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THE IDEAL TRANSFORM IN A GENERALIZED KRULL DOMAIN

ELBERT M. PIRTLE

Introduction. Let R be a commutative integral domain with identity and let L denote the quotient field of R. R is called a generalized Krull domain [5] if there is a family F of rank one valuation on L satisfying the following, which we label as (E).

(E)	(1)	Each $v \in F$ has rank one.
	(2)	$R = \bigcap\{R_v \mid v \in F\}$.
	(3)	$R_v = R_{P(v)},\ v \in F$.
	(4)	F is of finite character [5].

In this case, F is called the family of essential valuations of R.

In [6], Nagata defined the transform $T(A)$ of an ideal A of R as follows: $T(A) = \bigcup_{n=1}^{\infty} R : A^n$, where $R : B = \{x \in L \mid xB \subseteq R\}$ for any ideal B of R. Nagata [6] characterized the transform of an ideal A when R is a Krull domain and showed that when R is Krull, the transform of any ideal of R is the transform of a finitely generated ideal. Brewer in [1] characterized the transform of any finitely generated ideal when R is an arbitrary integral domain.

In §1 of this paper, we obtain a characterization of the transform of an arbitrary ideal of R, when R is a generalized Krull domain, that generalizes Nagata's and Brewer's results. This characterization provides the basis for a "transform algebra." §2 contains two examples to show how the results of §1 fail when the finite character assumption on F is dropped.

1. Let R be a generalized Krull domain with quotient field L and family F of essential valuations. For any nonzero ideal A of R and any $v \in F$, put $v(A) = \inf\{v(a) \mid a \in A\}$. Since F is of finite character, it follows that $v(A) \neq 0$ for only finitely many $v \in F$. We let $F_A = \{v \in F \mid v(A) \neq 0\}$.

THEOREM 1.1. *Let A be any nonzero ideal of R. Then* $T(A) = \bigcap\{R_w \mid w(A) = 0\}$.

PROOF. Let $x \in T(A)$. Then $xA^n \subseteq R$ for some n. So for $w \in F - F_A$

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we have $0 \leqq w(xA^n) = w(x)$, and $x \in \bigcap\{R_w \mid w \in F - F_A\}$. On the other hand, let $x \in \bigcap\{R_w \mid w \in F - F_A\}$. Let

$$s = \inf\{v(x) \mid v \in F_A\},$$

and let $r = \inf\{v(A) \mid v \in F_A\} > 0$. Choose an integer n so that $s + nr \geqq 0$. Then $w(xA^n) \geqq 0$ for all $w \in F$ and $xA^n \subseteq R$, i.e., $x \in T(A)$.

Theorem 1.1 above shows that the transform of any ideal in a generalized Krull domain is also a generalized Krull domain. It also provides the basis for a "transform algebra" in the sense of the next three propositions.

Let X be an indeterminate, and let F' denote the canonical extensions of elements of F to $L(X)$. Let G denote the family of $a(X)$-adic valuations on $L(X)$, where $a(X) \in L[X]$ is nonconstant, irreducible. Then $R[X]$ is a generalized Krull domain with $F' \cup G$ as family of essential valuations [5].

THEOREM 1.2. *Let A be any nonzero ideal of R. Then* $T(AR[X]) = T(A)[X]$.

PROOF. We observe that for any $v' \in F'$ we have $v'(AR[X]) = v(A)$, where $v' \in F'$ is the extension of $v \in F$. Also, for any $w \in G$, we have $w(AR[X]) = 0$. Thus

$$T(AR[X]) = (\bigcap\{R[X]_{v'} \mid v(A) = 0\}) \cap (\bigcap\{R[X]_w \mid w \in G\}).$$

Since $T(A) = \bigcap\{R_v \mid v(A) = 0\}$ is a generalized Krull domain,

$$T(A)[X] = (\bigcap\{R[X]_{v'} \mid v(A) = 0\}) \cap (\bigcap\{R[X]_w \mid w \in G\}).$$

It follows by induction that $T(AR[X_1, \cdots, X_n]) = T(A)[X_1, \cdots, X_n]$ for any finite number of indeterminates $X_1, \cdots, X_n$.

Now, let S be a multiplicative system in R. Then $R_S = \bigcap\{R_v \mid P(v) \cap S = \emptyset\}$, by the proof of Lemma 11 of [5], and is a generalized Krull domain. Now let A be any nonzero ideal of R.

THEOREM 1.3. $T(AR_S) = T(A)_S$.

PROOF. Let $H = \{v \in F \mid P(v) \cap S = \emptyset\}$. For any $v \in H$, $v(A) = v(AR_S)$. By 1.1 we have $T(AR_S) = \bigcap\{R_v \mid v \in H, v(AR_S) = 0\}$. Now $T(A) = \bigcap\{R_v \mid v(A) = 0\}$, so

$$T(A)_S = \bigcap\{R_v \mid v(A) = 0,\ P(v) \cap S = \emptyset\}.$$

Thus $T(A)_S = T(AR_S)$.

Let L' be any finite, algebraic extension of L. For any ring T such that $R \subseteq T \subseteq L$, we let T' denote the integral closure of T in L'.

Let F' denote the family of extensions of elements of F to L'. Then R' is a generalized Krull domain with F' as family of essential valuations [5]. Again, let A be any nonzero ideal of R.

THEOREM 1.4. $T(AR') = T(A)'$.

PROOF. $T(AR') = \bigcap \{R_{v'}' \mid v'(AR') = 0\}$. Now $v'(AR') = 0$ if and only if v' is the extension of some $u \in F$ such that $u(A) = 0$. Now, consider $T(A)'$. $T(A) = \bigcap \{R_u \mid u(A) = 0\}$. Thus

$$T(A)' = \bigcap\{R_{v'}' \mid v' \text{ is the extension of some } u \in F \text{ such that } u(A) = 0\},$$

and $T(AR') = T(A)'$.

NOTE. It has been brought to the author's attention that the results of this section, particularly Theorem 1.3 above, are closely related to the results of Arnold and Brewer in [2].

2. In this section we give two examples to show that some of the results of §1 break down when the finite character assumption on F is dropped.

EXAMPLE 2.1. In this example we show that the representation of the transform in Theorem 1.1 is not valid in domains satisfying (1)-(3) of (E).

Let R denote the ring of entire functions and let $\mathcal{C}$ denote the set of complex numbers. Let Z denote the additive group of integers. For $a \in \mathcal{C}$, define $v_a : R \to Z$ as follows. For $f \in R$, $f \neq 0$, if $a \in \mathcal{C}$ is a zero of order $n > 0$ then $v_a(f) = n$; if $a \in \mathcal{C}$ is not a zero of f, then $v_a(f) = 0$. We put $v_a(0) = +\infty$ for all $a \in \mathcal{C}$. Each v_a may be extended to a valuation on the quotient field L of R. Let F denote this family of valuations. Then R, F have the following properties: (i) each $v \in F$ is rank one, discrete; (ii) $R = \bigcap \{R_v \mid v \in F\}$; (iii) $R_v = R_{P(v)}$, $v \in F$; (iv) $P(v_a) = (z - a)R$, and hence is divisorial for each $v_a \in F$; (v) F is not of finite character. Thus R is a K domain [8]. It is also easy to verify that R has the following property: (*) Every nontrivial, rank one, discrete valuation on L which is nonnegative on R is equivalent to some $v \in F$.

Now let $\{z_n\}_{n=1}^{\infty}$ be a sequence of distinct complex numbers such that $\lim z_n = \infty$. Let $f(z)$ be an entire function such that z_n is a zero of order 1 for $n = 1, 2, \cdots$, and such that $f(z)$ has no other zeros, and let $h(z)$ be an entire function such that z_n is a zero of order n for $n = 1, 2, \cdots$, and such that these are the only zeros of $h(z)$. It is easy to see that $T(f) \subseteq R_v$ iff $v(f) = 0$. If $T(f)$ is an intersection of rank one, discrete valuation rings, then by (*) we have $T(f) = \bigcap \{R_v \mid v \in F$ and $T(f) \subseteq R_v\}$. However, $1/h(z) \in \bigcap \{R_v \mid v \in F, v(f) = 0\}$, but $1/h(z) \notin T(f)$ since $f^n(z)/h(z) \notin R$

for any n. Thus $T(f) \neq \bigcap \{R_v \mid v \in F, v(f) = 0\}$.

Thus if D is an integral domain with family F of valuations such that $D = \bigcap \{D_v \mid v \in F\}$ and $D_v = D_{P(v)}$, $v \in F$, then the transform of an ideal of D may not in general be computed in terms of the D_v, $v \in F$.

EXAMPLE 2.2. In this example, we show that the results of Theorem 1.3 need not hold in a domain satisfying conditions (1)–(3) of (E).

Let R be an almost Dedekind domain which is not Dedekind [3]. It follows from Corollary 28.4 of [4] that there is some maximal ideal M of R such that $T(M) = R$. Let $S = R - M$. Then $T(M)_S = R_S$, a rank one, discrete valuation ring. However, $T(MR_S) = L$, the quotient field of R, by Theorem 1.1, and $R_S < L$.

It follows from the definition of almost Dedekind domain that the family $\{M_\lambda\}$ of maximal ideals of R induce a family F of rank one, discrete valuations on L such that $R = \bigcap \{R_v \mid v \in F\}$ and $R_v = R_{P(v)}$, $v \in F$.

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UNIVERSITY OF MISSOURI, KANSAS CITY, MISSOURI 64110