 0+}(1/2h)[\mu(t+h) - \mu(t-h)]$ (which is true a.e.), then $\mu^{(')}(t) = (1/\pi) \lim_{\eta\to 0+} \operatorname{Im} \theta(t + i\eta)$.

(III) $\lim_{\eta \to 0+} \theta(t + i\eta)$ exists and is finite a.e.

(IV) $\eta|\theta(t + i\eta)|$ is uniformly bounded in any rectangle $0 < \eta < h$, $a \leqq t \leqq b$.

The following property is true for $R$-functions.

(V) If $\lim_{\eta\to 0+} \operatorname{Im} \theta(t + i\eta)$ exists and is finite at the point $t$, then $\mu^{(')}(t)$ exists, and $\mu^{(')}(t) = (1/\pi) \lim_{\eta\to 0+} \operatorname{Im} \theta(t + i\eta)$.

The following property involves an $R$-function and an $R_1$-function.

(VI) Suppose $\varphi(\lambda)$ is an $R_1$-function with spectral function $\mu(t)$, and suppose $\psi(\lambda)$ is an $R$-function with spectral function $\nu(t)$. Suppose at the point $t = t_0$ there exists a finite symmetric derivative

$$[d\mu(t)/(d)\nu(t)]_{t=t_0}$$
$$= \lim_{h\to 0+} \{[\mu(t_0 + h) - \mu(t_0 - h)]\, [\nu(t_0 + h) - \nu(t_0 - h)]^{-1}\} = k.$$

Suppose $\nu^{(')}(t_0)$ exists and $\nu^{(')}(t_0) \neq 0$. Then,

$$\lim_{\eta\to 0+} [\operatorname{Im} \varphi(t_0 + i\eta)/\operatorname{Im} \psi(t_0 + i\eta)] = k.$$

If $\theta(\lambda)$ is an $R$-function with spectral function $\mu(t)$, we define $Q_{a+}[\mu]$ to be the set of all points $t$ for which $\lim_{\eta\to 0+} \theta(t + i\eta)$ exists and is finite and nonreal. By $Q_s[\mu]$ we denote the complement of $Q_{a+}[\mu]$. As indicated by Kac [9], $Q_{a+}[\mu]$ is in a certain sense the carrier of the absolutely continuous part of $\mu$, and $Q_s[\mu]$ is the carrier of the singular part. We note that by property (III), $\lim_{\eta\to 0+} \operatorname{Im} \theta(t + i\eta) = 0$ a.e. on $Q_s[\mu]$, and therefore by property (V), $\mu^{(')}(t) = 0$ a.e. on $Q_s[\mu]$.

Let $Q_a[\mu]$ consist of all points $t$ for which $0 < \mu^{(')}(t) < \infty$. From properties (III) and (V) it follows that $Q_{a+}[\mu] \subset Q_a[\mu]$ and that $Q_a[\mu] \setminus Q_{a+}[\mu]$ has Lebesgue measure zero.

4. **Representation of $A$ as a multiplication operator.** Let $A_1$ be a simple closed symmetric operator in a Hilbert space $\mathfrak{H}_1$. Let $A_1$ have deficiency indices 1, 1 with respect to the nonreal number $\lambda_0$. Let $A$ be a minimal selfadjoint extension of $A_1$ in a Hilbert space $\mathfrak{H}$. Let $E(t)$ be the spectral function of $A$. Let $g_1, g_2$ be nonzero elements in $\mathfrak{M}_1(\lambda_0)$, $\mathfrak{M}_1(\bar{\lambda}_0)$, respectively. We define the matrix $S(t) =$

$\|\sigma_{jk}(t)\|_{j,k=1}^{2}$ by means of the equations $\sigma_{jk}(t) = (E(t)g_j, g_k)$, $j$, $k = 1, 2$. For each $t$, $S(t)$ is a Hermitian matrix. $S(t)$ is a nondecreasing function of $t$.

Let $\sigma(t) = \sigma_{11}(t) + \sigma_{22}(t)$. Let

$$\begin{aligned}\delta_{jk}(t) &= d\sigma_{jk}(t)/(d)\sigma(t)\\ &= \lim_{h\to 0+} \{[\sigma_{jk}(t+h) - \sigma_{jk}(t-h)]\,[\sigma(t+h) - \sigma(t-h)]^{-1}\}.\end{aligned}$$

The $\delta_{jk}(t)$ exist everywhere except on a set of $\sigma$-measure zero, and $0 \leqq \delta_{11}(t) \leqq 1$, $0 \leqq \delta_{22}(t) \leqq 1$, $\delta_{12}(t) = \delta_{21}(t)^{-}$, $\delta_{11}(t) + \delta_{22}(t) = 1$, $\delta_{11}(t)\delta_{22}(t) - \delta_{12}(t)\delta_{21}(t) \geqq 0$. By $L_S{}^2(-\infty, \infty)$ (or, more simply, $L_S{}^2$) we shall denote the Hilbert space consisting of all complex-valued vector functions $\vec{f}(t) = [f_1(t), f_2(t)]$ whose components are $\sigma$-measurable and for which

$$\|\vec{f}\|_S^2 = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} f_j(t) f_k(t)^{-} \delta_{jk}(t)\, d\sigma(t) < \infty.$$

Here $f_k(t)^{-}$ stands for the complex conjugate of $f_k(t)$. The inner product of two elements $\vec{f}, \vec{h} \in L_S{}^2$ is given by

$$(\vec{f}, \vec{h})_S = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} f_j(t) h_k(t)^{-} \delta_{jk}(t)\, d\sigma(t).$$

The symbol $\sum_{j,k=1}^{2} \int_{-\infty}^{\infty} f_j(t) h_k(t)^{-} d\sigma_{jk}(t)$ is often used for the integral above. By $T_S$ we denote the selfadjoint operator in $L_S{}^2$ consisting of multiplication by $t$. For further details concerning the space $L_S{}^2$ and the operator $T_S$, see Kac [9] or Dunford and Schwartz [5, XIII.5.9].

By Theorem 3, $\mathfrak{M}_1(\lambda_0) \dot{+} \mathfrak{M}_1(\bar{\lambda}_0)$ is a generating subspace for $A$. Hence, elements of the form $\sum_{j=1}^{m} E(\Delta_j)(a_j g_1 + b_j g_2)$ are dense in $\mathfrak{H}$. Without loss of generality, we can assume that the $\Delta_j$ are disjoint. Let the operator $W$ be defined by the equation

$$W\left[\sum_{j=1}^{m} E(\Delta_j)(a_j g_1 + b_j g_2)\right] = \left[\sum_{j=1}^{m} a_j \chi(\Delta_j),\ \sum_{j=1}^{m} b_j \chi(\Delta_j)\right],$$

where $\chi(\Delta_j)$ is the characteristic function of the interval $\Delta_j$. Then, it can be shown that $W$ is well defined. Suppose, for example, that $h = E(\Delta)f = E(\Delta')f' = h'$, where $f = ag_1 + bg_2$, $f' = a'g_1 + b'g_2$. Then, $Wh = [a\chi(\Delta), b\chi(\Delta)]$, $Wh' = [a'\chi(\Delta'), b'\chi(\Delta')]$. We wish to show that $\|Wh - Wh'\|_S^2 = 0$. Now,

$$Wh = [a\chi(\Delta - \Delta') + a\chi(\Delta \cap \Delta'), b\chi(\Delta - \Delta') + b\chi(\Delta \cap \Delta')],$$

$$Wh' = [a'\chi(\Delta' - \Delta) + a'\chi(\Delta \cap \Delta'),$$
$$b'\chi(\Delta' - \Delta) + b'\chi(\Delta \cap \Delta')],$$

so that

$$Wh - Wh' = [a\chi(\Delta - \Delta') - a'\chi(\Delta' - \Delta) + (a - a')\chi(\Delta \cap \Delta'),$$
$$b\chi(\Delta - \Delta') - b'\chi(\Delta' - \Delta) + (b - b')\chi(\Delta \cap \Delta')].$$

From this expression we can see that

$$\|Wh - Wh'\|_S^2 = (E(\Delta - \Delta')f, f) + (E(\Delta' - \Delta)f', f') + (E(\Delta \cap \Delta')(f - f'), (f - f')). \tag{2}$$

Since $E(\Delta)f = E(\Delta')f'$, we have that $E(\Delta - \Delta')f + E(\Delta \cap \Delta')f = E(\Delta' - \Delta)f' + E(\Delta \cap \Delta')f'$, and therefore $E(\Delta - \Delta')f = E(\Delta' - \Delta)f' = 0$, and $E(\Delta \cap \Delta')f = E(\Delta \cap \Delta')f'$. From these last equations and (2) it follows that $\|Wh - Wh'\|_S^2 = 0$ which is what we wished to prove. One may proceed similarly if $h$ and $h'$ involve more terms.

By arguments similar to the above we can also show that $W$ is linear and isometric. $W$ can therefore be extended to all of $\mathfrak{H}$. It can then be shown that $W$ is an isometry which maps $\mathfrak{H}$ onto $L_S{}^2$ and takes the spectral function of $A$ into the spectral function of $T_S$. Thus, $A$ and $T_S$ are unitarily equivalent via the operator $W$.

Now let the matrix $\mathrm{P}(t) = \|\rho_{jk}(t)\|_{j,k=1}^2$ be defined by the equations $\rho_{jk}(t) = \int_0^t (1 + s^2) d\sigma_{jk}(s)$. $\mathrm{P}(t)$ is a nondecreasing Hermitian matrix function of $t$. Let $\rho(t) = \rho_{11}(t) + \rho_{22}(t)$. $d\rho(t)$ and $d\sigma(t)$ are equivalent measures. Let $\Delta_{jk}(t) = d\rho_{jk}(t)/(d)\rho(t)$. The $\Delta_{jk}(t)$ have the same properties previously enumerated for the $\delta_{jk}(t)$. $L_P{}^2$ and $T_P$ are defined like $L_S{}^2$ and $T_S$. It is not difficult to see that the multiplication operator $T_S$ in $L_S{}^2$ is unitarily equivalent to the multiplication operator $T_P$ in $L_P{}^2$ under the map $[f_1(t), f_2(t)] \to [f_1(t)(t - i)^{-1}, f_2(t)(t - i)^{-1}]$. It turns out that in what follows it will be more convenient for us to use $L_P{}^2$ and $T_P$ rather than $L_S{}^2$ and $T_S$. This is because of the relation between the measures $d\rho_{jk}(t)$ and certain analytic functions $\Phi_{jk}(\lambda)$ to be defined.

Let us summarize the preceding results in the form of a theorem.

THEOREM 4. *Let $A_1$ be a simple closed symmetric operator in a Hilbert space $\mathfrak{H}_1$. Let $A_1$ have deficiency indices* 1, 1 *with respect to the nonreal number $\lambda$. Let $A$ be a minimal selfadjoint extension of $A_1$ in a Hilbert space $\mathfrak{H}$. Let $E(t)$ be the spectral function of $A$. Let $g_1$, $g_2$ be nonzero elements in $\mathfrak{M}_1(\lambda_0)$, $\mathfrak{M}_1(\bar{\lambda}_0)$, respectively. Let* *the matrix* $\mathrm{P}(t) = \|\rho_{jk}(t)\|_{j,k=1}^{2}$ *be defined by means of the equations* $\rho_{jk}(t) = \int_0^t (1 + s^2) d(E(s)g_j, g_k)$, $j, k = 1, 2$. *Let* $\rho(t) = \rho_{11}(t) + \rho_{22}(t)$, *and let* $\Delta_{jk}(t) = d\rho_{jk}(t)/(d)\rho(t)$. *Then, A is unitarily equivalent to the multiplication operator* $T_{\mathrm{P}}$ *in* $L_{\mathrm{P}}{}^2$, *where* $L_{\mathrm{P}}{}^2$ *is the Hilbert space consisting of all vector functions* $\vec{f}(t) = [f_1(t), f_2(t)]$ *whose components are* $\rho$*-measurable and for which*

$$\|\vec{f}\|_{\mathrm{P}}^2 = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} f_j(t) f_k(t)^- \Delta_{jk}(t) d\rho(t) < \infty .$$

*Under this equivalence,* $g_1 \leftrightarrow [(t-i)^{-1}, 0]$, *and* $g_2 \leftrightarrow [0, (t-i)^{-1}]$.

Remark. Let $\hat{g}_1(t) = [(t-i)^{-1}, 0]$, $\hat{g}_2(t) = [0, (t-i)^{-1}]$. Let $\hat{P}_j$ $(j = 1, 2)$ be the operator of orthogonal projection of $L_{\mathrm{P}}{}^2$ onto the space perpendicular to $\hat{g}_j$. Let $\hat{U}$ be the Cayley transform of $T_{\mathrm{P}}$, i.e., $(\hat{U}\hat{f})(t) = (t - \bar{\lambda}_0)(t - \lambda_0)^{-1}\hat{f}(t)$. Let the image of $\mathfrak{L}_1(\bar{\lambda}_0)$ in $L_{\mathrm{P}}{}^2$ be denoted by $\hat{\mathfrak{L}}_1(\bar{\lambda}_0)$. Then $\hat{\mathfrak{L}}_1(\bar{\lambda}_0)$ is the closed linear hull of the elements $\hat{P}_2\hat{g}_1$, $(\hat{P}_2\hat{U})\hat{P}_2\hat{g}_1$, $(\hat{P}_2\hat{U})^2\hat{P}_2\hat{g}_1$, $\cdots$, $(\hat{U}^{-1}\hat{P}_1)\hat{g}_2$, $(\hat{U}^{-1}\hat{P}_1)^2\hat{g}_2$, $(\hat{U}^{-1}\hat{P}_1)^3\hat{g}_2$, $\cdots$. Let $\hat{\mathfrak{D}}(A_1)$ be the image of $\mathfrak{D}(A_1)$ in $L_{\mathrm{P}}{}^2$, and let $\hat{A}_1$ be the image of $A_1$. Then, $\hat{\mathfrak{D}}(A_1)$ is the closure in the graph norm of $T_{\mathrm{P}}$ of the linear hull of the elements

$$(t-\lambda_0)^{-1}\hat{P}_2\hat{g}_1(t), (t-\lambda_0)^{-1}(\hat{P}_2\hat{U})\hat{P}_2\hat{g}_1(t), (t-\lambda_0)^{-1}(\hat{P}_2\hat{U})^2\hat{P}_2\hat{g}_1(t), \cdots,$$
$$(t-\bar{\lambda}_0)^{-1}\hat{P}_1\hat{g}_2(t), (t-\bar{\lambda}_0)^{-1}(\hat{P}_1\hat{U}^{-1})\hat{P}_1\hat{g}_2(t),$$
$$(t-\bar{\lambda}_0)^{-1}(\hat{P}_1\hat{U}^{-1})^2\hat{P}_1\hat{g}_2(t), \cdots .$$

$\hat{A}_1$ is the restriction of $T_{\mathrm{P}}$ to $\hat{\mathfrak{D}}(A_1)$.

Let $\rho^{(\prime)}(t) = d\rho(t)/(d)t = \lim_{h\to 0+}(1/2h)[\rho(t+h) - \rho(t-h)]$, and similarly for $\overset{(\prime)}{\rho_{jk}}(t)$. Let $\mathcal{E}$ be the set of points $t$ for which $\rho^{(\prime)}(t)$ exists, $0 < \rho^{(\prime)}(t) \leqq \infty$, and for which the $\Delta_{jk}(t)$ all exist. Then, $\rho(\sim\mathcal{E}) = 0$, where, $\sim\mathcal{E}$ stands for the complement of $\mathcal{E}$. (See Kac [9].) Let $\vartheta$ be the set of all points $t \in \mathcal{E}$ for which the rank of $\Delta(t) = \|\Delta_{jk}(t)\|_{j,k=1}^{2}$ is 2, i.e., for which $\det \Delta(t) = \Delta_{11}(t)\Delta_{22}(t) - \Delta_{12}(t)\Delta_{21}(t) > 0$. If $f \in L_{\mathrm{P}}{}^2$, one may show (see Kac [9]) that $\|\vec{f}\|_{\mathrm{P}}^2 = \int_{\mathcal{E}} |\tilde{f}_1(t)|^2 d\rho(t) + \int_{\vartheta} |\tilde{f}_2(t)|^2 d\rho(t)$, where $\vec{f}(t) = [f_1(t), f_2(t)]$, $\tilde{f}_1(t) = f_1(t)\Delta_{11}^{1/2}(t) + f_2(t)\Delta_{21}(t)\Delta_{11}^{-1/2}(t)$ on $\mathcal{E} \cap \{t : \Delta_{11}(t) > 0\}$, $\tilde{f}_1(t) = f_2(t)\Delta_{22}^{1/2}(t)$ on $\mathcal{E} \cap \{t : \Delta_{11}(t) = 0\}$, $\tilde{f}_2(t) = [\det \Delta(t)]^{1/2}\Delta_{11}^{-1/2}(t) f_2(t)$ on $\vartheta$. From this representation it follows that the multiplication operator $T_{\mathrm{P}}$ in $L_{\mathrm{P}}{}^2$ (and therefore $A$ in $\mathfrak{H}$) is unitarily equivalent to the multiplication operator in $L_\rho{}^2(\mathcal{E}) \oplus L_\rho{}^2(\vartheta)$. We summarize these results in the following theorem.

Theorem 5. *Let the hypotheses of Theorem 4 hold. Let* $\mathcal{E}$ *be the set of all points t for which* $\rho^{(\prime)}(t)$ *exists and* $0 < \rho^{(\prime)}(t) \leqq \infty$, *and for* *which the* $\Delta_{jk}(t)$ *all exist. Let* $\mathcal{G}$ *be the set of all points* $t \in \mathcal{E}$ *for which* $\det \Delta(t) > 0$. *Then,* $T_P$ *(and therefore* $A$*) is unitarily equivalent to the multiplication operator in* $L_\rho^2(\mathcal{E}) \oplus L_\rho^2(\mathcal{G})$. *Thus, the spectrum of* $A$ *is determined by the behavior of* $\rho$; *and if the* $\rho$*-measure of* $\mathcal{G}$ *is positive, the spectral multiplicity of* $A$ *is* 2, *but if the* $\rho$*-measure of* $\mathcal{G}$ *is zero, then the spectral multiplicity of* $A$ *is* 1.

5. **Spectrum and spectral multiplicity of** $A$. Let the hypotheses of Theorems 4 and 5 hold. Let $P_1$ be the operator of orthogonal projection onto $\mathfrak{H}_1$. If $R(\lambda)$ is the resolvent of $A$, then the operator $R_1(\lambda) = P_1R(\lambda)$, restricted to $\mathfrak{H}_1$, is called a *generalized resolvent* of $A_1$; and the operator $E_1(t) = P_1E(t)$, restricted to $\mathfrak{H}_1$, is called a *generalized spectral function* of $A_1$. If $\mathfrak{H} = \mathfrak{H}_1$, then, of course, $R_1(\lambda) = R(\lambda)$ is called simply a resolvent, and $E_1(t) = E(t)$ is called simply a spectral function. $R_1(\lambda)$ and $E_1(t)$ are related by the equation $R_1(\lambda) = \int_{-\infty}^{\infty} (t-\lambda)^{-1}dE_1(t)$.

Now let the matrix $\Phi(\lambda) = \|\Phi_{jk}(\lambda)\|_{j,k=1}^2$, $\lambda$ complex, be defined by the equations

$$\Phi_{jk}(\lambda) = \lambda(g_j, g_k) + (\lambda^2+1)(R_1(\lambda)g_j, g_k), \qquad j, k = 1, 2. \tag{3}$$

It can be checked that

$$\Phi_{jk}(\lambda) = \int_{-\infty}^{\infty} [(t-\lambda)^{-1} - t(1+t^2)^{-1}]\, d\rho_{jk}(t). \tag{4}$$

$\Phi_{jk}(\lambda)$ is analytic for $\lambda$ in the upper or lower half-planes.

From equation (4) and the properties of $R$-functions given in §3, we shall see that the behavior of $\rho_{jk}(t)$ is determined by the behavior of $\Phi_{jk}(\lambda)$ as $\lambda$ approaches the real axis (and the $\Delta_{jk}(t)$ are determined by the $\rho_{jk}(t)$). Hence, in order to compute $\rho(t)$ and the $\Delta_{jk}(t)$ and thus (by Theorem 5) study the spectrum and spectral multiplicity of $A$, we shall study the $\Phi_{jk}(\lambda)$ and their behavior as $\lambda$ approaches the real axis. Firstly we shall obtain expressions for the $\Phi_{jk}(\lambda)$ which can be used in studying this behavior.

Let $A_0$ be a fixed selfadjoint extension of $A_1$ in $\mathfrak{H}_1$. (Since $A_1$ has equal deficiency indices 1, 1, there exists such a selfadjoint extension $A_0$.) Let $\lambda_0$ have positive imaginary part. According to M. G. Kreĭn,

$$R_1(\lambda)f = R_0(\lambda)f - [\theta(\lambda) + Q(\lambda)]^{-1}(f, g(\bar{\lambda}))g(\lambda), \tag{5}$$

where $R_0(\lambda)$ is the resolvent of $A_0$,

$$g(\lambda) = g_1 + (\lambda - \lambda_0)R_0(\lambda)g_1, \tag{6}$$

$$Q(\lambda) = i \operatorname{Im} \lambda_0 + (\lambda - \lambda_0)(g_1, g(\bar{\lambda})), \tag{7}$$

and $\theta(\lambda)$ is an $R$-function. There is a one-one correspondence between selfadjoint extensions $A$ of $A_1$ and $R$-functions $\theta(\lambda)$. If $\theta(\lambda)$ is identically equal to a real constant, then $A$ is an extension of $A_1$ in $\mathfrak{H}_1$, i.e., $\mathfrak{H} = \mathfrak{H}_1$. $Q(\lambda)$ is a fixed $R$-function with positive imaginary part in the upper half-plane. It depends on $A_1$ and $A_0$ but not on $A$. It is true that $g(\lambda) \in \mathfrak{M}_1(\lambda)$. (The proofs of these facts may be found in Achieser and Glasman [1, Appendix I, §4], and in Gilbert [7].)

We shall now make particular selections for $g_1$, $g_2$ and $\lambda_0$ which will simplify our computations. Let $\lambda_0 = i$. Let $g_1 \in \mathfrak{M}_1(i)$, $\|g_1\| = 1$. Let

$$g_2 = g(-i) = g_1 - 2iR_0(-i)g_1. \tag{8}$$

Then, $g_2 \in \mathfrak{M}_1(-i)$, and it can be checked that $\|g_2\| = \|g_1\| = 1$.

Using equations (6) and (8), we can show that

$$(R_0(\lambda)g_2, g_2) = (R_0(\lambda)g_1, g_1), \tag{9}$$

$$(g(\lambda), g_1) = (g_2, g(\bar{\lambda})) = 1 + (\lambda - i)(R_0(\lambda)g_1, g_1), \tag{10}$$

$$(g_1, g(\bar{\lambda})) = (g(\lambda), g_2) = 1 + (\lambda + i)(R_0(\lambda)g_1, g_1). \tag{11}$$

From the above equations and equation (5) it follows that

$$(R_1(\lambda)g_1, g_1) = (R_1(\lambda)g_2, g_2), \tag{12}$$

and therefore, by equation (3), that

$$\Phi_{11}(\lambda) = \Phi_{22}(\lambda). \tag{13}$$

From equations (7) and (11) we see that

$$Q(\lambda) = \lambda + (\lambda^2 + 1)(R_0(\lambda)g_1, g_1). \tag{14}$$

Equation (14) implies that $Q(\lambda)$ is an $R$-function with spectral function $q(t) = (E_0(t)g_1, g_1)$, where $E_0(t)$ is the spectral function of $A_0$.

From equations (3), (5), (10), (11), (14) it can be seen that

$$\Phi_{11}(\lambda) = [\theta(\lambda)Q(\lambda) - 1][\theta(\lambda) + Q(\lambda)]^{-1}. \tag{15}$$

Using equations (5), (8), (10), (11), (14), we can show that the terms $(\lambda^2 + 1)(R_1(\lambda)g_1, g_2)$ and $(\lambda^2 + 1)(R_1(\lambda)g_2, g_1)$, which appear in $\Phi_{12}(\lambda)$ and $\Phi_{21}(\lambda)$ by equation (3), are given by

$$\begin{aligned}(\lambda^2 + 1)(R_1(\lambda)g_1, g_2) &= (\lambda + i)(\lambda - i)^{-1}Q(\lambda) - \lambda(\lambda + i)(\lambda - i)^{-1} \\ &\quad -2i(\lambda + i)(R_0(i)g_1, g_1) \\ &\quad - (\lambda + i)(\lambda - i)^{-1}[Q(\lambda) - i]^2[\theta(\lambda) + Q(\lambda)]^{-1},\end{aligned} \tag{16}$$

$$\begin{aligned}(\lambda^2+1)(R_1(\lambda)g_2,g_1) &= (\lambda-i)(\lambda+i)^{-1}Q(\lambda)-\lambda(\lambda-i)(\lambda+i)^{-1}\\ &\quad+2i(\lambda-i)(R_0(-i)g_1,g_1)\\ &\quad-(\lambda-i)(\lambda+i)^{-1}[Q(\lambda)+i]^2[\theta(\lambda)+Q(\lambda)]^{-1}.\end{aligned} \tag{17}$$

Since equation (4) shows that $\Phi_{11}(\lambda)$ and $\Phi_{22}(\lambda)$ are $R$-functions, it is true that $\rho_{jj}(t_2)-\rho_{jj}(t_1)=(1/\pi)\lim_{\eta\to0+}\int_{t_1}^{t_2}\operatorname{Im}\Phi_{jj}(t+i\eta)\,dt$, $j=1,2$, at continuity points $t_1$, $t_2$ of $\rho_{11}(t)$ and $\rho_{22}(t)$. Hence, by (13), $\rho_{11}(t_2)-\rho_{11}(t_1)=\rho_{22}(t_2)-\rho_{22}(t_1)$, and

$$\rho(t_2)-\rho(t_1)=2[\rho_{11}(t_2)-\rho_{11}(t_1)]=2[\rho_{22}(t_2)-\rho_{22}(t_1)].$$

Since

$$\begin{aligned}\Delta_{jk}(t) &= d\rho_{jk}(t)/(d)\rho(t)\\ &= \lim_{h\to0+}[\rho_{jk}(t+h)-\rho_{jk}(t-h)][\rho(t+h)-\rho(t-h)]^{-1},\end{aligned}$$

it follows that

$$\Delta_{11}(t)=\Delta_{22}(t)=1/2 \tag{18}$$

for all $t\in\mathcal{E}$. Also, by (4),

$$2\Phi_{11}(\lambda)=\int_{-\infty}^{\infty}[(t-\lambda)^{-1}-t(1+t^2)^{-1}]\,d\rho(t). \tag{19}$$

From equations (15) and (19) and the properties of $R$-functions, we see that the behavior of $\rho$, and therefore the spectrum of $T_{\mathrm{P}}$, is determined by the behavior of the imaginary part of the function $2\Phi_{11}(\lambda)=2[\theta(\lambda)Q(\lambda)-1][\theta(\lambda)+Q(\lambda)]^{-1}$ as $\lambda$ approaches the real axis. This behavior has been studied by McKelvey [**10**, §6], for the case that $A_1$ is a singular Sturm-Liouville operator and by Aronszajn [**2**] and Donoghue [**4**] for the case that $\theta(\lambda)$ is identically equal to a real constant.

Let us turn now to the computation of $\Delta_{12}(t)=\Delta_{21}(t)^{-}$.

Suppose that $t\in\mathcal{E}$ and that $0<\rho^{(\prime)}(t)<\infty$. From

$$\Delta_{12}(t)=\lim_{h\to0+}\{[\rho_{12}(t+h)-\rho_{12}(t-h)][\rho(t+h)-\rho(t-h)]^{-1}\} \tag{20}$$

we see that

$$\Delta_{12}(t)=\rho_{12}^{(\prime)}(t)/\rho^{(\prime)}(t) \tag{21}$$

and that $\rho_{12}^{(\prime)}(t)$ and $\rho_{21}^{(\prime)}(t)=\rho_{12}^{(\prime)}(t)^{-}$ exist and are finite. We already know that $\Delta_{11}(t)=\Delta_{22}(t)=1/2$, and therefore that $\rho_{11}^{(\prime)}(t)$ and $\rho_{22}^{(\prime)}(t)$ exist and are finite. If $\xi=[\xi_1,\xi_2]$, $\zeta=[\zeta_1,\zeta_2]$ are arbitrary vectors with complex components, then by (4),

$$(22) \qquad (\Phi(\lambda)\xi,\zeta) = \int_{-\infty}^{\infty} [(t-\lambda)^{-1} - t(1+t^2)^{-1}]\, d(\mathrm{P}(t)\xi,\zeta),$$

where $(\Phi(\lambda)\xi,\zeta) = \sum_{j,k=1}^{2}\Phi_{jk}(\lambda)\xi_k\bar{\zeta}_j$ and similarly for $(\mathrm{P}(t)\xi,\zeta)$. Since each $\rho_{jk}^{(\prime)}(t)$ exists and is finite, $d(\mathrm{P}(t)\xi,\zeta)/(d)t$ exists and is finite, for $d(\mathrm{P}(t)\xi,\zeta)/(d)t = \sum_{j,k=1}^{2}\rho_{jk}^{(\prime)}(t)\xi_k\bar{\zeta}_j$. Since (22) shows that $(\Phi(\lambda)\xi,\xi)$ is an $R$-function, by property (II) of $R$-functions,

$$(23) \qquad d(\mathrm{P}(t)\xi,\xi)/(d)t = (1/\pi) \lim_{\eta\to 0+} \operatorname{Im}(\Phi(\lambda)\xi,\xi),$$

where $\lambda = t + i\eta$. By (23) and polarization,

$$(24) \qquad d(\mathrm{P}(t)\xi,\zeta)/(d)t = (1/2\pi i) \lim_{\eta\to 0+} \sum_{j,k=1}^{2} [\Phi_{jk}(\lambda) - \Phi_{kj}(\lambda)^-]\, \xi_k\bar{\zeta}_j.$$

If we now take $\xi = [0,1]$, $\zeta = [1,0]$, we obtain

$$(25) \qquad \rho_{12}^{(\prime)}(t) = (1/2\pi i) \lim_{\eta\to 0+} [\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-].$$

From (19) and property (II) of $R$-functions we know that

$$(26) \qquad \rho^{(\prime)}(t) = (2/\pi) \lim_{\eta\to 0+} \operatorname{Im}\Phi_{11}(\lambda).$$

From (21), (25) and (26) it follows that if $t \in \mathcal{E}$ and $0 < \rho^{(\prime)}(t) < \infty$, then we can compute $\Delta_{12}(t)$ by the formula

$$(27) \qquad \Delta_{12}(t) = (1/4i) \lim_{\eta\to 0+} [\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-] \Big/ \lim_{\eta\to 0+} \operatorname{Im}\Phi_{11}(\lambda).$$

Suppose now that $t \in \mathcal{E}$ and that $\rho^{(\prime)}(t) = \infty$. Then, from (20) we see that

$$\lim_{h\to 0+} \{[\rho_{12}(t+h) - \rho_{12}(t-h)]\,[\rho(t+h) - \rho(t-h)]^{-1}\}$$

exists and is finite. If $\xi = [0,1]$, $\zeta = [1,0]$,

$$(28) \qquad \rho_{12}(t) = (\mathrm{P}(t)\xi,\zeta) = u(t) + iv(t),$$

where

$$u(t) = (1/4)(\mathrm{P}(t)(\xi+\zeta), \xi+\zeta) - (1/4)(\mathrm{P}(t)(\xi-\zeta), \xi-\zeta),$$

$$v(t) = (1/4)(\mathrm{P}(t)(\xi+i\zeta), \xi+i\zeta) - (1/4)(\mathrm{P}(t)(\xi-i\zeta), \xi-i\xi).$$

It follows that

$$k_1 = \lim_{h\to 0+} \{[u(t+h) - u(t-h)]\,[\rho(t+h) - \rho(t-h)]^{-1}\}$$

and

$$k_2 = \lim_{h\to 0+} \{[v(t+h) - v(t-h)]\,[\rho(t+h) - \rho(t-h)]^{-1}\}$$

exist and are finite, and that

(29) $$\Delta_{12}(t) = k_1 + ik_2.$$

Now, $u(t)$ is the spectral function of the $R_1$-function

$$\varphi_1(\lambda) = (1/4)(\Phi(\lambda)(\xi + \zeta), \xi + \zeta) - (1/4)(\Phi(\lambda)(\xi - \zeta), \xi - \zeta),$$

and $v(t)$ is the spectral function of the $R_1$-function

$$\varphi_2(\lambda) = (1/4)(\Phi(\lambda)(\xi + i\zeta), \xi + i\zeta) - (1/4)(\Phi(\lambda)(\xi - i\zeta), \xi - i\zeta).$$

$\rho(t)$ is the spectral function of the $R$-function $2\Phi_{11}(\lambda)$. Hence, by property (VI) of $R$-functions,

(30) $$k_1 = (1/2) \lim_{\eta\to 0+} [\operatorname{Im}\varphi_1(\lambda)/\operatorname{Im}\Phi_{11}(\lambda)],$$

(31) $$k_2 = (1/2) \lim_{\eta\to 0+} [\operatorname{Im}\varphi_2(\lambda)/\operatorname{Im}\Phi_{11}(\lambda)].$$

From (29), (30), (31) we see that

$$\begin{aligned}\Delta_{12}(t) &= (1/2) \lim_{\eta\to 0+} \{[\operatorname{Im}\varphi_1(\lambda) + i\operatorname{Im}\varphi_2(\lambda)]/\operatorname{Im}\Phi_{11}(\lambda)\}\\ &= (1/4i) \lim_{\eta\to 0+} \{[(\Phi(\lambda)\xi, \zeta) - (\Phi(\lambda)\zeta, \xi)^-]/\operatorname{Im}\Phi_{11}(\lambda)\}\\ &= (1/4i) \lim_{\eta\to 0+} \{[\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-]/\operatorname{Im}\Phi_{11}(\lambda)\}.\end{aligned}$$

Thus, if $t \in \mathcal{E}$ and $\rho^{(\prime)}(t) = \infty$, we can compute $\Delta_{12}(t)$ by means of the formula

(32) $$\Delta_{12}(t) = (1/4i) \lim_{\eta\to 0+} \{[\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-]/\operatorname{Im}\Phi_{11}(\lambda)\}.$$

Let us summarize the above results in the form of a theorem.

THEOREM 6. *Let the hypotheses of Theorems* 4 *and* 5 *hold with* $\lambda_0 = i$. *Let* $R_1(\lambda)$ *be the generalized resolvent of* $A_1$ *corresponding to the selfadjoint extension* A. *Let* $A_0$ *be a selfadjoint extension of* $A_1$ *in the space* $\mathfrak{H}_1$ *with resolvent* $R_0(\lambda)$. *Let* $g_1$ *have norm* 1, *and let* $g_2 = g_1 - 2iR_0(-i)g_1$. *Let*

(33) $$\Phi_{jk}(\lambda) = \lambda(g_j, g_k) + (\lambda^2 + 1)(R_1(\lambda)g_j, g_k), \qquad j, k = 1, 2.$$

*Then,* $\Delta_{11}(t) = \Delta_{22}(t) = 1/2$ *for all* $t \in \mathcal{E}$. *If* $t \in \mathcal{E}$ *and* $0 < \rho^{(\prime)}(t) < \infty$, *then*

(34) $\Delta_{12}(t) = \Delta_{21}(t)^- = (1/4i) \lim_{\eta\to 0+} [\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-] / \lim_{\eta\to 0+} \operatorname{Im}\Phi_{11}(\lambda).$

*If $t \in \mathcal{E}$ and $\rho^{(\prime)}(t) = \infty$, then*

(35) $\Delta_{12}(t) = \Delta_{21}(t)^- = (1/4i) \lim_{\eta\to 0+} \{[\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-] / \operatorname{Im}\Phi_{11}(\lambda)\}.$

*The expressions* $(\lambda^2 + 1)(R_1(\lambda)g_1, g_2)$ *and* $(\lambda^2 + 1)(R_1(\lambda)g_2, g_1)$ *in equations* (33) *for* $\Phi_{12}(\lambda)$ *and* $\Phi_{21}(\lambda)$ *are given in terms of* $Q(\lambda)$ *and* $\theta(\lambda)$ *by equations* (16) *and* (17). *Here* $Q(\lambda) = \lambda + (\lambda^2 + 1)(R_0(\lambda)g_1, g_1)$, *and* $\theta(\lambda)$ *is the R-function corresponding to A in Kreĭn's formula* (5). $\Phi_{11}(\lambda)$ *is given in terms of* $Q(\lambda)$ *and* $\theta(\lambda)$ *by equation* (15).

We now wish to evaluate the limits in (34) and (35). This can be simplified by the following remarks. We first observe that equations (33), (16) and (17) show that

$$\begin{aligned}
&\lim_{\eta\to 0+} [\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-] \\
&\quad = \lim_{\eta\to 0+} \{(\lambda + i)(\lambda - i)^{-1}Q(\lambda) - (\bar\lambda + i)(\bar\lambda - i)^{-1}Q(\lambda)^- \\
&\qquad - (\lambda + i)(\lambda - i)^{-1}[Q(\lambda) - i]^2[\theta(\lambda) + Q(\lambda)]^{-1} \\
&\qquad + (\bar\lambda + i)(\bar\lambda - i)^{-1}[Q(\lambda)^- - i]^2[\theta(\lambda)^- + Q(\lambda)^-]^{-1}\}.
\end{aligned} \tag{36}$$

Since $Q - [Q - i]^2[\theta + Q]^{-1} = 2i + \theta - [\theta + i]^2[\theta + Q]^{-1}$, it follows from (36) that

$$\begin{aligned}
&\lim_{\eta\to 0+} [\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-] \\
&\quad = \lim_{\eta\to 0+} \{(\lambda + i)(\lambda - i)^{-1}\theta(\lambda) - (\bar\lambda + i)(\bar\lambda - i)^{-1}\theta(\lambda)^- \\
&\qquad - (\lambda + i)(\lambda - i)^{-1}[\theta(\lambda) + i]^2[\theta(\lambda) + Q(\lambda)]^{-1} \\
&\qquad + (\bar\lambda + i)(\bar\lambda - i)^{-1}[\theta(\lambda)^- + i]^2[\theta(\lambda)^- + Q(\lambda)^-]^{-1}\}.
\end{aligned} \tag{37}$$

Equations (36) and (37) can be used in equation (34).

Suppose that $t \in \mathcal{E}$ and $\rho^{(\prime)}(t) = \infty$. From (19) and property (II) of $R$-functions it follows that $\lim_{\eta\to 0+}\operatorname{Im}\Phi_{11}(\lambda) = \infty$. Using (33), (16) and (17), we see, then, that

$$\begin{aligned}
&\lim_{\eta\to 0+} \{[\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-] / \operatorname{Im}\Phi_{11}(\lambda)\} \\
&\quad = \lim_{\eta\to 0+} (\{(\lambda + i)(\lambda - i)^{-1}[\Omega_{11}(\lambda) + 2i\Omega_{12}(\lambda) - \Omega_{22}(\lambda)] \\
&\qquad - (\bar\lambda + i)(\bar\lambda - i)^{-1}[\Omega_{11}(\lambda)^- + 2i\Omega_{12}(\lambda)^- - \Omega_{22}(\lambda)^-]\} \\
&\qquad \cdot \{\operatorname{Im}\Phi_{11}(\lambda)\}^{-1}),
\end{aligned} \tag{38}$$

where we have put $\Omega_{11}(\lambda) = \theta(\lambda)Q(\lambda)[\theta(\lambda) + Q(\lambda)]^{-1}$, $\Omega_{12}(\lambda) = \Omega_{21}(\lambda) = Q(\lambda)[\theta(\lambda) + Q(\lambda)]^{-1}$, $\Omega_{22}(\lambda) = -[\theta(\lambda) + Q(\lambda)]^{-1}$. Then, $\Phi_{11}(\lambda) = \Omega_{11}(\lambda) + \Omega_{22}(\lambda)$. For arbitrary complex numbers $\xi_1, \xi_2$ it can be checked that $\sum_{j,k=1}^{2}\Omega_{jk}(\lambda)\xi_k\bar{\xi}_j$ is an $R$-function. It follows that $\Omega_{11}(\lambda)$ and $\Omega_{22}(\lambda)$ are $R$-functions, and $\Omega_{12}(\lambda) = \Omega_{21}(\lambda)$ is an $R_1$-function. Hence, by property (IV) of $R_1$-functions, $|\eta\Omega_{11}(\lambda)|$, $|\eta\Omega_{12}(\lambda)|$ and $|\eta\Omega_{22}(\lambda)|$, where $\lambda = t + i\eta$, are all bounded with respect to $\eta$. If we factor $(\lambda + i)(\lambda - i)^{-1}$ out of (38), put $\lambda = t + i\eta$ and take a limit as $\eta \to 0+$, we obtain

$$\begin{aligned}
&\lim_{\eta\to 0+} \{[\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-]/\operatorname{Im}\Phi_{11}(\lambda)\} \\
&\quad = 2i(t+i)(t-i)^{-1} \\
&\qquad \cdot \lim_{\eta\to 0+} \{[\operatorname{Im}\Omega_{11}(\lambda) + 2i\operatorname{Im}\Omega_{12}(\lambda) - \operatorname{Im}\Omega_{22}(\lambda)]\,[\operatorname{Im}\Phi_{11}(\lambda)]^{-1}\} \\
&\qquad + 4(t+i)(t-i)^{-1}(t^2+1)^{-1} \\
&\qquad \cdot \lim_{\eta\to 0+} \{[\eta\Omega_{11}(\lambda)^- + 2i\eta\Omega_{12}(\lambda)^- - \eta\Omega_{22}(\lambda)^-]\,[\operatorname{Im}\Phi_{11}(\lambda)]^{-1}\}.
\end{aligned} \tag{39}$$

Using the boundedness of the $|\eta\Omega_{jk}(\lambda)|$ and the fact that $\lim_{\eta\to 0+}\operatorname{Im}\Phi_{11}(\lambda) = \infty$, we see that for $t \in \mathcal{E}$ and $\rho^{(\prime)}(t) = \infty$,

$$\begin{aligned}
&\lim_{\eta\to 0+} \{[\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-]/\operatorname{Im}\Phi_{11}(\lambda)\} \\
&\quad = 2i(t+i)(t-i)^{-1} \\
&\qquad \cdot \lim_{\eta\to 0+} \{[\operatorname{Im}\Omega_{11}(\lambda) + 2i\operatorname{Im}\Omega_{12}(\lambda) - \operatorname{Im} Q_{22}(\lambda)]\,[\operatorname{Im}\Phi_{11}(\lambda)]^{-1}\}.
\end{aligned} \tag{40}$$

Now let $\sigma_{jk}(t) = (1/\pi)\lim_{\eta\to 0+}\int_0^t \operatorname{Im}\Omega_{jk}(s + i\eta)\,ds$. Then the $\sigma_{jk}(t)$ are spectral functions of the $\Omega_{jk}(\lambda)$. The matrix $\|\sigma_{jk}(t)\|_{j,k=1}^{2}$ is a nondecreasing Hermitian matrix function of $t$. Let $\sigma(t) = \sigma_{11}(t) + \sigma_{22}(t)$. Then, $\rho(t_2) - \rho(t_1) = 2[\sigma(t_2) - \sigma(t_1)]$ at continuity points $t_1$, $t_2$ of $\rho$ and $\sigma$. It follows that $\rho$ and $\sigma$ are equivalent measures. Let $\delta_{jk}(t) = d\sigma_{jk}(t)/(d)\sigma(t)$. The $\delta_{jk}(t)$ exist and are finite $\sigma$-almost everywhere, and $0 \leqq \delta_{11}(t) \leqq 1$, $0 \leqq \delta_{22}(t) \leqq 1$, $\delta_{12}(t) = \delta_{21}(t)$, $\delta_{11}(t) + \delta_{22}(t) = 1$, $\det\|\delta_{jk}(t)\| \geqq 0$. Since $\rho$ and $\sigma$ are equivalent measures, we can assume without loss of generality that $\mathcal{E}$ was chosen so that the $\delta_{jk}(t)$ exist and are finite on $\mathcal{E}$. (*Note.* The $\sigma_{jk}(t)$ and $\delta_{jk}(t)$ here are to be distinguished from the $\sigma_{jk}(t)$ and $\delta_{jk}(t)$ of §4.)

Now suppose that $t \in \mathcal{E}$, $\rho^{(\prime)}(t) = \infty$, $\delta_{22}(t) > 0$. Since $\sigma_{22}^{(\prime)}(t) = \delta_{22}(t)\rho^{(\prime)}(t)/2 = \infty$, it follows from property (II) of $R$-functions that

$$\lim_{\eta\to 0+} \operatorname{Im} \Omega_{22}(t+i\eta) = \infty. \tag{41}$$

Hence, we write (40) in the form

$$\begin{aligned} &\lim_{\eta\to 0+} \{[\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-]/\operatorname{Im}\Phi_{11}(\lambda)\} \\ &\quad = 2i(t+i)(t-i)^{-1} \\ &\quad\cdot \lim_{\eta\to 0+} \{[(\operatorname{Im}\Omega_{11}/\operatorname{Im}\Omega_{22}) + 2i(\operatorname{Im}\Omega_{12}/\operatorname{Im}\Omega_{22}) - 1] \\ &\qquad\qquad \cdot [(\operatorname{Im}\Omega_{11}/\operatorname{Im}\Omega_{22}) + 1]^{-1}\}. \end{aligned} \tag{42}$$

Since $\Omega_{12} = -Q\Omega_{22}$ and $\Omega_{11} = Q + Q^2\Omega_{22}$, we see that

$$\operatorname{Im}\Omega_{12}/\operatorname{Im}\Omega_{22} = -\operatorname{Im} Q \operatorname{Re}\Omega_{22}/\operatorname{Im}\Omega_{22} - \operatorname{Re} Q, \tag{43}$$

$$\begin{aligned} \operatorname{Im}\Omega_{11}/\operatorname{Im}\Omega_{22} = {} & \operatorname{Im} Q/\operatorname{Im}\Omega_{22} \\ & + 2\operatorname{Re} Q \operatorname{Im} Q \operatorname{Re}\Omega_{22}/\operatorname{Im}\Omega_{22} \\ & + (\operatorname{Re} Q)^2 - (\operatorname{Im} Q)^2. \end{aligned} \tag{44}$$

Since $|\operatorname{Re}\Omega_{22}| \leqq |\Omega_{22}| = |\theta + Q|^{-1} \leqq (\operatorname{Im} Q)^{-1}$, it follows that

$$|\operatorname{Re}\Omega_{22} \operatorname{Im} Q| \leqq 1. \tag{45}$$

From (41) it follows that $\lim_{\eta\to 0+}|\theta + Q|^{-1} = \infty$, and hence,

$$\lim_{\eta\to 0+} |\theta + Q| = 0, \tag{46}$$

$$\lim_{\eta\to 0+} \operatorname{Im} Q = 0. \tag{47}$$

Since

$$\begin{aligned} \delta_{jk}(t) = \delta_{22}(t) \lim_{h\to 0+} \{&[\sigma_{jk}(t+h) - \sigma_{jk}(t-h)] \\ &\cdot [\sigma_{22}(t+h) - \sigma_{22}(t-h)]^{-1}\}, \end{aligned}$$

one sees that

$$\lim_{h\to 0+} \{[\sigma_{jk}(t+h) - \sigma_{jk}(t-h)]\,[\sigma_{22}(t+h) - \sigma_{22}(t-h)]^{-1}\}$$

exists and is finite, and hence, by property (VI) of $R$-functions, $\lim_{\eta\to 0+}[\operatorname{Im}\Omega_{jk}/\operatorname{Im}\Omega_{22}]$ exists and is finite. From this fact and (43), (45), (41) one concludes that $\lim_{\eta\to 0+}\operatorname{Re} Q(t+i\eta)$ exists and is finite (which we denote by $\operatorname{Re} Q(t)$), and that

$$\lim_{\eta\to 0+} [\operatorname{Im}\Omega_{12}/\operatorname{Im}\Omega_{22}] = -\operatorname{Re} Q(t). \tag{48}$$

From (44), (41), (47), (45), it is seen that

(49) $$\lim_{\eta\to 0+} [\operatorname{Im} \Omega_{11}/\operatorname{Im} \Omega_{22}] = [\operatorname{Re} Q(t)]^2.$$

Thus, if $t \in \mathcal{E}, \rho^{(\prime)}(t) = \infty, \delta_{22}(t) > 0$,

(50) $$\begin{aligned}&\lim_{\eta\to 0+} \{[\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-]/\operatorname{Im} \Phi_{11}(\lambda)\}\\ &\quad = 2i(t+i)(t-i)^{-1}[\operatorname{Re} Q(t) - i]\,[\operatorname{Re} Q(t) + i]^{-1}.\end{aligned}$$

Now suppose that $t \in \mathcal{E}$, $\rho^{(\prime)}(t) = \infty$, $\delta_{22}(t) = 0$. Then, $\delta_{11}(t) = 1$, $\delta_{12}(t) = \delta_{21}(t) = 0$. Since $\sigma_{11}^{(\prime)}(t) = \delta_{11}(t)\rho^{(\prime)}(t)/2 = \infty$,

(51) $$\lim_{\eta\to 0+} \operatorname{Im} \Omega_{11}(t + i\eta) = \infty.$$

Since

$$\delta_{jk}(t) = \lim_{h\to 0+} \{[\sigma_{jk}(t+h) - \sigma_{jk}(t-h)]\,[\sigma_{11}(t+h) - \sigma_{11}(t-h)]^{-1}\}$$

exists and is finite, property (VI) of $R$-functions shows that $\delta_{jk}(t) = \lim_{\eta\to 0+} [\operatorname{Im} \Omega_{jk}/\operatorname{Im} \Omega_{11}]$. Hence,

(52) $$\lim_{\eta\to 0+} [\operatorname{Im} \Omega_{12}/\operatorname{Im} \Omega_{11}] = \lim_{\eta\to 0+} [\operatorname{Im} \Omega_{22}/\operatorname{Im} \Omega_{11}] = 0.$$

From (40) and (52) it follows that if $t \in \mathcal{E}$, $\rho^{(\prime)}(t) = \infty$, $\delta_{22}(t) = 0$, then

(53) $$\lim_{\eta\to 0+} \{[\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-]/\operatorname{Im} \Phi_{11}(\lambda)\} = 2i(t+i)(t-i)^{-1}.$$

We are now in a position to prove the following theorem.

**Theorem** 7. *Let the hypotheses of Theorems* 4, 5, 6 *hold. Let $\theta(\lambda)$ have the spectral function $\mu(t)$, and let $Q(\lambda)$ have the spectral function $q(t)$. Then,*

$$Q_{a+}[q] \cap Q_{a+}[\mu] \cap \mathcal{E} \subset \mathcal{G},$$

*and*

$$\mathcal{G} \backslash \{Q_{a+}[q] \cap Q_{a+}[\mu] \cap \mathcal{E}\}$$

*has $\rho$-measure zero.*

**Proof.** We shall show that if $t \in Q_{a+}[q] \cap Q_{a+}[\mu] \cap \mathcal{E}$, then $\det \Delta(t) > 0$, and if $t \in \mathcal{E}\backslash(Q_{a+}[q] \cap Q_{a+}[\mu])$, then $\det \Delta(t) = 0$ a.e. with respect to $\rho$-measure.

Suppose $t \in Q_{a+}[q] \cap Q_{a+}[\mu]$. Then, if $\lambda = t + i\eta$, $\lim_{\eta\to 0+} Q(\lambda)$ and $\lim_{\eta\to 0+} \theta(\lambda)$ exist and are finite and nonreal. Let $\lim_{\eta\to 0+} Q(\lambda) = a + bi$, $b > 0$, and let $\lim_{\eta\to 0+} \theta(\lambda) = c + di$, $d > 0$. From (15),

(54) $\quad \operatorname{Im}\Phi_{11} = [(|\theta|^2 + 1)\operatorname{Im} Q + (|Q|^2 + 1)\operatorname{Im}\theta]\,|\theta + Q|^{-2}.$

Hence, if $t \in Q_{a+}[q] \cap Q_{a+}[\mu]$,

$$\lim_{\eta\to 0+} \operatorname{Im}\Phi_{11}(\lambda) = [(c^2 + d^2 + 1)b + (a^2 + b^2 + 1)d]\,[(a + c)^2 + (b + d)^2]^{-1}. \tag{55}$$

From (55), (19) and property (V) of $R$-functions, it follows that if $t \in Q_{a+}[q] \cap Q_{a+}[\mu]$, then $\rho^{(\prime)}(t)$ exists, and $0 < \rho^{(\prime)}(t) < \infty$. Therefore, if $t \in Q_{a+}[q] \cap Q_{a+}[\mu] \cap \mathcal{E}$, we can use equation (34) to compute $\Delta_{12}(t)$. From (36) we obtain

$$\begin{aligned}\lim_{\eta\to 0+} &[\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-] \\ &= (t + i)(t - i)^{-1}[(c^2 + d^2 - 1)2bi + (a^2 + b^2 - 1)2di + 4(ad - bc)] \\ &\qquad \cdot [(a + c)^2 + (b + d)^2]^{-1}.\end{aligned} \tag{56}$$

By (34), (55), (56), we have

$$\begin{aligned}\Delta_{12}(t) = \Delta_{21}(t)^- &= (1/2)(t + i)(t - i)^{-1} \\ &\cdot [(c^2 + d^2 - 1)b + (a^2 + b^2 - 1)d + 2i(bc - ad)] \\ &\cdot [(c^2 + d^2 + 1)b + (a^2 + b^2 + 1)d]^{-1}.\end{aligned} \tag{57}$$

From (18) and (57) it is seen that if $t \in Q_{a+}[q] \cap Q_{a+}[\mu] \cap \mathcal{E}$, then

$$\det\Delta(t) = bd[(a + c)^2 + (b + d)^2]\,[(c^2 + d^2 + 1)b + (a^2 + b^2 + 1)d]^{-2} > 0.$$

This means that $Q_{a+}[q] \cap Q_{a+}[\mu] \cap \mathcal{E} \subset \mathcal{G}$.

We now wish to consider points $t$ in $\mathcal{E}\backslash(Q_{a+}[q] \cap Q_{a+}[\mu]) = \mathcal{E} \cap (Q_s[q] \cup Q_s[\mu]) = (\mathcal{E} \cap Q_s[q]) \cup (\mathcal{E} \cap Q_s[\mu])$. Let $A = \mathcal{E} \cap Q_s[q]$ and $B = \mathcal{E} \cap Q_s[\mu]$. We shall show that $\det\Delta(t) = 0$ for all $t \in A$ except on a set of $\rho$-measure zero and that $\det\Delta(t) = 0$ for all $t \in B$ except on a set of $\rho$-measure zero. This will prove the theorem.

We know by the properties of $R$-functions that $\lim_{\eta\to 0+}\operatorname{Im} Q(t + i\eta) = 0$ for almost all $t$ in $A$ and that $\lim_{\eta\to 0+}\operatorname{Re} Q(t + i\eta)$ exists and is finite for almost all $t$ in $A$. Let $A_1$ consist of all $t \in A$ for which $0 < \rho^{(\prime)}(t) < \infty$, and let $A_\infty$ consist of all $t \in A$ for which $\rho^{(\prime)}(t) = \infty$. Let $A_1{}'$ consist of all $t \in A_1$ for which $\lim_{\eta\to 0+}\operatorname{Im} Q(t + i\eta) = 0$, $\lim_{\eta\to 0+}\operatorname{Re} Q(t + i\eta)$ exists and is finite, and $\lim_{\eta\to 0+}\operatorname{Re}\Phi_{11}(t + i\eta)$ exists and is finite. (Since $0 < \rho^{(\prime)}(t) < \infty$ on $A_1$, we already know that $\lim_{\eta\to 0+}\operatorname{Im}\Phi_{11}(t+i\eta)$ exists and is finite and positive on $A_1$.) Let $A_1'' = A_1 \backslash A_1'$. By the properties of $R$-functions, $A_1''$ has Lebesgue measure zero. Since $\rho^{(\prime)}(t) < \infty$ on $A_1$, it follows as in the proof of Lemma 5 of Kac [9] that $A_1''$ also has $\rho$-measure zero. Hence, we calculate $\det \Delta(t)$ for $t \in A_1'$ and for $t \in A_\infty$.

On $A_1'$ we can calculate $\Delta_{12}(t)$ by means of (34). Writing $\Phi_{11} = Q - [Q^2+1][\theta + Q]^{-1}$ and using the fact that $\lim_{\eta\to 0+}\operatorname{Im} Q = 0$, $\lim_{\eta\to 0+}\operatorname{Re} Q$ exists and is finite, $\lim_{\eta\to 0+}\Phi_{11}$ exists and is finite, we see that $\lim_{\eta\to 0+}[\theta + Q]^{-1}$ exists and is finite and that

$$\begin{aligned} &\lim_{\eta\to 0+} \operatorname{Im}\Phi_{11}(t+i\eta) \\ &\qquad = -\{[\operatorname{Re} Q(t)]^2 + 1\} \lim_{\eta\to 0+} \operatorname{Im}[\theta(\lambda) + Q(\lambda)]^{-1}, \end{aligned} \tag{58}$$

where we have put

$$\operatorname{Re} Q(t) = \lim_{\eta\to 0+} \operatorname{Re} Q(t+i\eta).$$

$\lim_{\eta\to 0+}\operatorname{Im}[\theta(\lambda)+Q(\lambda)]^{-1} \neq 0$ because $\lim_{\eta\to 0+}\operatorname{Im}\Phi_{11}(\lambda) > 0$. From (36) it follows that

$$\begin{aligned} &\lim_{\eta\to 0+} [\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-] \\ &\qquad = -2i(t+i)(t-i)^{-1}[\operatorname{Re} Q(t) - i]^2 \lim_{\eta\to 0+} \operatorname{Im}[\theta(\lambda)+Q(\lambda)]^{-1}. \end{aligned} \tag{59}$$

From (34), (58), (59) one obtains

$$\begin{aligned} \Delta_{12}(t) &= \Delta_{21}(t)^- \\ &= (1/2)(t+i)(t-i)^{-1}[\operatorname{Re} Q(t) - i][\operatorname{Re} Q(t) + i]^{-1}. \end{aligned} \tag{60}$$

By (18) and (60) we see that $\det \Delta(t) = 1/4 - 1/4 = 0$ for $t \in A_1'$.

On $A_\infty$ we can calculate $\Delta_{12}(t)$ by means of (35). If $t \in A_\infty$ and $\delta_{22}(t) > 0$, then by (35) and (50) we see that (60) is again true, and therefore $\det \Delta(t) = 0$. If $t \in A_\infty$ and $\delta_{22}(t) = 0$, then by (35) and (53) we see that

$$\Delta_{12}(t) = \Delta_{21}(t)^- = (1/2)(t+i)(t-i)^{-1}. \tag{61}$$

Hence, $\det \Delta(t) = 1/4 - 1/4$ in this case also.

We have thus shown that $\det \Delta(t) = 0$ for all $t \in A$ except on the set $A_1''$ of $\rho$-measure zero.

On $B$ we know that $\lim_{\eta\to 0+}\operatorname{Im}\theta(t+i\eta) = 0$ a.e. and that $\lim_{\eta\to 0+}\operatorname{Re}\theta(t+i\eta)$ exists and is finite a.e. Let $B_1$ consist of all $t \in B$ for which $0 < \rho^{(\prime)}(t) < \infty$, and let $B_\infty$ consist of all $t \in B$ for which $\rho^{(\prime)}(t) = \infty$. Let $B_1'$ consist of all $t \in B_1$ for which $\lim_{\eta\to 0+} \mathrm{Im}\, \theta(t + i\eta) = 0$, $\lim_{\eta\to 0+} \mathrm{Re}\, \theta(t + i\eta)$ exists and is finite, and $\lim_{\eta\to 0+} \mathrm{Re}\, \Phi_{11}(t + i\eta)$ exists and is finite. ($\lim_{\eta\to 0+} \mathrm{Im}\, \Phi_{11}(t + i\eta)$, of course, exists and is finite and positive on $B_1$.) Let $B_1'' = B_1 \backslash B_1'$. As in the case of $A_1''$, $B_1''$ has $\rho$-measure zero. Therefore, we calculate $\det \Delta(t)$ for $t \in B_1'$ and $t \in B_\infty$.

On $B_1'$ we calculate $\Delta_{12}(t)$ by means of (34). Writing $\Phi_{11}$ in the form $\Phi_{11} = \theta - [\theta^2 + 1][\theta + Q]^{-1}$, we see that

$$\lim_{\eta\to 0+} [\theta(\lambda) + Q(\lambda)]^{-1}$$

exists and is finite on $B_1'$, and

$$\lim_{\eta\to 0+} \mathrm{Im}\, \Phi_{11}(t + i\eta) = -\{[\mathrm{Re}\, \theta(t)]^2 + 1\} \lim_{\eta\to 0+} [\theta(\lambda) + Q(\lambda)]^{-1}. \tag{62}$$

From (37) it follows that

$$\begin{aligned} &\lim_{\eta\to 0+} [\Phi_{12}(\lambda) - \Phi_{21}(\lambda)^-] \\ &\quad = -2i(t + i)(t - i)^{-1}[\mathrm{Re}\, \theta(t) + i]^2 \lim_{\eta\to 0+} [\theta(\lambda) + Q(\lambda)]^{-1}. \end{aligned} \tag{63}$$

From (34), (62), (63) one obtains

$$\begin{aligned} \Delta_{12}(t) &= \Delta_{21}(t)^- \\ &= (1/2)(t + i)(t - i)^{-1}[\mathrm{Re}\, \theta(t) + i]\,[\mathrm{Re}\, \theta(t) - i]^{-1}. \end{aligned} \tag{64}$$

Equations (18) and (64) imply that $\det \Delta(t) = 0$ for $t \in B_1'$.

If $t \in B_\infty$, then the fact that $\det \Delta(t) = 0$ follows from (50) and (53) just as it does for $t \in A_\infty$.

Thus, we have shown that if $t \in B$, then $\det \Delta(t) = 0$ except on the set $B_1''$ of $\rho$-measure zero. This completes the proof of the theorem.

COROLLARY 1. *A has spectral multiplicity* 2 *if and only if the* $\rho$-*measure of* $Q_{a+}[q] \cap Q_{a+}[\mu] \cap \mathcal{E}$ *is positive.*

The corollary follows immediately from Theorems 5 and 7.

COROLLARY 2. *A has spectral multiplicity* 2 *if and only if the Lebesgue measure of* $Q_a[q] \cap Q_a[\mu]$ *is positive.*

PROOF. Following Kac [9, Theorem 5], one may show that the $\rho$-measure of $Q_{a+}[q] \cap Q_{a+}[\mu] \cap \mathcal{E}$ is positive if and only if the Lebesgue measure of $Q_a[q] \cap Q_a[\mu]$ is positive. The corollary then follows from Corollary 1.

COROLLARY 3. *If q or* $\mu$ *is singular, then A has spectral multiplicity* 1.

PROOF. If $q$, say, is singular, then $q^{(\prime)}(t) = 0$ a.e. Therefore, the Lebesgue measure of $Q_a[q]$ is zero, and the Lebesgue measure of $Q_a[q] \cap Q_a[\mu]$ is zero. Hence, by Corollary 2, $A$ has spectral multiplicity 1.

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