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# $\lambda(n)$-CONVEX FUNCTIONS

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ABSTRACT. $\lambda(n)$-convex functions include as special cases the classical convex and generalized convex functions. Relationships between convexity and $\lambda(n)$-convexity are noted for certain types of $\lambda(n)$-convexity, and some relationships among various types of $\lambda(n)$-convexity are derived. In addition it is shown that for a disconjugate linear homogeneous differential equation solutions to the corresponding differential inequality are $\lambda(n)$-convex for all values of the "ordered partition" $\lambda(n)$ of the positive integer $n$.

**Introduction.** A real valued function $s$ is said to be convex with respect to an "$n$-parameter family" $F$ of functions on an interval $I$ of the real numbers if whenever there is an $f \in F$ such that $s - f$ has $n$ zeros on $I$, then $s - f$ is nonpositive on the interval between the last two of the zeros and changes sign only at each of the other zeros except the first. In this paper we consider convexity in the case that $s - f$ has $n$ zeros *counting multiplicity*, give a definition for this new type of convexity and, under appropriate assumptions on $F$, establish some relationships among various types of convex functions where $s - f$ has at least one double zero.

Functions convex with respect to $n$-parameter families of functions are often called *generalized convex functions*. An extensive bibliography on this subject is found in the interesting recent article [2] by J. H. B. Kemperman.

All functions considered are real valued, $f^{(j)}(x)$ denotes the $j$th derivative of $f$ at $x$ and $I^{\circ}$ denotes the interior of the interval $I$ of the real numbers. In the subsequent discussion we assume that $n \geqq 3$. $F$ denotes an $n$-parameter family unless specified otherwise.

We also remark here that there is an obvious analogous idea of "concave functions" obtained by replacing "$>$" by "$<$" in (1) and (5). It is clear that each result for convex functions has an analog for concave functions, and where convenient we shall use these results for concavity.

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1. **Convex functions.** A family $F$ of real valued functions defined on the interval $I$ of the real numbers is said to be an *$n$-parameter family* on $I$ in case for any $n$ points $x_1 < x_2 < \cdots < x_n$ in $I$ and any set $\{y_i\}$ of $n$ real numbers there is a unique $f \in F$ satisfying $f(x_i) = y_i$ for $i = 1, 2, \cdots, n$. The definition of convex function with respect to $F$ can be given more precisely as follows:

Let $F$ be an $n$-parameter family on $I$. A function $s$ defined on $I$ is said to be *convex* with respect to $F$ on $I$ in case for any $n$ points $x_1 < x_2 < \cdots < x_n$ in $I$ and any $f \in F$, if $f(x_i) = s(x_i)$ for $i = 1, 2, \cdots, n$, then

$$(1) \quad (-1)^{n+i-1}(s(x) - f(x)) \geqq 0 \quad \text{on } (x_{i-1}, x_i) \quad \text{for } i = 2, 3, \cdots, n.$$

LEMMA 1.1. *An $F$ convex function on $I$ is continuous on $I^0$ if $F \subset C(I)$.*

PROOF. Let $x_{n-1} \in I^0$ and pick $x_1 < x_2 < \cdots < x_{n-1} < x_n < x_{n+1}$ all in $I^0$. Let $u, v \in F$ with $u(x_i) = s(x_i)$ and $v(x_{i+1}) = s(x_{i+1})$ for $i = 1, 2, 3, \cdots, n$. Then $u(x) \geqq s(x)$ for all $x_{n-1} < x < x_n$ and $v(x) \leqq s(x)$ for all $x_n < x < x_{n+1}$. Hence $s(x_{n-1} + 0) = s(x_{n-1})$. Similarly $s(x_{n-1} - 0) = s(x_{n-1})$.

We state here two results due to Tornheim [5].

LEMMA 1.2. *Let $F \subset C(I)$ and let $f$ and $g$ be distinct members of $F$. If $f(x) = g(x)$ at $n - 1$ points in $I^0$, then $f - g$ changes signs at each of those $n - 1$ points.*

LEMMA 1.3. *Let $s$ be an $F$ convex function and suppose that every $f \in F$ has a derivative at each point of $I^0$. Then $s$ has a derivative at each point of $I^0$.*

The next lemma considers the case that $s$ and $f$ intersect at $n - 1$ points and are tangent at one of those points. The resulting behavior of $f - s$ can be predicted as follows: Let $f(x) = s(x)$ for $x = x_j$, $j = 1, 2, \cdots, n - 1$, and $f'(x_i) = s'(x_i)$ where $i$ is a fixed integer between 1 and $n - 1$. Let $z_j = x_j$ for $j = 1, 2, \cdots, i - 1$, $z_i$ be any point strictly between $x_{i-1}$ and $x_i$ and $z_{j+1} = x_j$ for $j = i, i + 1, \cdots, n - 1$. Let $g(x) = s(x)$ for $x = z_j$, $j = 1, 2, \cdots, n$. Then the sign of $g(x) - s(x)$ is determined in each of the intervals $(z_j, z_{j+1})$ for $j = 1, 2, \cdots, n - 1$. Now let $z_i \rightarrow z_{i+1}$ with each $z_j$ fixed for $j \neq i$. Then if $j < i$, take the sign of $f(x) - s(x)$ in $(x_j, x_{j+1})$ to be the sign of $g(x) - s(x)$ in $(x_j, x_{j+1})$, and if $j > i$ take the sign of $f(x) - s(x)$ in $(x_{j-1}, x_j)$ to be the sign of $g(x) - s(x)$ in $(z_{j-1}, z_j) = (x_j, x_{j+1})$. See Figure 1.

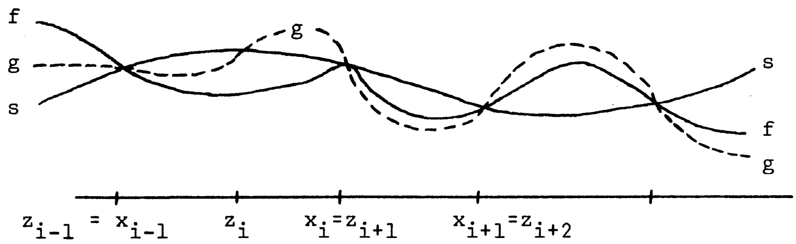


FIGURE 1

By Lemma 1.2 $f - g$ changes sign at each point of intersection of $f$ and $g$ if there are $n - 1$ such points. We need a condition for the next lemma which is somewhat stronger, namely that

(2) if $f, g \in F$ intersect at $n - 1$ points and are tangent at one of those $n - 1$ points, then $f$ and $g$ are identical.

LEMMA 1.4. *Let $F \subset C^1(I)$ and let $F$ satisfy condition* (2). *Suppose $s$ is an $F$ convex function on $I$ and that, for some $f \in F$, $s(x) = f(x)$ at the points $x_1 < x_2 < \cdots < x_{n-1}$ in $I^\circ$ and in addition $s'(x_i) = f'(x_i)$ for some $1 \leqq i \leqq n - 1$. Then $(s(x) - f(x))(-1)^{n+i-1} \geqq 0$ for $x_{i-1} < x < x_{i+1}$. ($x_0$ = left endpoint of $I$ and $x_n$ = right endpoint of $I$.)*

PROOF. Suppose $n$ is odd. We argue the case that $i = 1$ since the cases $i = 2, 3, \cdots, n$ are entirely similar to this case. Suppose the lemma is false. Then there is a point $x'$ in $(x_0, x_2)$ so that $f(x') < s(x')$. Pick $g \in F$ such that $g(x) = s(x)$ for $x = x_1, x_2, \cdots, x_{n-1}$ and $x'$. See Figure 2. $g(x') = s(x') > f(x')$ so that, if $x' > x_1$, $f(x) < g(x) \leqq s(x)$ for $x_1 < x < x'$. Hence $f'(x_1) \leqq g'(x_1) \leqq s'(x_1) = f'(x_1)$ so $f'(x_1) = g'(x_1)$, and hence by (2) $g(x) = f(x)$ for all $x$ in $I$. This is impossible. If there is point $x' < x_1$ such that $f(x') < s(x')$, choose $g$ as before. Then $g(x') = s(x') > f(x')$, so $g(x) < f(x)$ for $x' < x < x_1$. Also since $s$ is convex, $g(x) \leqq s(x)$ for $x' < x < x_1$. Hence $s'(x_1) \leqq g'(x_1) \leqq f'(x_1)$, so again $f$ and $g$ are identical, and we have arrived at a contradiction. The argument for $n$ even is entirely similar.

Actually condition (2) is stronger than needed in the sense that the lemma is valid for each specific $i$, $1 \leqq i \leqq n$, under the assumption that two functions from $F$ which intersect at $n - 1$ points in $I$ and are tangent at the $i$th of those points are identical. In the next section we show that the sign of $f - s$ is determined in all the intervals $(x_j, x_{j+1})$ in Lemma 1.4 and not just in $(x_{i-1}, x_i)$ and $(x_i, x_{i+1})$.

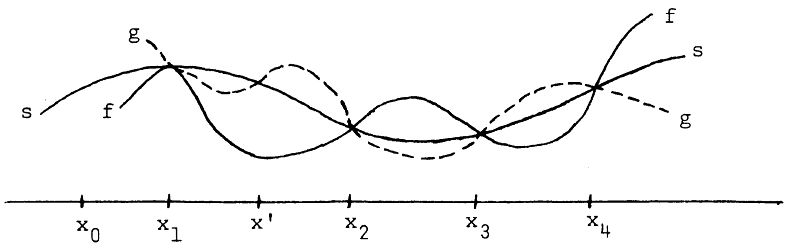


FIGURE 2

2. **$\lambda(n)$-convex functions.** $I$ is an interval of the real numbers, $n$ is a positive integer larger than 2 and $F \subset C^j(I)$ for $j$ $(>0)$ large enough so that the following definitions are possible:

Let $\lambda(n) = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ where $k$, $\lambda_1, \cdots, \lambda_k$ are positive integers satisfying $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. Denote by $P(n)$ the set of all such "ordered partitions" $\lambda(n)$ of $n$. The number $|\lambda(n)| = k$ is called the "length of $\lambda(n)$." The family $F$ is said to be a $\lambda(n)$-parameter family on $I$ in case for every choice of the $k$ points $x_1 < x_2 < \cdots < x_k$ in $I$ and every set $\{y_i^j\}$ of $n$ real numbers there is a unique $f \in F$ satisfying

$$(3) \qquad f^{(j)}(x_i) = y_i^j, \qquad j = 0, 1, \cdots, \lambda_i - 1, \; i = 1, 2, \cdots, k.$$

The following definition is a generalization of the definition of convex function as given in §1. A function $s$ is said to be $\lambda(n)$-*convex* with respect to $F$ on $I$ (or simply $\lambda(n)$-*convex*) in case for every choice of $k$ points $x_1 < x_2 < \cdots < x_k$ from $I$ and for every $f$ in $F$ satisfying

$$(4) \qquad f^{(j)}(x_i) = s^{(j)}(x_i), \qquad j = 0, 1, \cdots, \lambda_i - 1, \; i = 1, 2, \cdots, k,$$

we have

$$(5) \quad (-1)^{M(i)}(s(x) - f(x)) \geqq 0 \quad \text{for } x_{i-1} < x < x_i, \; i = 2, 3, \cdots, k,$$

where $M(i) = n + \lambda_1 + \lambda_2 + \cdots + \lambda_{i-1}$.

We shall now see that the mentioned generalization of Lemma 1.4 follows from Lemmas 1.2 and 1.4.

THEOREM 2.1. *Let $F \subset C^1(a, b)$ be an $n$-parameter family on $(a, b)$ and also a $\lambda(n)$-parameter family on $(a, b)$ where $\lambda(n)$ is some fixed ordered partition of $n$ having length $n - 1$. If $s$ is convex with respect to $F$ on $(a, b)$ then $s$ is also $\lambda(n)$-convex with respect to $F$ on $(a, b)$.*

PROOF. Suppose that $\lambda_j = 2$, and let $f \in F$ satisfy $f(x_1) = s(x_i)$ for $i = 1, 2, \cdots, n - 1$ and $f'(x_j) = s'(x_j)$. Then by Lemma 1.4 $f(x) - s(x)$ does not change sign in $(x_{j-1}, x_{j+1})$. If $s(x) = f(x)$ for

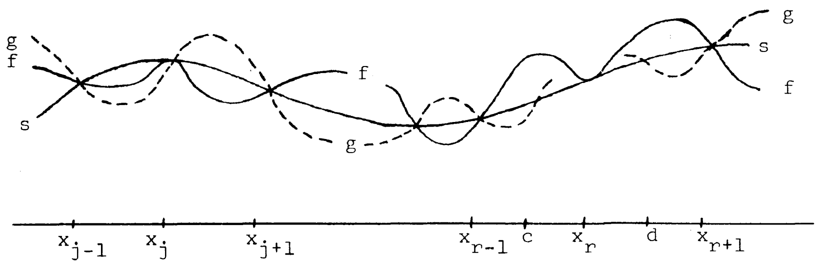


FIGURE 3

some $x \neq x_i$, $i = 1, 2, \cdots, n-1$, then from the definition $s(x) = f(x)$ for all $x \in (x_1, x_{n-1})$. It is a routine matter to check that the sign of $f - s$ in $(x_{j-1}, x_j)$ and $(x_j, x_{j+1})$ is the sign of $f - s$ needed in the above definition. Hence it suffices to show that $f - s$ changes sign only at $x_j$. Suppose the contrary, and consider the case that $s'(x_r) = f'(x_r)$ for $j < r < n-1$ and $s'(x_i) \neq f'(x_i)$ for $j < i < r$. Pick points $c$ and $d$ in $(a, b)$ satisfying $x_{r-1} < c < x_r < d < x_{r+1}$ and pick $g \in F$ satisfying $g(x) = s(x)$ at the $n$ points $x_1, x_2, \cdots, x_{r-1}, c, d, x_{r+1}, \cdots, x_{n-1}$. See Figure 3.

Then $f - s$ and $g - s$ have opposite signs in each of the intervals $(x_j, x_{j+1}), \cdots, (x_{r-1}, c)$ and have the same sign in $(c, d)$ since $g - s$ changes sign at $c$ and $f - s$ does not. $f$ and $g$ are distinct members of $F$ (they differ at both $c$ and $d$) which intersect at $n-2$ points. Since $f - s = 0$ at both $c$ and $d$, and $f - s$ has the same sign at $c$ and $d$, either $f$ and $g$ intersect at two points in $(c, d)$ or else $g - f$ does not change sign in $(c, d)$. The first possibility would contradict the distinctness of $f$ and $g$ (they would meet in $n$ points), and the second possibility would imply that $g - f = 0$ at $x_r$, but $g - f$ does not change sign at $x_r$. Then $g$ and $f$ would meet in $n-1$ points, so by Lemma 1.2 $g - f$ would change sign at $x_r$. We must therefore agree that no such $r$ exists. A similar argument resolves the case $1 < r < j$ also, and hence the theorem is proved.

We remark here that the conclusion of Theorem 2.1 is valid for $[a, b]$ (resp. for $[a, b)$; for $(a, b]$) if $\lambda_1 \neq 2$ and $\lambda_{n-1} \neq 2$ (if $\lambda_1 \neq 2$; if $\lambda_{n-1} \neq 2$). As in Lemma 1.4 the "uniqueness" and not the "existence" of elements of $F$ satisfying (3) for $\lambda(n)$ is what we require in Theorem 2.1.

THEOREM 2.2. *Let $F$ be a $n$-parameter and a $\lambda(n)$-parameter family on $I$ for some $\lambda(n) \in P(n)$ having length $n-1$ with $\lambda_1 \neq 2$ and $\lambda_{n-1} \neq 2$. If $s$ is $\lambda(n)$-convex with respect to $F$ on $I$, then $s$ is convex with respect to $F$ on $I$.*

When the previous two theorems are combined we have the following

COROLLARY. *Let the n-parameter family F be a $\mu(n)$ and a $\nu(n)$-parameter family on $(a, b)$ with both $\mu(n)$ and $\nu(n)$ having length $n-1$, $\mu_1 \neq 2 \neq \nu_1$, $\mu_{n-1} \neq 2 \neq \nu_{n-1}$. Then the function s is $\mu(n)$-convex with respect to F on $(a, b)$ iff s is $\nu(n)$-convex with respect to F on $(a, b)$.*

In this corollary one would like to delete the condition that $F$ should be an $n$-parameter family on $(a, b)$. However, the validity of the corollary in case this is done remains an open question.

Before giving the proof of Theorem 2.2 we state a property of $\lambda(n)$-convex functions which is used in the proof.

LEMMA 2.3. *Let s be a $\lambda(n)$-convex function with respect to the $\lambda(n)$ and n-parameter family F on I. Let $|\lambda(n)| = n-1$ with $\lambda_j = 2$ for some fixed value of j, $1 < j < n-1$. Let $f \in F$ with $f(x_i) = s(x_i)$ for $i = 1, 2, \cdots, n-1$ and $f'(x_j) = s'(x_j)$ where $x_1 < x_2 < \cdots < x_{n-1}$ are $n-1$ points of I. Then $s(x) - f(x) \geqq 0$ for $x \in I \cap (x_{n-1}, +\infty)$ and $(-1)^n(s(x) - f(x)) \geqq 0$ for $x \in I(-\infty, x_1)$. Moreover, if equality holds for $x = w > x_{n-1}$ (or $x = z < x_1$), then $f(x) = s(x)$ for all $x_1 \leqq x \leqq w$ $(z \leqq x \leqq x_{n-1})$.*

PROOF. Suppose $f(x) > s(w)$ for $w > x_{n-1}$. Pick $g \in F$ such that $g(x_i) = f(x_i)$, $i = 1, 2, \cdots, n-2$, $g'(x_j) = f'(x_j) = s'(x_j)$ and $g(w) = s(w)$. Now $g(w) < f(w)$ implies that $g(x) < f(x)$ for $x_{n-2} < x < w$. But $s(x)$ is $\lambda(n)$-convex, so $g(x) > s(x)$ for $x_{n-2} < x < w$. Hence $s(x_{n-1}) \leqq g(x_{n-1}) < f(x_{n-1}) = s(x_{n-1})$, and this is impossible. Similarly, $(-1)^n(s(x) - f(x)) \geqq 0$ for $x < x_1$.

If $s(x) = f(w)$ for some $w > x_{n-1}$, then by replacing $x_{n-1}$ by $w$ and using the $\lambda(n)$-convexity of $s$ we get $s(x) \leqq f(x)$ for $x_{n-2} < x < w$, and hence, by what we just proved we must have $s(x) \leqq f(x) \leqq s(x)$ for $x_{n-1} \leqq x \leqq w$. But then by choosing a new set of $n$ points $z_1 < z_2 < \cdots < z_{n-1}$ with $z_j, \cdots, z_{n-1}$ in $[x_{n-1}, w]$ and varying $z_2, \cdots, z_{j-1}$ among $x_1, \cdots, x_{n-1}$ we get $s(x) = f(x)$ for all $x_1 \leqq x \leqq w$. Similarly, if $s(z) = f(z)$ for some $z < x_1$.

PROOF OF THEOREM 2.2. Let $\lambda_r = 2$ for some $r$ strictly between 1 and $n-1$. Suppose that $n-r$ is odd and that $f - s$ is not of the correct sign in $(x_{r-1}, x_r)$, i.e., $f - s > 0$ in $(x_{r-1}, x_r)$. Then since $f$ and $s$ cannot be tangent at $x_r$ or at $x_{r+1}$, by Lemma 2.5 $f - s < 0$ in $(x_r, x_{r+1})$ and $f - s > 0$ in $(x_{r+1}, x_{r+2})$. Pick $g \in F$ satisfying $g(x_i) = s(x_i)$ for $i = 1, 2, \cdots, r, r+2, r+3, \cdots, n$ and $g'(x_r) = s'(x_r)$. $n - r$ is odd, so $n - (r+2)$ is odd, and hence $g < s$ in $(x_{r+1}, x_{r+2})$. Since

$g < s < f$ in $(x_{r-1}, x_r)$, by Lemma 1.2 we must have $g > f$ in $(x_r, x_{r+2})$. But $f > s > g$ in $(x_{r+1}, x_{r+2})$, and we have a contradiction. So $f - s$ is of the correct sign in $(x_r, x_{r+1})$, and thus it remains to show that $f - s$ changes sign at each of the points $x_i$ for $i = 2, 3, \cdots, n - 1$. Suppose not. Let $x_j$ be the maximum of the points $x_i$, $1 < i < r$, at which $f - s$ fails to change sign. Pick $g \in F$ satisfying $g(x_i) = s(x_i)$ for $i = 1, 2, \cdots, r, r + 2, \cdots, n$ and $g'(x_r) = s'(x_r)$. Again since $n - r$ is odd, we have $f < g < s$ in $(x_{r-1}, x_r)$. $g - f$ has $n - 1$ zeros and must change sign at each of these zeros, but $g - s$ and $f - s$ change sign at each of the points $x_{j+1}, x_{j+2}, \cdots, x_{r-1}$, and hence, $g - s$ and $f - s$ are of the same sign in $(x_j, x_{j+1})$ with $g$ between $s$ and $f$ on that interval.

Then since $g - s$ changes sign at $x_j$ and $f - s$ does not, $f - g$ does not change sign at $x_j$ which is in violation of Lemma 1.4. Hence $f - s$ changes sign at each $x_i$, $i = 2, 3, \cdots, r$. A similar agrument shows that $f - s$ also changes sign at $x_i$, $i = r + 2, r + 3, \cdots, n - 1$. The argument in the case that $n - j$ is even is analogous to the above argument for $n - j$ odd and will be omitted. This proves the theorem.

3. **Further aspects of convexity.** In this section we shall introduce a stronger form of $\lambda(n)$-convexity and use it to extend the theorems of §2. It is not difficult to show that if $s$ is convex on $I$ and if $f(x_i) = s(x_i)$ for $i = 1, 2, \cdots, n$, then (1) holds for $i = 1$ and $i = n + 1$. (Here $x_0$ denotes the left endpoint of $I$ and $x_{n+1}$ denotes the right endpoint of $I$.) In fact Hartman in [1] requires (1) to hold for $i = 1$ and $i = n + 1$ in his definition of convexity. This being the case, one could with reasonable justification take the definition of $\lambda(n)$-convexity to require that (5) should hold for $i = 1$ and $i = k + 1$ also.

Definition. A function $s$ defined on an interval $I$ is said to be $\lambda(n)$-*convex with respect to the $\lambda(n)$-parameter family $F$ on $I$ in case for any $k$ points $x_1 < x_2 < \cdots < x_k$ in $I$, if $f \in F$ satisfies (4), then (5) holds for $i = 1, 2, \cdots, k + 1$ where $x_0$ denotes the left endpoint of $I$ and $x_{k+1}$ denotes the right endpoint of $I$.

Remark. It is easy to show that if $s$ is $\lambda(n)$-convex with respect to $F$ and if $f \in F$ satisfies (4) where $\lambda_1 = 1$ (or $\lambda_k = 1$), then (5) also holds for $i = 1$ (or $i = k + 1$).

It is easy to show as in Lemma 2.3 that if $\lambda_1 = 1$ and $\lambda_k = 1$, then $\lambda(n)$-convexity and $\lambda(n)$-*convexity are equivalent. However, if $\lambda(n) = (1, 2)$ $\lambda(n)$-*convexity requires that $s(x) \geqq f(x)$ for $x > x_2$, whereas in $\lambda(n)$-convexity it is possible to have $s(x) < f(x)$ for all $x > x_2$.

Theorem 2.1 remains true for $\lambda(n)$-*convexity, since it is easy to show that $\lambda(n)$-convexity implies $\lambda(n)$-*convexity if $|\lambda(n)| = n - 1$

under the assumption of convexity. Theorem 2.2 is true for $\lambda(n)$-* convexity as it stands, and can in fact be extended to include the cases that $\lambda_1 = 2$ and $\lambda_{n-1} = 2$. Let $s$ be $(1, \cdots, 1, 2)$-*convex and suppose that $f - s$ has zeros at $x_1 < x_2 < \cdots < x_n$ in $I$. If $f'(x_n) = s'(x_n)$, then by Lemma 2.3 we conclude that $f = s$ on $[x_1, x_{n-1}]$. If $f(d) < s(d)$ for some point $d$ in $(x_{n-1}, x_n)$, then pick $g \in F$ so that $g(x_i) = s(x_i)$, $i = 1, 2, \cdots, n-2$, $g(d) = s(d)$ and $g'(d) = s'(d)$. Then $g - f$ has at least two zeros in $[x_{n-1}, x_n]$, so $g = f$ which is impossible. Hence $f(x) = s(x)$ for all $x \in (x_1, x_n)$, and (1) is satisfied.

If $f(x) < s(x)$ for all $x \in (x_{n-1}, x_n)$, then pick $g \in F$ such that $g(x_i) = s(x_i) = f(x_i)$ for $i = 1, 2, \cdots, n-2$ and $n$, and $g'(x_n) = s'(x_n) < f'(x_n)$. Then $g(x_{n-1}) < s(x_{n-1}) = f(x_{n-1})$ implies that $g - f$ has at least one zero in $(x_{n-2}, x_n)$, and hence $g = f$. This is impossible. Hence we must have $s(x) < f(x)$ for $x \in (x_{n-1}, x_n)$. (We have without loss of generality assumed that $s - f$ has exactly $n$ zeros in $I$.) $f'(x_n) \neq s'(x_n)$ implies that $f'(x_{n-1}) \neq s'(x_{n-1})$, so $f - s$ changes sign at $x_{n-1}$. Let $x_j$, $1 < j < n-1$, be the maximum of the set of points $x_2, x_3, \cdots, x_{n-2}$ at which $f - s$ does not change sign. Pick $g \in F$ satisfying $g(x_i) = s(x_i) = f(x_i)$ for $i = 1, 2, \cdots, n-1$ and $g'(x_{n-1}) = s'(x_{n-1}) < f'(x_{n-1})$. Then $g - s$ changes sign at each of the points $x_i$, $i = 1, 2, \cdots, n-2$; in particular $g - s$ changes sign at $x_j$. Now $g - s < 0$ and $f - s < 0$ in $(x_{n-2}, x_{n-1})$, so by definition of $j$, $f - s$ and $g - s$ have the same sign in $(x_j, x_{j+1})$. But $g - s$ changes sign at $x_j$, and $f - s$ does not, so $f - g$ does not change sign at $x_j$. This violates Lemma 1.2. Clearly the argument for the case $\lambda_1 = 2$ is just a reflection of the above argument in the line $x = 0$. Hence we have proved

THEOREM 3.1. *Let $F$ be an $n$-parameter and a $\lambda(n)$-parameter family on $I$ for some $\lambda(n) \in P(n)$ having length $n - 1$. If $s$ is $\lambda(n)$-*convex with respect to $F$ on $I$, then $s$ is convex with respect to $F$ on $I$.*

The difficulty in attempting to extend Theorem 2.1 to include the cases $\lambda_1 = 2$ and $\lambda_{n-1} = 2$ is illustrated by taking $n = 3$ and considering the partition $(1, 2)$. If $s$ is $(1, 2)$ convex and $s - f$ has zeros at $x_1$, $x_2$ and $x_3$, then it is possible to have $s(x) = f(x)$ for all $x \in (x_1, x_2)$ and $s(x) > f(x)$ for $x \in (x_2, x_3)$ with $s'(x_3) = f'(x_3)$.

LEMMA 3.2. *Let $F$ be a $(2)$-parameter and a $(1, 1)$-parameter family on $I$. Then convexity and $(2)$-*convexity with respect to $F$ on $I$ are equivalent.*

PROOF. Let $s$ be convex and let $f(x_1) = s(x_1)$ and $f'(x_1) = s'(x_1)$. If there is an $x_2 > x_1$ such that $f(x_2) > s(x_2)$, pick $g \in F$

such that $g(x_1) = f(x_1)$ and $g(x_2) = s(x_2)$. Then by convexity $s(x) < g(x) < f(x)$ for all $x \in (x_1, x_2)$. Hence $g'(x_1) = f'(x_1)$ which implies that $f = g$. This is impossible. Therefore, $f(x) \leqq s(x)$ for $x > x_1$. Similarly, $f(x) \leqq s(x)$ for $x < x_1$. Next suppose that $s$ is (2)-*convex and that $f \in F$ with $f(x_1) = s(x_1)$ and $f(x_2)$. Also suppose that $f(c) < s(c)$ for some $c \in (x_1, x_2)$. Then pick $g \in F$ such that $g(c) = s(c)$ and $g'(c) = s'(c)$. $g(x) > s(x)$ for all $x$ in $I$, so $g(x_1) > f(x_1)$ and $g(x_2) > f(x_2)$. Hence $f - g$ has 2 zeros in $[x_1, x_2)$ which is not possible since $f \neq g$. Therefore, $f(x) > s(x)$ for all $x \in (x_1, x_2)$ as was asserted.

Using this lemma as a basic tool we shall establish a generalization of Theorem 2.1. For convenience in stating results we shall denote by $[\mu(n)]$ the set of all ordered partitions $\lambda(n)$ with the property that $\lambda(n)$ is obtained from $\mu(n)$ by formally replacing a 2 in $\mu(n)$ by a pair of 1's, i.e., there is a $j$, $1 < j < k$, with $\mu_j = 2$ such that $\mu_i = \lambda_i$ for $i < j$, $\lambda_j = \lambda_{j+1} = 1$ and $\lambda_{i+1} = \mu_i$ for $i = j + 1, \cdots, k$. Also let $\|\lambda(n)\|$ = maximum of the numbers $\lambda_i$ for $i = 1, 2, \cdots, k$.

THEOREM 3.2. *Let $F$ be a $\lambda(n)$-parameter family on an interval $(a, b)$ for $\lambda(n) = \mu(n)$ and for all $\lambda(n) \in [\mu(n)]$. Let $\|\mu(n)\| \leqq 2$. If $s$ is $\lambda(n)$-convex with respect to $F$ on $(a, b)$ for all $\lambda(n) \in [\mu(n)]$, then $s$ is also $\mu(n)$-convex with respect to $F$ on $(a, b)$.*

PROOF. Suppose that $g \in F$ with $f = g$ satisfying (4) for $\lambda(n)$ replaced by $\mu(n)$. Pick $x_r$ to be the maximum of the points $x_i$, $1 \leqq i \leqq k$, for which the corresponding $\lambda_i = 2$. Consider the family $G$ of all restrictions to $(x_{r-1}, x_{r+1})$ of members $f \in F$ satisfying (4) for $i \neq r$. Then $G$ is a (2)-parameter and a (1, 1)-parameter family on $(x_{r-1}, x_{r+1})$, and $s$ is (1, 1)-convex (or concave) with respect to $G$ on $(x_{r-1}, x_{r+1})$. By Lemma 3.2 $s$ is (2)-*convex (concave) with respect to $G$. Also by the remark at the beginning of this section, $g - s$ must change sign at each of the points $x_i$ for $r < i < n$. This shows that (5) is satisfied for $i \geqq r$. At each point $x_j$ for $j < r$ if $\lambda_j = 1$, $g - s$ changes sign, and if $\lambda_j = 2$, $g - s$ does not change sign. To establish this, and hence to show that (5) holds, consider $x_{r-1}$. If $\lambda_{r-1} = 1$, apply the result from the remark at the beginning of §3 to show that $g - s$ changes sign at $x_{r-1}$. If $\lambda_{r-1} = 2$, apply the same argument as at $x_r$ using Lemma 3.2. Continue this process at $x_{r-2}, \cdots, x_2$ to establish the claim and the theorem.

COROLLARY. *Let $F$ be a $\lambda(n)$-parameter family on $(a, b)$ for all $\lambda(n)$ with $\|\lambda(n)\| \leqq 2$. If $s$ is convex with respect to $F$ on $(a, b)$ then $s$ is also $\lambda(n)$-convex with respect to $F$ on $(a, b)$ for all $\lambda(n)$ with $\|\lambda(n)\| = 2$.*

PROOF. Note that if $\nu(n) \in [\mu(n)]$, then $|\nu(n)| = |\mu(n)| + 1$. The corollary will then follow from Theorem 3.2 by induction on the length of $\lambda(n)$ beginning with $|\lambda(n)| = n - 1$.

In Theorem 3.2 and its corollary we have shown that under the given conditions $\lambda(n)$-convexity for partitions of a fixed length $k$ implies convexity for some partitions with length $k - 1$. We are of course interested in implications in the reverse direction as in Theorems 2.2 and 3.1. Suppose that $\mu(n) \in P(n)$ with $\|\mu(n)\| = 2$ and let $s$ be $\lambda(n)$-convex for all $\lambda(n)$ such that $\mu(n) \in [\lambda(n)]$. Is $s$ $\mu(n)$-convex? In the most general case this question remains unresolved. We can however in some restricted cases answer the question affirmatively. For instance, suppose that $\mu(n) = (\mu_1, \mu_2, \cdots, \mu_k)$ where $\mu_1 = \mu_k = 2$ and if $i < j$ with $\mu_i = 2 = \mu_j$, then $i + 2 < j$ and $\mu_{i+1} = \mu_{i+2} = 1$. We are of course assuming that $\|\mu(n)\| = 2$. One can then prove that $s$ is $\mu(n)$-convex by using Theorem 3.1 and the remark at the beginning of this section. Rather than give a general proof to this result, we indicate the method of proof by considering the special case $\mu(15) = (2, 1, 1, 2, 1, 1, 1, 2, 1, 1, 2)$. We begin by considering $(x_8, x_{11})$. Here $s$ is (2)-*convex with respect to the subfamily of $F$ whose elements satisfy (4) for $1 \leqq i \leqq 8$ and $i = 11$. This gives us the correct signs for $f - s$ on $(x_8, x_{11})$. Similarly $s$ is (2)-*convex on $(x_5, x_8)$, so $f - s$ has the correct signs on $(x_5, x_{11})$ and hence also on $(x_4, x_{11})$ by the remark at the beginning of this section. Now again the signs are shown to be correct on $(x_1, x_4)$ using Lemma 3.2. By separating each pair of 2's in $\mu(n)$ by at least two 1's we are able to use either convexity or *convexity. If $\mu(n)$ contained a pair of adjacent 2's or a pair of 2's separated by only one 1, we would need the equivalence of convexity and *convexity in cases where $\lambda_1 = 2$, and $\lambda_k = 2$, and this we do not have.

4. **Linear differential inequalities.** In this section we consider the differential equations

$$L_n y = 0 \tag{6}$$

and

$$L_n y = 1, \tag{7}$$

and the differential inequality

$$L_n y \geqq 0 \tag{8}$$

on an interval $(a, b)$ of the real numbers where $L_n y = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y$ and each $a_i(x)$ is continuous on $(a, b)$. We will also employ the assumption that (6) is disconjugate on $(a, b)$. This implies that the set of solutions to (6) on $(a, b)$ is a $\lambda(n)$-parameter family on $(a, b)$ for all $\lambda(n)$. See [3].

THEOREM 4.1. *Let* (6) *be disconjugate on* $(a, b)$ *and let* $g \in C^n(a, b)$ *be a solution of* (8) *on* $(a, b)$. *Then* $g$ *is* $\lambda(n)$*-convex with respect to the solution set of* (6) *on* $(a, b)$ *for all* $\lambda(n) \in P(n)$.

COMMENT. P. Hartman [1, p. 137] proved this theorem for the case of convexity, i.e., $\lambda_i = 1$ for each $i = 1, 2, \cdots, n$. We will give a modification of his proof that will include all $\lambda(n)$.

PROOF. First we shall prove the theorem for solutions of (7). The theorem will then follow immediately by application of a mean value theorem due to Pólya. See [4, Theorem III, p. 313] or [1, Theorem II, p. 136].

Suppose that $h$ is a solution of (7) and

$$(9) \qquad h^{(j)}(x_i) = 0, \qquad i = 1, 2, \cdots, k \text{ and } j = 0, 1, \cdots, \lambda_i - 1,$$

where $\lambda(n) = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ and $x_1 < x_2 < \cdots < x_k$ are points of $(a, b)$. If $g$ is a solution of (3.3) and $y$ is a solution of (3.1) so that $g - y$ satisfies the conditions imposed on $h$ in (9), then by Pólya's mean value theorem $g(x) = y(x) + h(x)(L_n g)(\epsilon)$, and hence the sign of $h(x)$ determines the sign of $g(x) - y(x)$. Thus it will be sufficient to show that

$$(10) \qquad (-1)^{M(i)} h(x) > 0 \qquad \text{for } x_{i-1} < x < x_i,\ i = 2, 3, \cdots, k,$$

where $M(i) = n + \lambda_1 + \cdots + \lambda_{i-1}$.

We consider first the case that $\lambda(n)$ has length 1, i.e., $\lambda(n) = (n)$. Then by (7) and (9) $h^{(n)}(x_1) = 1$, and since $h$ has at most $n$ zeros counting multiplicity (see [4, p. 317]), we must have $h(x) \neq 0$ for $x \neq x_1$. Therefore, $(x - x_1)^n h(x) > 0$ for $x \neq x_1$. Next let $h^*$ be a solution to (7) with $h^{*(j)}(x_1) = 0$, $j = 0, 1, \cdots, n - 2$, and $h^*(x_2) = 0$ where $x_1 < x_2$. Suppose that $h^*(x) > 0$ for $x \in (x_1, x_2)$. Then $h^{(n-1)}(x_1) = 0 < h^{*(n-1)}(x_1)$ implies that $h^*(x) - h(x) > 0$ for $x - x_1 > 0$ and small, and since $h^*(x) - h(x)$ is a solution to (6) with $n - 1$ zeros counting multiplicity, we must have $h^*(x) > h(x)$ for $x > x_1$. But then $0 < h(x_2) < h^*(x_2) = 0$ which is impossible. Therefore, $h^*(x) < 0$ for $x \in (x_1, x_2)$, i.e., $(-1)^{n+n-1} h^*(x) = -h^*(x) > 0$ for $x \in (x_1, x_2)$.

Before we continue the proof, let us define an ordering in the set $P(n)$ of all ordered partitions $\lambda(n)$ of $n$ as follows: if $\zeta(n)$ and $\eta(n)$ are elements of $P(n)$ we will say that $\zeta(n) > \eta(n)$ provided either $|\zeta(n)| > |\eta(n)|$ or else $|\zeta(n)| = |\eta(n)|$, $\zeta_j > \eta_j$ and $\zeta_i = \eta_i$ for all $i > j$. This is a total order on $P(n)$. We continue the proof of the theorem by finite induction on $P(n)$. We have shown that if $\lambda(n) = (n-1, 1)$ and (9) holds, then (10) also holds. Suppose (9) implies (10) for all $\lambda(n) < \mu(n)$ where $\mu(n) \in P(n)$. Let (9) hold with $h$ replaced by $h^*$ and $\lambda(n)$ replaced by $\mu(n)$. First we consider the case that $\mu(n)$ is the smallest member of $P(n)$ of length $k$. Then $\mu(n) = (n-k+1, 1, 1, \cdots, 1)$, i.e., $\mu_1 = n-k+1$ and $\mu_i = 1$ for $i = 2, 3, \cdots, k$. Then $\nu(n) = (n-k+2, 1, 1, \cdots, 1)$ satisfies $\nu(n) < \mu(n)$. Choose $h$ so that (9) holds for $\lambda(n)$ replaced by $\nu(n)$. Then by (10) $(-1)^{n+\mu_1+\cdots+\mu_{i-1}}h(x) > 0$ for $x_{i-1} < x < x_i$, $i = 2, 3, \cdots, k-1$. Suppose $h^*(x) < 0$ in $(x_{k-1}, x_k)$ fails. Then since $h^*(x)$ may have at most $n$ zeros counting multiplicity, we must have $h^*(x) > 0$ in $(x_{k-1}, x_k)$. Now $h(x)$, $h^*(x)$ and $h(x) - h^*(x)$ change sign at each $x_i$, $i = 2, 3, \cdots, k-1$. Also $h^*(x_k) = 0 > h(x_k)$, so $h(x) < h^*(x) < 0$ for $x_{k-1} < x < x_k$, and hence $h^*(x)$ lies between $h(x)$ and 0 on each of the intervals $(x_{i-1}, x_i)$ and in particular for $(x_1, x_2)$ we have $h^*(x)$ between $h(x)$ and zero. But then $0 = h^{(n-k+1)}(x_1) = h^{(n-k+1)}(x_1) = 0$ which is impossible. Hence $h^*(x) < 0$ for $x_{k-1} < x < x_k$ must hold, and therefore (10) holds for $h^*$ and $\mu(n)$.

In the remaining possibilities for $\mu(n)$ we may pick an integer $p$, $2 < p < k$, so that $\mu_p \neq 1$ and $\mu_i = 1$ if $i > p$. Define $\nu(n) = (\nu_1, \cdots, \nu_k)$ by $\nu_p = \mu_p - 1$, $\nu_{p-1} = \mu_{p-1} + 1$ and $\nu_i = \mu_i$ for all other values of $i$ from 1 through $k$. Suppose that (9) holds for $h$ replaced by $h^*$ and $\lambda(n)$ replaced by $\mu(n)$. Also suppose that $((-1)^{2n-\mu_k})h^*(x) < 0$ for $x_{k-1} < x < x_k$. Pick $h$ to satisfy (9) with $\lambda(n)$ replaced by $\nu(n)$. Then (10) holds for $\lambda(n) = \nu(n)$. Hence $h(x)$ and $h^*(x)$ have opposite signs on each of the intervals $(x_{i-1}, x_i)$ for $i = p+1, \cdots, k$, and they have the same sign on $(x_{p-1}, x_p)$. But close to $x_p$, $h^*(x)$ lies between 0 and $h(x)$ since $h^{(i)}(x_p) \neq h^{*(i)}(x_p) = 0$ for $i = \mu_p$ while close to $x_{p-1}h(x)$ lies between 0 and $h^*(x)$ since $h^{*(i)}(x_{p-1}) \neq h^{(i)}(x_{p-1}) = 0$ for $i = \mu_{p-1} + 1$. Hence $h(x) - h^*(x)$ has at least one zero in $(x_{p-1}, x_p)$. Then $h(x) - h^*(x)$ would be a nontrivial solution of (6) having at least $n$ zeros in $(a, b)$, and that is impossible. Therefore, $((-1)^{2n-\mu_k})h^*(x) > 0$ for $x_{k-1} < x < x_k$, and since $h^*(x)$ has exactly $n$ zero counting multiplicity, (10) must hold for $\lambda(n)$ replaced by $\mu(n)$. So by induction the theorem follows.

It is clear that a solution to the strict inequality is, under the given assumptions, a strict $\lambda(n)$-convex function for all $\lambda(n) \in P(n)$.

5. **Remarks.** The definitions and theorems in this paper suggest a number of questions regarding $\lambda(n)$-convex functions and $\lambda(n)$-parameter families. Some of these have been noted explicitly in the text, and many others are implicit in the discussions. At this stage to appreciate these concepts fully one should have a collection of

examples of $\lambda(n)$-parameter families and $\lambda(n)$-convex functions which exhibit some of the behavior which is noted at places in this paper especially in connection with the ideas of §3. These examples are indeed difficult to find. One also expects that the relationships among the $\lambda(n)$-parameter families for various values of the ordered partition $\lambda(n)$ of $n$ (for instance see [3]) will play a basic role in determining relationships among various types of $\lambda(n)$-convex functions.

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