ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 2, Number 1, Winter 1972

REMARKS ON HOMOLOGICAL DIMENSIONS

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1. **Introduction.** It has been noticed by the author and others that there are many ways in which the homological dimension of a module can be defined. The author has noticed this particularly with regard to algebraic geometry and specifically with regard to the following three theorems:

a. *A local ring is regular if and only if it is a local ring of finite global dimension.*

b. *The quotient ring of a regular local ring at any prime ideal is also a regular local ring.*

c. *Regular local rings are unique factorization domains.*

It is, therefore, the purpose of this article to present four of the more well-known definitions of homological dimension and to show that these four are equivalent if one restricts oneself to the class of finitely generated modules over Noetherian local rings.

It will be assumed throughout this article that R is a commutative ring with unit. It will be clear, however, that one could assume that R is not necessarily commutative if one would consistently use left R-modules or right R-modules throughout.

2. The first definition of homological dimension that will be presented is due to Cartan and Eilenberg [1].

DEFINITION. An R-module A has homological dimension n (n an integer $\geqq 0$) if and only if

(i) there is a projective resolution of A of the form $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$, and

(ii) there does not exist such a projective resolution for A with fewer terms.

In this case we write $dh_R(A) = n$. If no such projective resolution exists for A with a finite number of terms we write $dh_R(A) = \infty$.

Another definition of homological dimension is due to Kaplansky [2]. Here it is said first that two R-modules A and B are *equivalent* ($A \sim B$) if and only if there exist projective R-modules P and P' such that $A \oplus P \cong B \oplus P'$. This is easily seen to be an equivalence relation if one observes that the direct sum of projective R-modules is again a projective R-module.

Received by the editors March 16, 1970.
AMS 1970 *subject classifications.* Primary 14A05, 13D05.

If A is an R-module we let $\bar{A}$ denote the equivalence class containing A. We let $\mathscr{R}(A)$ denote any fixed member of the class $\bar{K}$ where K is the kernel of some epimorphism of a projective module P onto A. $\mathscr{R}(A)$ is well defined because of the following remark.

Remark 1. If P and P' are projective modules which map homomorphically onto a module A, and if K and K' are the kernels of these mappings respectively, then $P \oplus K' \cong P' \oplus K$.

Proof. Suppose $0 \to K \to P \xrightarrow{\alpha} A \to 0$ and $0 \to K' \to P' \xrightarrow{\beta} A \to 0$ are the given exact sequences. Since P and P' are projective there exist homomorphisms $g: P \to P'$ and $f: P' \to P$ such that $\alpha = \beta g$ and $\beta = \alpha f$.

Define $\theta = 1_p - fg$ and $\psi = 1_{p'} - gf$. Now define $\gamma: P \oplus K' \to P' \oplus K$ and $\delta: P' \oplus K \to P \oplus K'$ by $\gamma(p, k') = (-g(p) + k', \ \theta(p) + f(k'))$ and $\delta(p', k) = (-f(p') + k, \psi(p') + g(k))$. Computation then shows that $\gamma\delta$ and $\delta\gamma$ are identity mappings.

The above remark shows that it makes sense to define $\mathscr{R}^0(A) = A$ and $\mathscr{R}^n(A) = \mathscr{R}(\mathscr{R}^{n-1}(A))$, when n is a positive integer.

Remark 2. If $K \sim M$, then K is projective if and only if M is projective.

Proof. There exist projective modules P and P' such that $K \oplus P \cong M \oplus P'$.

Remark 2 makes it possible to make the following definition:

Definition (Kaplansky). An R-module A has homological dimension n (written $d_R(A) = n$) if n is the least integer i such that $\mathscr{R}^i(A)$ is projective. We write $d_R(A) = \infty$ in case $\mathscr{R}^n(A)$ is not projective for all nonnegative integers n.

Notice that the symbol $dh_R(A)$ has been used for the Cartan and Eilenberg type of homological dimension, whereas $d_R(A)$ has been used for the Kaplansky type.

Theorem 1. *For any R-module A we have $d_R(A) = dh_R(A)$.*

Proof. Suppose that $dh_R(A) = n < \infty$. Then there is a projective resolution $0 \to P_n \xrightarrow{f_n} \cdots \to P_0 \xrightarrow{f_0} A \to 0$. Let $K_i = \operatorname{Ker} f_i$, for $0 \leqq i \leqq n$. Then we have $K_i \sim \mathscr{R}^{i+1}(A)$ for $0 \leqq i \leqq n$. But $K_{n-1} \cong P_n$ so that $\mathscr{R}^n(A)$ is projective. This shows that $d_R(A) \leqq dh_R(A)$.

Now suppose $d_R(A) = n < \infty$. By definition there exist projective modules P_i and epimorphisms f_i such that $0 \to \mathscr{R}^{i+1}(A) \to P_i \xrightarrow{f_i} \mathscr{R}^i(A) \to 0$ is exact for each $i = 0, 1, \cdots, n-1$. Since $\mathscr{R}^n(A)$ is projective we define $P_n = \mathscr{R}^n(A)$. This leads to a projective resolution $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to A \to 0$, which implies that $dh_R(A) \leqq n$. Thus, $dh_R(A) = d_R(A)$ in all cases.

3. The next definition of homological dimension is found in Samuel [5].

Definition. If M is a finitely generated module over a Noetherian local ring $(R, \mathcal{M})$, then M has homological dimension n if n is the least integer d such that $\mathrm{Tor}^R_{d+1}(M, R/\mathcal{M}) = (0)$. If the torsion modules are all nonzero, then it is said that M has infinite homological dimension. One writes $Dh_R(M) = n$ or $Dh_R(M) = \infty$, as the case may be.

Theorem 2. *Let $(R, \mathcal{M})$ be a Noetherian local ring and let M be a finitely generated R-module. If $Dh_R(M) \leqq n$ and if $L_{n-1} \to \cdots \to L_1 \to L_0 \to M \to 0$ is an exact sequence where the L_i are finitely generated free R-modules, then L_n, where $L_n = \mathrm{Ker}\,(L_{n-1} \to L_{n-2})$, is free and finitely generated over R, and $0 \to L_n \to L_{n-1} \to \cdots \to L_0 \to M \to 0$ is exact.*

Proof. It is obvious that L_n is finitely generated and that $0 \to L_n \to L_{n-1} \to \cdots \to L_1 \to L_0 \to M \to 0$ is exact. The shift formula for Tor shows that $\mathrm{Tor}^R_{n+1}(M, R/\mathcal{M})$ is equal to $\mathrm{Tor}_1{}^R(L_n, R/\mathcal{M})$. Then $\mathrm{Tor}^R_{n+1}(M, R/\mathcal{M}) = (0)$ since $Dh_R(M) \leqq n$. Thus $\mathrm{Tor}_1{}^R(L_n, R/\mathcal{M}) = (0)$, so that L_n is free.

The above theorem shows that to say $Dh_R(M) = n$ means that there exists a free resolution for M where each term is finitely generated and the minimal length of such resolutions is $n + 1$ terms.

Recall that Kaplansky [3], proved that any projective module over a local ring is free. This means that Samuel's definition of homological dimension could be rephrased in terms of minimal length projective resolutions with finitely generated terms. It will be seen later that it is not necessary to stipulate 'with finitely generated terms.'

4. The fourth and last method of defining homological dimension that we shall examine is the 'syzygy' method. Historically, this concept was introduced by Hilbert in the case of polynomial rings, and was extended to the case of regular local rings by Serre. The relevant material is related by Zariski and Samuel in [**6**] and by Nagata in [4]. We shall follow the exposition of Nagata in the discussion of the syzygy concept.

Again, Nagata is concerned only with finitely generated modules over Noetherian local rings. Let $u_1, \cdots, u_n$ be a minimal base for the module M over the Noetherian local ring $(R, \mathcal{M})$. Define

$$N = \left\{ \sum_{i=1}^{n} a_i U_i \mid \sum_{i=1}^{n} a_i u_i = 0,\ a_i \in R \right\},$$

where the U_i are indeterminates over R, and call N a *relation module of* M.

Recall from the theory of local rings (see (5.1) in [4]) that $u_1, \cdots, u_n$

is a minimal basis of M over R if and only if $u_1 + \mathcal{M}M, \cdots, u_n + \mathcal{M}M$ is a vector basis of $M/\mathcal{M}M$ over $R/\mathcal{M}$. With this in mind it is easy to see that N will be well defined up to isomorphism, and thus N is called *the* relation module of M. In the case where $u_1, \cdots, u_n$ is a generating set and not necessarily a minimal basis of M, N is called the relation module of the u_i.

Definition. If M is a finitely generated module over a Noetherian local ring $(R, \mathcal{M})$, then the 0th syzygy of M is M itself, and when the ith syzygy ($i \geqq 0$) is defined the $(i+1)$st syzygy of M is the relation module of the ith syzygy of M. One writes $\operatorname{syz}_R{}^0 M = M$ or $\operatorname{syz}_R{}^{i+1} M = \operatorname{syz}_R{}^i (\operatorname{syz}_R{}^i M)$ as the case may be.

Remark 3. With the above notation let F be the free R-module $\sum_{i=1}^n RU_i$. Then $N = \operatorname{syz}_R{}^1 M \subseteq \mathcal{M}F$. By induction, it is easy to verify that for any positive integer n, there is a free R-module F_n such that $\operatorname{syz}_R{}^n M \subseteq \mathcal{M}F_n$.

Remark 4. If A is any ideal of a Noetherian local ring $(R, \mathcal{M})$, then $A \cong \operatorname{syz}_R{}^1 (R/A)$.

Proof. Observe that R is a free R-module.

We are now ready to present Nagata's definition of homological dimension.

Definition. Let M be a finitely generated module over the Noetherian local ring $(R, \mathcal{M})$. Then the homological dimension of M is n (written $Hd_R(M) = n$) if $n+1$ is the smallest integer i such that $\operatorname{syz}_R{}^i M = (0)$. We write $Hd_R(M) = \infty$ if $\operatorname{syz}_R{}^i(M) \neq (0)$ for all positive integers i.

Suppose now that R and M are as above and that $Hd_R(M) = n$. Then for each $i = 1, \cdots, n+1$ there exists a free R-module F_i and homomorphisms α_i and β_i such that the sequence $0 \to \operatorname{syz}_R{}^i M \xrightarrow{\alpha_i} F_i \xrightarrow{\beta_i} \operatorname{syz}_R{}^{i-1} M \to 0$ is exact. $Hd_R(M) = n$ implies that $\operatorname{syz}_R{}^{n+1} M = (0)$ so that $F_{n+1} \cong \operatorname{syz}_R{}^n M$. By defining $f_1 = \beta_1$ and $f_i = \alpha_{i-1}\beta_i$ for $i \geqq 2$, one has the free resolution $0 \to F_{n+1} \xrightarrow{f_{n+1}} F_n \to \cdots \to F_1 \xrightarrow{f_1} M \to 0$. This proves the following theorem:

Theorem 3. *If M is a finitely generated module over a Noetherian local ring $(R, \mathcal{M})$, then $Dh_R(M) \leqq Hd_R(M)$.*

Remark 5. Let F be a free module mapping homomorphically onto M, where M and F are finitely generated over the Noetherian local ring R. Assume that $0 \to K \to F \xrightarrow{f} M \to 0$ is exact. Then $K = \operatorname{Ker} f$ is isomorphic to the direct sum of $\operatorname{syz}_R{}^1 M$ and a free R-module.

Proof. May be found in [4].

Theorem 4. *If M is a finitely generated module over a Noetherian local ring R, then $Hd_R(M) \leqq Dh_R(M)$.*

PROOF. Assume that $Dh_R(M) = n < \infty$. Then there is a free resolution $0 \to L_n \xrightarrow{g_n} L_{n-1} \to \cdots \to L_0 \xrightarrow{g_0} M \to 0$, with the L_i finitely generated over R, and there is no such resolution with fewer terms. As has been done previously one may use the syzygy method to construct a free resolution $\cdots \to F_n \xrightarrow{f_n} F_{n-1} \to \cdots \to F_0 \xrightarrow{f_0} M \to 0$, where $\mathrm{syz}_R{}^{i+1} M$ equals $\mathrm{Ker} f_i$, for $i \geqq 0$. In order to show that $Hd_R(M) \leqq n$, one needs to prove that $\mathrm{syz}_R{}^{n+1}M = \mathrm{Ker} f_n = (0)$.

Suppose we let $K_i = \mathrm{Ker}\, g_i$ for $i \geqq 0$. Then Remark 5 shows that $K_0 \cong N_0 \oplus \mathrm{syz}_R{}^1M$ where N_0 is a free R-module. The relation module of $N_0 \oplus \mathrm{syz}_R{}^1M$ equals the relation module of $\mathrm{syz}_R{}^1M$ because N_0 is free. This means that K_1 is isomorphic to $N_1 \oplus \mathrm{syz}_R{}^2M$ where N_1 is some free R-module. Proceeding in this fashion by induction one has that K_n is isomorphic to $N_n \oplus \mathrm{syz}_R{}^{n+1}M$ with N_n a free R-module. But $K_n = \mathrm{Ker}\, g_n = (0)$, so that $\mathrm{syz}_R{}^{n+1}M = (0)$. Thus, we have shown that $Hd_R(M) \leqq n = Dh_R(M)$.

Let M be a finitely generated R-module, where $(R, \mathscr{M})$ is a Noetherian local ring. Then a minimal length projective resolution from the class of all projective resolutions of M is no longer than a minimal length projective resolution of M from the class of projective resolutions with finitely generated terms. Hence, we have that $dh_R(M) = d_R(M) \leqq Dh_R(M) = Hd_R(M)$ for any finitely generated module M over a Noetherian local ring R.

Suppose now that $d_R(M) = n < \infty$. Then there is a projective resolution of the form $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$. We have that $\mathrm{Tor}^R_{n+1}(M, R/\mathscr{M}) = \mathrm{Tor}_1{}^R(P_n, R/\mathscr{M}) = 0$ by the shift formula for Tor and from the fact that P_n is projective. Thus $Dh_R(M) \leqq n$. The following theorem has now been completely proved.

THEOREM 5. *When M is a finitely generated module over a Noetherian local ring R, then $d_R(M) = dh_R(M) = Dh_R(M) = Hd_R(M)$.*

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