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PARALLELISM IN NEAR-RINGS

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Various concepts, such as planarity [2], affinity [4], incidence [5] and projectivity [1], which lead to geometrical interpretations, have been introduced in near-rings and near-fields. In this paper it is shown that one can also define parallelism in near-rings as soon as one assumes the existence of nonzero constants. This concept will be discussed and completely described in the case, when N is an ordered near-ring.

1. **Definitions.** A near-ring $(N, +, \cdot)$ is a set N with two binary operations, such that

(i) $(N, +)$ is a group with additive identity 0,

(ii) $(N, \cdot)$ is a semigroup, and

(iii) $n_1(n_2 + n_3) = n_1n_2 + n_1n_3$ for all $n_1, n_2, n_3 \in N$.

An element $k \in N$ is called constant, if $0k = k$. K denotes the set of all constant elements, which turns out to be an invariant sub-near-ring of N (see [3]).

If N is a ring or a near-field, then $K = \{0\}$. Therefore the case $K \neq \{0\}$ is somewhat typical for near-rings and we will therefore assume this, throughout this paper.

K is said to form a base (for equality), if $kn_1 = kn_2$, for all $k \in K$, implies $n_1 = n_2$ (cf. [6]).

For each $n \in N$ one can form its graph $\Gamma(n)$ in $K \times K$ by defining $\Gamma(n) := \{(k, kn), k, kn \in K\}$. $n_1 \neq n_2$ implies $\Gamma(n_1) \neq \Gamma(n_2)$ iff K forms a base.

We will call two graphs $\Gamma(n_1)$ and $\Gamma(n_2)$ v-parallel, if there exists a $k_v \in K$, which depends only on n_1 and n_2, such that

$$kn_2 = k_v + kn_1 \quad \text{for all } k \in K. \tag{1}$$

Similarly we call $\Gamma(n_1)$ and $\Gamma(n_2)$ h-parallel, if there exists a $k_h \in K$ such that

$$kn_2 = (k + k_h)n_1 \quad \text{for all } k \in K. \tag{2}$$

If K forms a base, then each $n \in N$ can be regarded as a function f_n mapping K into K by defining $f_n(k) := kn$ (see [6]). Then two graphs are v-parallel (h-parallel), if the corresponding functions are vertical (horizontal) translations of each other.

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We will call two elements $n_1, n_2 \in N$ v-parallel (h-parallel), if their associated graphs are v-parallel (h-parallel) and denote this by $n_1 \parallel_v n_2$ or $n_1 \parallel_h n_2$, respectively.

If the elements k_v and k_h in (1) and (2), respectively, are unique, then we define

$$v(n_1, n_2) := k_v \tag{3}$$

and

$$h(n_1, n_2) := k_h. \tag{4}$$

Formulas involving these v- and h-functions shall be understood in this way: they are valid, if all expressions are defined. Notice, that these functions are only defined for parallel n_1, n_2.

2. **Basic properties of the parallelity relation.**

THEOREM 1. *$\parallel_v$ is an equivalance relation in N. If $n_1 \parallel_v n_2$, then $n_1 + n \parallel n_2 + n$ and $nn_1 \parallel_v nn_2$ for all $n \in N$. Moreover, $v(n, n) = 0$; $v(n_1, n_2) = -v(n_2, n_1)$; $v(n_1, n_2) + v(n_2, n_3) = v(n_1, n_3)$; $v(n_1 + n, n_2 + n) = v(n_1, n_2)$; $v(nn_1, nn_2) = v(n_1, n_2)$.*

THEOREM 2. *If $(K, +)$ is abelian, then $\parallel_h$ is an equivalence relation in N. Moreover, $h(n, n) = 0$, $h(n_1, n_2) = -h(n_2, n_1)$ and $h(n_1, n_2) + h(n_2, n_3) = h(n_1, n_3)$.*

The proofs of these basic theorems are straightforward and therefore omitted.

THEOREM 3. *$\parallel_v$ is a congruence relation in $(N, +)$ with $kn_2 = k_v + kn_1$, $kn_2{}' = k_v{}' + kn_1{}'$, $k(n_1 + n_1{}') = \bar{k}_v + k(n_2 + n_2{}')$ implying $\bar{k}_v = k_v + k_v{}'$ if and only if $(K, +)$ is abelian.*

PROOF. Let $n_1 \parallel_v n_2$ and $n_1{}' \parallel_v n_2{}'$. Then $kn_2 = k_v + kn_1$ and $kn_2{}' = k_v{}' + kn_1{}'$ for all $k \in K$. Then $k(n_2 + n_2{}') = k_v + kn_1 + k_v{}' + kn_1{}'$. If $(K, +)$ is abelian, then this is equal to $k_v + k_v{}' + k(n_1 + n_1{}')$, which proves the first part of the theorem. Now let k', k'' be arbitrary elements $\in K$. Define $n_2 := n_1{}' = k' = n_1$ and $n_2{}' := k'' + n_1{}'$. Then $n_1 \parallel_v n_2$ and $n_1{}' \parallel_v n_2{}'$. If $\parallel_v$ is a congruence relation, then $n_1 + n_1{}' \parallel n_2 + n_2{}'$, therefore $k(n_2 + n_2{}') = \bar{k}_v + k(n_1 + n_1{}')$, $kk' + kk'' + kn_1{}' = \bar{k}_v + kk' + kn_1{}'$, which gives us $k' + k'' = \bar{k}_v + k'$. Now $k_v = 0$, $k_v{}' = k''$. By assumption $\bar{k}_v = k_v + k_v{}' = k''$, which proves $k' + k'' = k'' + k'$. Therefore $(K, +)$ is abelian.

THEOREM 4. *Let $n_1 \parallel_h n_2$ and $n_1{}' \parallel_h n_2{}'$ with $kn_2 = (k + k_h)n_1$ and $kn_2{}' = (k + k_h{}')n_1{}'$. if $k_h = k_h{}'$, then $n_1 + n_1{}' \parallel_h n_2 + n_2{}'$. If $k_h{}' = 0$, then $n_1n_1{}' \parallel_h n_2n_2{}'$.*

This theorem follows immediately from the equations $k(n_2 + n_2') = (k + k_h)n_1 + (k + k_h')n_1' = (k + k_h)(n_1 + n_1')$ and $k(n_2 n_2') = (kn_2)n_2' = ((k + k_h)n_1)n_2' = ((k + k_h)n_1 + k_h')n_1' = (k + k_h)(n_1 n_1')$.

One can define $n_1 \parallel n_2 :\Longleftrightarrow n_1 \parallel_v n_2$ and $n_1 \parallel_h n_2$. It would be interesting to examine, how far classes of $\parallel$-parallel graphs characterize somewhat like classes of "parallel lines". Theorems 1 and 2 imply

THEOREM 5. *If $(K, +)$ is abelian, then $\parallel$ is an equivalence relation in N.*

Of geometric interest is the analogue to one of Euclid's axioms:

THEOREM 6. *Let $n \in N$, $(k_1, k_2) \in K \times K$. Then there exists an element $n_1 \in N$ which is v-parallel to n and whose graph $\Gamma(n_1)$ contains (k_1, k_2). There exists exactly one n_1 if and only if K forms a base.*

PROOF. The first assertion follows if one takes n_1 as $n_1 := k_2 - k_1 n + n$. Now let n_1' be v-parallel to n with $k_1 n_1' = k_2$. From $n_1' \parallel_v n_1$, it follows that $kn_2' = k_v + kn_1$ for all $k \in K$. If we take $k = k_1$, we get

$$k_2 = k_1 n_1' = k_v + k_1 n_1 = k_v + k_2$$

and therefore $k_v = 0$. But $kn_1' = kn_1$ for all $k \in K$ implies $k(n_1 - n_1') = 0$, therefore $n_1 - n_1' = 0$.

Conversely, let n_1 and n_1' be arbitrary $\in N$ with $kn_1 = kn_1'$ for all $k \in K$. Thus $n_1 \parallel_v n_2$ and $k_1 n_1 = k_1 n_1' = k_2$ for $k_1 \in K$. By hypothesis, $n_1 = n_1'$ and therefore K forms a base.

The analogous theorem for h-parallelity does not hold in general.

3. Horizontal translations.

DEFINITION. If $n \in N$ has the property that for all $k_h \in K$ there exists an element $n_1 \in N$ with $kn_1 = (k + k_n)n$, for all $k \in K$, then we will say that n *admits translations*. Let T be the set of all elements in N which admit translations.

PROPOSITION 1. *T is a right invariant sub-near-ring of N with $K \subseteq T$.*

The proof is quite obvious.

PROPOSITION 2. (i) *$I := \{n \in N : kn = 0$ for all $k \in K\}$ is an ideal (kernel of a near-ring homomorphism) in N.*

(ii) *$I = \{0\} \Longleftrightarrow K$ forms a base.*

(iii) *In $\bar{N} := N/I$ the constants form a base.*

(iv) *In each coset of $\bar{N}$ exists at most one constant element of N.*

(v) *The constant cosets are exactly those containing a constant element of N.*

(vi) card $\bar{N} \geqq$ card K.

These statements are either contained in [6] or their proof is straightforward.

Proposition 3. *Let t be any element of T, I as above and ϕ the natural homomorphism $N \to N/I = \bar{N}$. Then $\phi(t)$ admits translations in $\bar{N}$.*

Proof. Let C_t denote the coset containing t. Because of Proposition 2 (v), each constant coset of $\bar{N}$ is the ϕ-image of a $k \in K$. Let $C_{k_0} = \phi(K_0)$ denote such a constant coset. We have to find a coset C_n fulfilling $C \cdot C_n = (C + C_{k_0})C_t$ for all constant cosets C. Denote by n the element with $kn = (k + k_0)t$. Then $\phi(n) =: C_n$ fulfills the above equation, as desired.

Proposition 4. *If N contains an identity, then $T = N$.*

The proof is essentially the same as in Lemma 5 of [7].

Let T^* be the set of all elements fulfilling the condition for $\|_h$ analogous to that in Theorem 6. One can easily see that $T^* \subseteq T$. Equality does not hold in general. Now we will determine the sets T^* and T in the case when N is a fully ordered near-ring (see [8]).

Theorem 7. *Let N be (fully) ordered. Then*

$$T = \{n \in N : kn = 0n \text{ for all } k \in K\} =: A.$$

Proof. In [8] it was proved that in a fully ordered near-ring with $K \neq \{0\}$, $kn = 0n$ is valid for all $k \geqq 0$, $k \in K$. If $n \in T$, $k_1 n \neq 0n$ for some $k_1 < 0$, then there exists $n_1 \in N$ fulfilling $kn = (k + |k_1|)n_1 = (k - k_1)n_1$. Then $0n = (-k_1)n_1$, $k_1 n = 0n_1$. But $k_1 n \neq 0n$, therefore $(-k_1)n_1 \neq 0n_1$, $-k_1 > 0$, which is impossible. This proves $T \subseteq A$. $A \subseteq T$ is obvious.

Proposition 5. *If N is fully ordered, then N/I is also fully ordered (I as in Proposition 2).*

This follows from the fact that I turns out to be a convex ideal and from [8].

Theorem 8. *If N is ordered, then the set of all elements of $N/I = \bar{N}$ admitting translations coincides with the set of all constant elements of $\bar{N}$.*

Proof. Proposition 5 insures that $\bar{N}$ is ordered. Therefore each element $\bar{t}$ of $\bar{N}$ admitting translations fulfills $\bar{k}\,\bar{t} = \bar{0}\,\bar{t}$, therefore $\bar{k}(\bar{t}) = \bar{k}(\bar{0}\,\bar{t})$. In $\bar{N}$ the constants form a base (Proposition 2 (iii)), therefore $\bar{t} = \bar{0}\,\bar{t}$, which proves that $\bar{t}$ is constant. The opposite direction follows from Proposition 1.

THEOREM 9. *If N is fully ordered, then* $\phi(T) = \bar{K}$ = *set of constant elements of* $\bar{N}$.

This follows from the preceding theorem and Proposition 3.

THEOREM 10. *If N is fully ordered, then* $T^* = \emptyset$.

Take for any element $n \in N$ the pair (k_1, k_2) with $k_2 \neq 0n$.

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