ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 1, Number 3, Summer 1971

THE BOUNDARY OF A VERTICALLY CONNECTED CUBE IS TAME

L. D. LOVELAND[1]

A subset X of E^3 is defined to be *vertically connected* if the intersection of each vertical line with X is a connected set. The following question, stated here in terms of the above definition, appears in §9.3 of [2]:

Is the boundary of a vertically connected crumpled cube in E^3 a tame 2-sphere?

In this note we present an affirmative answer to the question.

We define a crumpled cube in E^3 to be the union of a 2-sphere in E^3 with either of its complementary domains. Thus if C is a crumpled cube in E^3, then the closure of the complementary set $(E^3 - C)$ is also a crumpled cube which we shall denote by C^*. Notice that C and C^* are not homeomorphic. The boundary of a crumpled cube C, denoted by $\mathrm{Bd}\, C$, is a 2-sphere; and the set $C - \mathrm{Bd}\, C$ is called the interior of C and is denoted by $\mathrm{Int}\, C$. We shall make use of J. W. Cannon's $*$-taming set theory for crumpled cubes in E^3 [4]. A subset X of E^3 is defined to be a $*$-taming set for crumpled cubes in E^3 if X is closed and satisfies the following condition: if C is a crumpled cube in E^3 such that $X \subset C$ and $\mathrm{Bd}\, C$ is locally tame modulo X, then $\mathrm{Bd}\, C$ is tame from $\mathrm{Int}\, C^*$. Thus if C^* is a compact crumpled cube, X is a $*$-taming set in C $(=(C^*)^*)$, and $\mathrm{Bd}\, C$ is locally tame modulo X, then C^* is a 3-cell.

The properties of $*$-taming sets which we shall need are given in [4] and [5]. In particular we shall use the following results:

(1) The closed countable union of $*$-taming sets is a $*$-taming set [4].

(2) The closed union of nondegenerate vertical intervals in E^3 is a $*$-taming set [5, Theorem 4].

(3) Each tame nondegenerate subcontinuum of a 2-sphere in E^3 is a $*$-taming set [4].

(4) Theorem 1 of [5] which is stated in the second paragraph of the following proof.

THEOREM. *If a crumpled cube C in E^3 is vertically connected, then the boundary of C is a tame 2-sphere.*

Received the the editors February 24, 1970 and, in revised form, March 21, 1970.

AMS 1970 *subject classifications*. Primary 55A30, 57A10.

Key words and phrases. Tame 2-spheres, tame surfaces, embeddings in E^3, surfaces in E^3, $*$-taming sets, vertically connected.

[1]This research was partially supported by NSF-GP-8454.

PROOF. The fact that C is a 3-cell comes directly from Theorem 4 of [5] since the set C^* is the union of nondegenerate vertical rays. We shall show that the boundary S of C is tame from Int C^* by exhibiting C as the countable union of $*$-taming sets (see statement (1) above).

Let P be a vertical plane and choose a coordinate system for E^3 such that P is the x-z plane. For each real number r we denote the plane parallel to P which contains $(0, r, 0)$ by $P(r)$, and we let $X(r) = P(r) \cap C$. For each positive number t we let $X^t(r)$ be the union of all components of $X(r)$ having diameter no smaller than t, and we let $X^t = \{X^t(r) \mid r \text{ is a real number}\}$. Cannon proved [5, Theorem 1] that, for each i, $X^{1/i}$ is a $*$-taming set. We shall obtain a $*$-taming set X in S such that $C = (\bigcup_{i=1}^{\infty} X^{1/i}) \cup X$. Once this is accomplished it will follow that C is a $*$-taming set [4], and consequently that S is tame from Int C^*.

For each point x of E^3 let $L(x)$ denote the vertical line containing x, let $X = \{x \in S \mid L(x) \cap \text{Int } C = \emptyset \text{ and } L(x) \cap C \neq \emptyset\}$, and let π denote the vertical projection of E^3 onto a horizontal plane Q where Q lies below C. Let $A = \{x \in Q \mid L(x) \cap \text{Int } C \neq \emptyset\}$, let $B = \{x \in Q \mid L(x) \cap C = \emptyset\}$, and let $R = \pi(X)$. It is clear that A and B are disjoint open sets, that $Q = A \cup B \cup R$, and that R is the closed set $Q - (A \cup B)$. Our object is to show that R is a simple closed curve because X would then lie on the vertical cylinder through R and would consequently be tame. We shall prove that R is a simple closed curve and use this fact later to show that X is a $*$-taming set.

In order to show that R is a simple closed curve it suffices to show that $R = \text{Bd } A$ and that the bounded, open subset A of Q is also connected, simply connected, and 0-ulc [6]. (This result is also given in [7, p. 167].) For $p \in R$ we note that each point of $\pi^{-1}(p)$ is arcwise accessible from Int C. Since π is continuous this means that R is arcwise accessible from A at p and implies that $R \subset \text{Bd } A$. In a similar manner we use the fact that Int C is arcwise connected to show that A is arcwise connected. Thus $R = \text{Bd } A$ and A is connected. To show that A is simply connected we let J be a simple closed curve in A, and we let D denote the disk in Q bounded by J. Let H be a vertical annulus whose boundary components are J and another simple closed curve above C, and let $U = H \cap \text{Int } C$. Clearly U is an open subset of H, and we claim that U contains a simple closed curve K that is essential (bounds no disk) in H. One way to obtain K is to note that if fxg is a short horizontal arc in U, then $U - L(x)$ is connected and hence arcwise connected. (The set $U - L(x) = (\pi \mid \text{Int } C)^{-1}(J - x)$ is connected because $J - x$ is connected and $\pi \mid \text{Int } C$ is an open map with connected point inverses.) Thus the simple closed curve K can be selected to lie in the union of fxg with an arc from f to g which lies in $U - L(x)$.

We desire to show that $D \subset A$, so we suppose that there exists a point z in Int D such that z is not in A. Either z belongs to B or to R, and if $z \in B$ there must exist a point of R in Int D, since R separates A from B in Q. Thus we may assume that z lies in R. Our contradiction will be that K can be shrunk to a point in $E^3 - L(z)$. This is an impossibility since K is a generator for the infinite cyclic fundamental group $\pi_1(E^3 - L(z))$. Since $L(z) \cap C$ lies in S we may use [1] to obtain a 2-sphere S' such that $L(z) \cap S' = L(z) \cap C$, S' is locally polyhedral modulo $L(z) \cap C$, and $K \subset \text{Int } S'$. The crumpled cube $C' = S' \cup \text{Int } S'$ has a boundary S' which is locally tame modulo the $*$-taming set $L(z)$ [5, Theorem 4], and $L(z)$ lies in $E^3 - \text{Int } C'$. Thus, from the definition of $*$-taming sets, we see that C' is a 3-cell. Now we know that K can be shrunk to a point in Int C', and consequently K contracts in $E^3 - L(z)$. It follows that A is simply connected.

We now show that A is 0-ulc. If this is not the case there must exist a positive number ξ and a sequence $\{p_i, q_i\}_{i=1}^{\infty}$ of pairs of points of A such that, for each positive integer i,

(1) $0 < d(p_i, q_i) < 1/i$ and

(2) p_i and q_i do not lie in an arc in A of diameter less than ξ. We shall show that the existence of such a number ξ and such a sequence of pairs of points of A leads to a contradiction.

For each positive integer i, let p_i' and q_i' denote points of $L(p_i) \cap \text{Int } C$ and $L(q_i) \cap \text{Int } C$, respectively. Choosing subsequences if necessary, we find that we lose no generality in assuming that the sequences $\{p_i\}$, $\{q_i\}$, $\{p_i'\}$, and $\{q_i'\}$ all converge, say to points p, q, p', and q', respectively, with $p = q \in R$ and $\{p', q'\} \subset L(p) \cap C = L(q) \cap C$.

It is well known that Int C is 0-ulc and that each point of Bd C is arcwise accessible from Int C [8, p. 66]. Thus, for i sufficiently large, there exist arcs P and Q in C, each of diameter less than $\xi/2$, joining p_i' and q_i' to p' and q', respectively. The set $P \cup Q \cup (L(p) \cap C)$ clearly contains an arc α from p_i' to q_i' which, of necessity, lies in an $\xi/2$-neighborhood of $L(p) \cap C$. Since Int C is 0-ulc, the arc α can be approximated by an arc β from p_i' to q_i' which lies in Int C and also in an $\xi/2$-neighborhood of $L(p) \cap C$. The set $\pi(\beta)$ lies in A, has diameter less than ξ, and contains an arc from p_i to q_i. This is in contradiction to the properties of the sequence $\{p_i, q_i\}$, and our claim that A is 0-ulc is established.

Now X is known to be tame for it lies in a vertical cylinder generated by the simple closed curve R. Since X is vertically connected and projects under π to the connected set R, it is clear that X is connected. This insures that X is a taming set [3], and consequently X is also a $*$-taming set [4]. We note that $C = X \cup (\bigcup_{i=1}^{\infty} X^{1/i})$, and the result follows.

References

1. R. H. Bing, *Approximating surfaces with polyhedral ones,* Ann. of Math. (2) **65** (1957), 456–483. MR **19**, 300.

2. C. E. Burgess and J. W. Cannon, *Embeddings of surfaces in E^3*, Rocky Mt. J. Math. **1** (1971), 259–344.

3. J. W. Cannon, *Characterizations of taming sets on 2-spheres,* Trans. Amer. Math. Soc. **147** (1970), 289–299.

4. ——, *$*$-taming sets for crumpled cubes.* I: *Basic properties,* Trans. Amer. Math. Soc. (to appear).

5. ——, *$*$-taming sets for crumpled cubes.* II: *Horizontal sections in closed sets,* Trans. Amer. Math. Soc. (to appear).

6. R. L. Moore, *A characterization of Jordan regions by properties having no reference to their boundaries,* Proc. Nat. Acad. Sci. U.S.A. 4 (1918), 364–370.

7. M. H. A. Newman, *Elements of the topology of plane sets of points,* 2nd ed., Cambridge Univ. Press, Cambridge, 1951. MR **13**, 483.

8. R. L. Wilder, *Topology of manifolds,* Amer. Math. Soc. Colloq. Publ., vol. 32, Amer. Math. Soc., Providence, R. I., 1963. MR **32** #440.

Utah State University, Logan, Utah 84321