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SPECTRAL REPRESENTATION OF SELFADJOINT DILATIONS OF SYMMETRIC OPERATORS WITH PIECEWISE C^2 SPECTRAL FUNCTIONS

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ABSTRACT. Let A be a simple closed symmetric operator with deficiency index (1, 1) in a Hilbert space $\mathfrak{H}$. Suppose A has a selfadjoint extension A_0 in $\mathfrak{H}$ for which $\rho_0(t) = (E_0(t)g_0, g_0)$ is piecewise C^2, where $E_0(t)$ is the spectral function of A_0, and g_0 is an element in a deficiency subspace of A. Under this assumption, a spectral representation is given for all the selfadjoint extensions and minimal selfadjoint dilations of A. The procedure used is a generalization of that used when A is a Sturm-Liouville operator on $[0, \infty)$ in the limit point case at ∞. The spectral representation clarifies the nature of the spectrum and spectral multiplicity of A^+.

1. Introduction. Let A be a simple closed symmetric operator with deficiency index (1, 1) in the Hilbert space $\mathfrak{H}$. If A^+ is a selfadjoint operator in a Hilbert space $\mathfrak{H}^+$ such that $\mathfrak{H} \subset \mathfrak{H}^+$ and $A \subset A^+$, then A^+ is called a *selfadjoint extension* of A wherever $\mathfrak{H} = \mathfrak{H}^+$, and A^+ is called a *selfadjoint dilation* whenever $\mathfrak{H}$ is properly contained in $\mathfrak{H}^+$. A^+ is called a *minimal* selfadjoint dilation if A^+ is not reduced by any nontrivial subspace of $\mathfrak{H}^+ \ominus \mathfrak{H}$. It is the purpose of this article to present an expansion theorem (Theorem 1) and a spectral representation theorem (Theorem 2) for the selfadjoint extensions and dilations of A. These theorems are analogs of the eigenfunction expansion and spectral representation theorems which can be proved when A is a Sturm-Liouville differential operator on $[0, \infty)$ in the limit point case at ∞. (See, for example, Štraus [7].) In the spectral representation theorem a spectral matrix corresponding

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to A^+ is provided. The spectral representation theorem clarifies the nature of the spectrum and spectral multiplicity of A^+. It includes the representations given in [2], [3], [4].

Let $\lambda_0 = \xi_0 + i\eta_0$ be a complex number with positive imaginary part, and let g_0 be an element of norm 1 in the deficiency subspace of A corresponding to $\bar{\lambda}_0$. We shall assume in this article that A has a selfadjoint extension A_0 with spectral function $E_0(t)$ for which $\rho_0(t) = (E_0(t)g_0, g_0)$ is twice continuously differentiable everywhere except possibly at a countable set $\{t_k\}$ with no finite limit points and with $\rho_0[t_k] \neq 0$ for each k, where $\rho_0[t_k]$ is the jump in ρ_0 at t_k. While this is a stringent condition on A, it should be kept in mind that a spectral representation is being obtained for *all* the selfadjoint extensions and minimal selfadjoint dilations of A. There exist symmetric operators A which have this property and which do not come from Sturm-Liouville operators in the manner indicated by Štraus [7]. The purpose of the condition is to enable one to provide an analog of a basis for the solutions of $Af = \lambda f$. Since A is assumed to be simple, A_0 is unitarily equivalent to the multiplication operator in $L^2_{\rho_0}(-\infty, \infty)$.

If A^+ is a selfadjoint dilation of A, then the operator $R(\lambda)$ defined by the equation $R(\lambda)f = PR^+(\lambda)f$, $f \in \mathfrak{H}$, is called a generalized resolvent of A (corresponding to A^+). Here $R^+(\lambda)$ is the resolvent of A^+, and P is the operator of orthogonal projection of $\mathfrak{H}^+$ onto $\mathfrak{H}$. If A^+ is a selfadjoint extension, then $R(\lambda) = R^+(\lambda)$ is called a resolvent of A (corresponding to A^+). The operator $E(t)$ defined by the equation $E(t)f = PE^+(t)f$, $f \in \mathfrak{H}$, is called a spectral function of A (corresponding to A^+). The Stieltjes inversion formula states that $E(t)$ and $R(\lambda)$ are related by the equation

$$\begin{aligned}
&([(1/2)\{E^+(\beta) + E^+(\beta+0)\} - (1/2)\{E^+(\alpha) + E^+(\alpha+0)\}]f, h) \\
&\quad = ([(1/2)\{E(\beta) + E(\beta+0)\} - (1/2)\{E(\alpha) + E(\alpha+0)\}]f, h) \\
&\quad = (2\pi i)^{-1} \lim_{\eta \to 0+} \int_\alpha^\beta [(R(\lambda)f, h) - (R(\bar{\lambda})f, h)]\, d\xi
\end{aligned}$$

for all f, $h \in \mathfrak{H}$ and all α, β. Here $\lambda = \xi + i\eta$. We obtain our expansion theorem by evaluating the limit on the right for f, h in a certain linear manifold S which is dense in $\mathfrak{H}$. In the case that A is a Sturm-Liouville operator this can be done by expressing $(R(\lambda)f, h)$ in terms of an analytic basis for the solutions of the equation $Af = \lambda f$ and in terms of a fundamental solution constructed by use of this basis. See Štraus [7]. In our case, suppose that $R_0(\lambda)$ is the resolvent of A_0. Let $g(\lambda) = g_0 + (\lambda - \lambda_0) R_0(\lambda) g_0$, $Q(\lambda) = i \operatorname{Im} \lambda_0 + (\lambda - \lambda_0)(g_0, g(\bar{\lambda}))$. It is known that $R(\lambda) = R_0(\lambda)$

$-[\theta(\lambda)+Q(\lambda)]^{-1}(\cdot, g(\bar{\lambda}))g(\lambda)$ where $\theta(\lambda)$ is analytic for $\operatorname{Im}\lambda \neq 0$ and has nonnegative imaginary part in the upper half-plane. $\theta(\lambda)$ depends on A^+, but $R_0(\lambda)$, $Q(\lambda)$ and $g(\lambda)$ depend only on A_0. There is a one-one correspondence between the operators A^+ and the functions $\theta(\lambda)$. We define two linear functionals $D_1(f; \lambda)$ and $D_2(f; \lambda)$ on S by means of the equations

$$D_1(f;\lambda) = (f, g_0) + (\lambda - \bar{\lambda}_0)\int_{-\infty}^{\infty} \operatorname{Re}[(t-\lambda)^{-1}]\,F(t)d\rho_0(t),$$

$$D_2(f;\lambda) = (\lambda - \bar{\lambda}_0)\int_{-\infty}^{\infty} \operatorname{Im}[(t-\lambda)^{-1}]\,F(t)d\rho_0(t)/\operatorname{Im} Q(\lambda)\,.$$

Here $F(t)$ is the transform of f in $L^2_{\rho_0}(-\infty, \infty)$. Then, $D_1(f; \lambda)$ can be defined continuously across the real axis except at the t_k, and $D_2(f; \lambda)$ can be defined continuously across the real axis on the set E_2 of all t for which $\rho_0'(t) > 0$. $D_1(f; \lambda)$ and $D_2(f; \lambda)$ depend only on A_0 and not on A^+; they take the place of the analytic basis for the solutions of $Af = \lambda f$ in the case of a Sturm-Liouville operator, and the bilinear functional $D_2(f; \lambda)[D_2(h; \bar{\lambda})]^- \, i \operatorname{Im} Q(\lambda)$ takes the place of the fundamental solution. (Here $[\ ]^-$ denotes complex conjugate.) $(R(\lambda)f, h) = (R_0(\lambda)f, h) - [\theta(\lambda) + Q(\lambda)]^{-1}(f, g(\bar{\lambda}))(g(\lambda), h)$ can be expressed in terms of $D_1(f; \lambda)$, $D_2(f; \lambda)$ and $D_2(f; \lambda)[D_2(h; \bar{\lambda})]^- \, i \operatorname{Im} Q(\lambda)$, and the limit on the right in the Stieltjes inversion formula can then be evaluated much as is done by Štraus [7]. One must proceed somewhat differently, however, when the interval (α, β) contains points t_k at which $\rho_0[t_k] \neq 0$ and points t at which $\rho_0'(t) = 0$. (The latter set of points is designated by E_3.)

The expansion theorem involves two nondecreasing functions $\rho_{11}(\xi)$ and $\tau_{11}(\xi)$, defined for all real ξ, and a nondecreasing matrix function $\rho^k(\xi) = (\rho^k_{uv}(\xi))^2_{u,v=1}$ defined for ξ in $I_k = (t_k, t_{k+1})$ for each k. These functions are determined by means of the formulas

$$\rho_{11}(\xi) = \rho^k_{11}(\xi) = \lim_{\eta\to 0+} (1/\pi)\int_0^{\xi} \Phi_{11}(\lambda)d\sigma,$$

$$\tau_{11}(\xi) = \lim_{\eta\to 0+} (1/\pi)\int_0^{\xi} \operatorname{Im}\Psi_{11}(\lambda)d\sigma,$$

$$\rho^k_{22}(\xi) = \lim_{\eta\to 0} (1/\pi)\int_{a_k}^{\xi} \operatorname{Im}\Phi_{22}(\lambda)d\sigma,$$

$$\rho^k_{12}(\xi) = \rho^k_{21}(\xi) = \lim_{n\to\infty} (1/\pi)\int_{a_k}^{\xi} \operatorname{Im}\Phi_{12}(\sigma + i\eta_n)d\sigma,$$

where $\lambda = \sigma + i\eta$, a_k is an arbitrary point in I_k, $\{\eta_n\}$ is a sequence approaching zero, and

$$\Phi_{11}(\lambda) = -[\theta(\lambda) + Q(\lambda)]^{-1},$$

$$\Phi_{12}(\lambda) = \Phi_{21}(\lambda) = i\,\Phi_{11}(\lambda)\operatorname{Im} Q(\lambda),$$

$$\Phi_{22}(\lambda) = -\Phi_{11}(\lambda)[\operatorname{Im} Q(\lambda)]^2 + i \operatorname{Im} Q(\lambda),$$

$$\Psi_{11}(\lambda) = -\{Q_d(\lambda)[\theta(\lambda) + Q_c(\lambda)] - 1\}\Phi_{11}(\lambda),$$

$$Q_d(\lambda) = [i\eta_0 + (\lambda - \lambda_0)] \sum_k \rho_0[t_k] + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0) \sum_k (t_k - \lambda)^{-1} \rho_0[t_k],$$

$$Q_c(\lambda) = Q(\lambda) - Q_d(\lambda).$$

Let $\mathfrak{H}_1$ consist of all sequences $\{a(t_k)\}_k$ for which $\sum_k |a(t_k)|^2 \tau_{11}[t_k] < \infty$. Let $\mathfrak{H}_2 = \sum_k L^2_{\rho^k}(E_2 \cap I_k)$, where $L^2_{\rho^k}(E_2 \cap I_k)$ consists of all vector functions $[F_1(\xi), F_2(\xi)]$ whose components are measurable with respect to $\sigma^k(\xi) = \rho_{11}(\xi) + \rho^k_{22}(\xi)$ on $E_2 \cap I_k$ and such that

$$\int_{E_2 \cap I_k} \sum_{u,v=1}^{2} F_u(\xi)[F_v(\xi)]^- d\rho^k_{uv}(\xi) < \infty .$$

Let $\mathfrak{H}_3$ consist of all functions $F(\xi)$ for which $\int_{E_3} |F(\xi)|^2 d\rho_{11}(\xi) < \infty$. Then the spectral representation theorem says that A^+, if it is minimal, is unitarily equivalent to the multiplication operator in $\mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \mathfrak{H}_3$.

2. **Auxiliary propositions.** In the following lemmas, $\lambda = \xi + i\eta$ is a number in the complex plane, Re stands for real part, Im stands for imaginary part, and P stands for the Cauchy principal value of an integral.

LEMMA 1. *Suppose that $\rho(t)$ is a bounded nondecreasing function defined on the whole real axis and that $F(t)$ is a bounded continuously differentiable function also defined on the whole real axis. Then the Cauchy integral $\int_{-\infty}^{\infty} (t - \lambda)^{-1} F(t) d\rho(t)$ is defined and analytic for λ in the upper and lower halves of the complex plane.*

(I) *If $\rho(t)$ is twice continuously differentiable on the open interval (a, b), the following statements are true:*

$$\text{(I.A)} \quad \int_{-\infty}^{\infty} \operatorname{Re}[(t - \lambda)^{-1}] F(t) d\rho(t) = \int_{-\infty}^{\infty} (t - \xi)[(t - \xi)^2 + \eta^2]^{-1} F(t) d\rho(t)$$

can be extended continuously across (a, b), and

$$(1) \quad \lim_{\eta \to 0\pm} \int_{-\infty}^{\infty} \operatorname{Re}[(t - \lambda)^{-1}] F(t) d\rho(t) - \mathrm{P} \int_{-\infty}^{\infty} (t - \xi)^{-1} F(t) d\rho(t)$$

for ξ in (a, b). The convergence is uniform on any bounded closed subinterval $[a_1, b_1]$ of (a, b); indeed,

$$\left| \int_{-\infty}^{\infty} \operatorname{Re}[(t-\lambda)^{-1}]\, F(t)d\rho(t) - \mathrm{P} \int_{-\infty}^{\infty} (t-\xi)^{-1} F(t)d\rho(t) \right| = O(|\eta|) \tag{2}$$

uniformly for $a_1 \leqq \xi \leqq b_1$ *as* $\eta \to 0\pm$.

(I.B)

$$\int_{-\infty}^{\infty} \operatorname{Im}[(t-\lambda)^{-1}]\, F(t)d\rho(t) = \int_{-\infty}^{\infty} \eta[(t-\xi)^2+\eta^2]^{-1} F(t)d\rho(t)$$

can be extended continuously from the upper (lower) half-plane down (up) to (a, b), *and*

$$\lim_{\eta\to 0\pm} \int_{-\infty}^{\infty} \operatorname{Im}[(t-\lambda)^{-1}]\, F(t)d\rho(t) = \pm\pi F(\xi)\rho'(\xi) \quad \text{for } \xi \text{ in } (a, b). \tag{3}$$

The convergence is uniform on any bounded closed subinterval $[a_1, b_1]$ *of* (a, b); *indeed,*

$$\left| \int_{-\infty}^{\infty} \operatorname{Im}[(t-\lambda)^{-1}]\, F(t)d\rho(t) - [\pm\pi F(\xi)\rho'(\xi)] \right| = O(|\eta| \log |\eta|^{-1}) \tag{4}$$

uniformly for $a_1 \leqq \xi \leqq b_1$ *as* $\eta \to 0\pm$.

(I.C)

$$\int_{-\infty}^{\infty} (t-\lambda)^{-1} F(t)d\rho(t)$$

can be extended continuously from the upper (lower) half-plane down (up) to (a, b), *and*

$$\lim_{\eta\to 0\pm} \int_{-\infty}^{\infty} (t-\lambda)^{-1} F(t)d\rho(t) = \mathrm{P} \int_{-\infty}^{\infty} (t-\xi)^{-1} F(t)d\rho(t) \pm i\pi F(\xi)\rho'(\xi). \tag{5}$$

The convergence is uniform on any bounded closed subinterval of (a, b), *and the order of approach is the same as in* (4).

(II) *If* $\rho(t)$ *is constant on* (a, b) *or if* $F(t) \equiv 0$ *on* (a, b), *then* $\int_{-\infty}^{\infty} (t-\lambda)^{-1} F(t)d\rho(t)$ *is analytic across* (a, b), *and estimates* (2) *and* (4) *are valid for any bounded closed subinterval* $[a_1, b_1]$ *of* (a, b) *with the order of approach now being* $O(|\eta|)$ *in both cases.*

PROOF. Without loss of generality we can assume that $F(t)$ is real; for, if $F(t)$ is not real, we separate $F(t)$ into real and imaginary parts and deal with the parts separately.

PROOF OF (I.A). For ξ, t in (a, b), let $g(t) = F(t)\rho'(t)$, let $f(\xi, t) = [g(t) - g(\xi)](t-\xi)^{-1}$ if $t \neq \xi$, and let $f(\xi, t) = g'(t)$ if $t = \xi$. Then, $f(\xi, t)$ is a continuous function of (ξ, t) for ξ, t in (a, b), and $g(t) = g(\xi) + (t-\xi)f(\xi, t)$ for ξ, t in (a, b).

Suppose that a_0, b_0 are two arbitrary but fixed numbers in (a, b) and that $a < a_0 < \xi < b_0 < b$. Then,

$$
\begin{aligned}
\mathrm{P}\int_{-\infty}^{\infty} (t-\xi)^{-1}F(t)d\rho(t) = & \int_{-\infty}^{a_0} (t-\xi)^{-1}F(t)d\rho(t) \\
& + \int_{b_0}^{\infty} (t-\xi)^{-1}F(t)d\rho(t) \\
& + g(\xi)\log|(b_0-\xi)(a_0-\xi)^{-1}| \\
& + \int_{a_0}^{b_0} f(\xi, t)dt\,.
\end{aligned} \tag{6}
$$

From this equation it is evident that $\mathrm{P}\int_{-\infty}^{\infty} (t-\xi)^{-1}F(t)d\rho(t)$ is a continuous function of ξ for $a < \xi < b$. If $\lambda = \xi + i\eta$, $a_0 < \xi < b_0$, $|\eta| > 0$,

$$
\begin{aligned}
\int_{-\infty}^{\infty} & \mathrm{Re}[(t-\lambda)^{-1}]F(t)d\rho(t) \\
&= \int_{-\infty}^{a_0} (t-\xi)[(t-\xi)^2+\eta^2]^{-1}F(t)d\rho(t) \\
&\quad + \int_{b_0}^{\infty} (t-\xi)[(t-\xi)^2+\eta^2]^{-1}F(t)d\rho(t) \\
&\quad + (1/2)g(\xi)\log\{[(b_0-\xi)^2+\eta^2][(a_0-\xi)^2+\eta^2]^{-1}\} \\
&\quad + \int_{a_0}^{b_0} f(\xi, t)dt - \eta^2\int_{a_0}^{b_0} [(t-\xi)^2+\eta^2]^{-1}f(\xi, t)dt.
\end{aligned} \tag{7}
$$

From equations (6) and (7) we see that if $a < \xi_0 < b$ and if $\eta \neq 0$,

$$
\lim_{\lambda\to\xi_0} \int_{-\infty}^{\infty} \mathrm{Re}[(t-\lambda)^{-1}]F(t)d\rho(t) = \mathrm{P}\int_{-\infty}^{\infty} (t-\xi_0)^{-1}F(t)d\rho(t)\,.
$$

From this equation and the continuity of $\mathrm{P}\int_{-\infty}^{\infty} (t-\xi)^{-1}F(t)d\rho(t)$ it follows that $\int_{-\infty}^{\infty} \mathrm{Re}[(t-\lambda)^{-1}]F(t)d\rho(t)$ can be extended continuously across (a, b) and that equation (1) is true. Suppose that $[a_1, b_1]$ is a bounded closed subinterval of (a, b) and that a_0, b_0 are chosen so that $a < a_0 < a_1 < b_1 < b_0 < b$. From equations (6) and (7) we see that

$$\int_{-\infty}^{\infty} \mathrm{Re}[(t - \xi \pm i\eta)^{-1}] F(t)d\rho(t) - \mathrm{P}\int_{-\infty}^{\infty} (t - \xi)^{-1}F(t)d\rho(t)$$

$$= -\eta^2 \int_{-\infty}^{a_0} [(t - \xi)^2 + \eta^2]^{-1}(t - \xi)^{-1}F(t)d\rho(t)$$

$$- \eta^2 \int_{b_0}^{\infty} [(t - \xi)^2 + \eta^2]^{-1}(t - \xi)^{-1}F(t)d\rho(t)$$

$$+ (1/2)g(\xi) \log \{[1 + \eta^2/(b_0 - \xi)^2] [1 + \eta^2/(a_0 - \xi)^2]^{-1}\}$$

$$- \eta^2 \int_{a_0}^{b_0} [(t - \xi)^2 + \eta^2]^{-1}f(\xi, t)dt\,.$$

If we assume that $a_1 \leqq \xi \leqq b_1$ and estimate each of the terms on the right in the above equation, we see that estimate (2) is correct.

PROOF OF (I.B). If $\lambda = \xi + i\eta$, $\eta > 0$, and if $a < a_0 < \xi < b_0 < b$, then

$$\int_{-\infty}^{\infty} \mathrm{Im}\, [(t - \lambda)^{-1}] F(t)d\rho(t)$$

$$= \eta \int_{-\infty}^{a_0} [(t - \xi)^2 + \eta^2]^{-1}F(t)d\rho(t)$$

$$+ \eta \int_{b_0}^{\infty} [(t - \xi)^2 + \eta^2]^{-1}F(t)d\rho(t) \tag{8}$$

$$+ g(\xi)\{\tan^{-1}[\eta^{-1}(b_0 - \xi)] + \tan^{-1}[\eta^{-1}(\xi - a_0)]\}$$

$$+ \eta \int_{a_0}^{b_0} (t - \xi)[(t - \xi)^2 + \eta^2]^{-1}f(\xi, t)dt\,.$$

From this expression we see that if $\lambda = \xi + i\eta$, $\eta > 0$, and if $a < \xi_0 < b$, then

$$\lim_{\lambda \to \xi_0} \int_{-\infty}^{\infty} \mathrm{Im}\, [(t - \lambda)^{-1}] F(t)d\rho(t) = \pi F(\xi_0)\rho'(\xi_0)\,.$$

Since $\pi F(\xi)\rho'(\xi)$ is continuous for $a < \xi < b$, we see that

$\int_{-\infty}^{\infty} \operatorname{Im}[(t-\lambda)^{-1}]F(t)d\rho(t)$ can be extended continuously from the upper half-plane down to the real axis, and equation (3) is valid with the plus signs. Suppose now that $a < a_0 < a_1 \leqq \xi \leqq b_1 < b_0 < b$. Then it can be seen from equation (8) that estimate (4) is valid with the plus sign. Since

$$\int_{-\infty}^{\infty} \operatorname{Im}[(t-\bar{\lambda})^{-1}]F(t)d\rho(t) = -\int_{-\infty}^{\infty} \operatorname{Im}[(t-\lambda)^{-1}]F(t)d\rho(t),$$

it follows immediately that (3) and (4) are also valid with the minus sign.

(I.C) follows immediately from (I.A) and (I.B). Statement (II) is not difficult to check. This completes the proof of Lemma 1.

The following lemma is a generalization of Lemma 5 of Štraus [7].

LEMMA 2. *Suppose that* $\Psi(\lambda) = \Psi(\xi + i\eta)$ *is continuous for* $a \leqq \xi \leqq b, 0 \leqq |\eta| \leqq \eta_0$, *and that* $|\Psi(\xi + i\eta) - \Psi(\xi)| = O(|\eta| \log |\eta|^{-1})$ *uniformly for* $a \leqq \xi \leqq b$ *as* $|\eta| \to 0$. *Suppose also that* $\Phi(\lambda)$ *is continuous in the upper half-plane and that* $\int_a^b |\Phi(\xi + i\eta)|d\xi = O(\log \eta^{-1})$ *as* $\eta \to 0+$. *Suppose, finally, that for a fixed point* a_0 *the family of functions* $\rho(\xi, \eta) = (1/\pi)\int_{a_0}^{\xi} \operatorname{Im} \Phi(\sigma + i\eta)d\sigma$ *is of uniformly bounded variation in* ξ *for* ξ *in* $[a, b]$ *and for* $0 < \eta \leqq \eta_0$, *and that* $\rho(\xi) = \lim_{\eta\to 0+} \rho(\xi, \eta)$ *exists for each* ξ *in* $[a, b]$. *Then,*

$$\lim_{\eta\to 0+} (2\pi i)^{-1} \int_a^b [\Phi(\xi + i\eta)\Psi(\xi + i\eta) - [\Phi(\xi + i\eta)]^{-}\Psi(\xi - i\eta)]\, d\xi$$

$$= \int_a^b \Psi(\xi)d\rho(\xi).$$

REMARK 1. Suppose $\{\eta_n\}$ is a decreasing sequence tending to zero. The lemma is still true if in the last supposition and the conclusion we replace η by η_n and $\lim_{\eta\to 0}$ by $\lim_{n\to\infty}$.

REMARK 2. The lemma is still true if we replace the last supposition by the supposition that $\operatorname{Im}\Phi(\lambda) \geqq 0$ in the upper half-plane and $\lim_{\eta\to 0+} \rho(\xi, \eta)$ exists for each ξ in $[a, b]$. (In this case it follows that $\rho(\xi, \eta)$ is of uniformly bounded variation in ξ for ξ in $[a, b]$ and for $0 < \eta \leqq \eta_0$.)

REMARK 3. The lemma is true if we replace all the assumptions about $\Phi(\lambda)$ by the assumption that $\Phi(\lambda)$ is analytic in the upper half-plane with nonnegative imaginary part. (For the original assumptions then follow. See Štraus [7, Lemmas 3 and 4].)

REMARK 4. The lemma is true if we assume that $\Phi(\lambda)$ is analytic in the upper half-plane with nonnegative imaginary part and $\Psi(\lambda)$ is analytic in a neighborhood of $[a, b]$.

PROOF OF LEMMA 2. We can write

$$\int_a^b [\Phi(\xi + i\eta)\Psi(\xi + i\eta) - [\Phi(\xi + i\eta)]^- \Psi(\xi - i\eta)]\, d\xi$$

$$= \int_a^b \Psi(\xi) d_\xi \rho(\xi, \eta) + \int_a^b [\Psi(\xi + i\eta) - \Psi(\xi)]\Phi(\xi + i\eta) d\xi$$

$$- \int_a^b [\Psi(\xi - i\eta) - \Psi(\xi)]\, [\Phi(\xi + i\eta)]^- d\xi\, .$$

Now,

$$\left| \int_a^b [\Psi(\xi + i\eta) - \Psi(\xi)]\Phi(\xi + i\eta) d\xi \right|$$

$$\leqq K\eta(\log \eta^{-1}) \int_a^b |\Phi(\xi + i\eta)| d\xi$$

$$\leqq K\eta(\log \eta^{-1})^2 \quad \text{as}\ \eta \to 0+.$$

Hence,

$$\lim_{\eta \to 0+} \int_a^b [\Psi(\xi + i\eta) - \Psi(\xi)]\, \Phi(\xi + i\eta) d\xi = 0.$$

Similarly,

$$\lim_{\eta \to 0+} \int_a^b [\Psi(\xi - i\eta) - \Psi(\xi)]\, [\Phi(\xi + i\eta)]^- d\xi = 0\, .$$

On the other hand,

$$\lim_{\eta \to 0+} \int_a^b \Psi(\xi) d_\xi \rho(\xi, \eta) = \int_a^b \Psi(\xi) d\rho(\xi)\ ,$$

by the Helly-Bray theorem. (See Widder [8, Chapter I, Theorem 16.4].) This completes the proof of Lemma 2.

In the following lemmas we shall assume that A is a simple closed symmetric operator with deficiency index (1, 1) in the Hilbert space $\mathfrak{H}$. Let $\lambda_0 = \xi_0 + i\eta_0$ be a complex number with positive imaginary part, and let g_0 be an element of norm 1 in the deficiency subspace of A corresponding to $\bar{\lambda}_0$. Suppose that A_0 is a selfadjoint extension of A in $\mathfrak{H}$. Since A is simple, g_0 is a generating element for A. Hence, A_0 is unitarily equivalent to the multiplication operator in $L^2_{\rho_0}(-\infty, \infty)$, where $\rho_0(t) = (E_0(t)g_0, g_0)$ and $E_0(t)$ is the spectral function of A_0. $E_0(t)$ is assumed to be continuous on the left.

Throughout this paper we shall assume that $\rho_0(t)$ is twice continuously differentiable everywhere except possibly at a countable set $\{t_k\}$ with no finite limit points. We also assume that at each t_k,

$\rho_0[t_k] \neq 0$, *where* $\rho_0[t_k]$ *denotes the jump in* ρ_0 *at* t_k. We shall assume that the $\{t_k\}$ are indexed in order of growth. If there is a first one, it is denoted by t_1, and in this case we take $t_0 = -\infty$. If there is a last t_k, say t_n, then we take $t_{n+1} = +\infty$. If $\rho_0 \in C^2$ everywhere, we take $t_0 = -\infty$, $t_1 = +\infty$. We shall denote the set of finite numbers t_k by E_1, and we shall denote the interval (t_k, t_{k+1}) by I_k.

Let E_2 be the set of points for which $\rho_0 \in C^2$ and $\rho_0{}' > 0$. Then, $E_2 = \bigcup_m J_m$, where $\{J_m\}$ is a collection of disjoint open intervals. Let E_3 be the set of zeros of $\rho_0{}'$. We note that $E_1 \cup E_2 \cup E_3 = (-\infty, \infty)$.

Let S be the set of elements $f \in \mathfrak{H}$ whose transforms $F(t)$ in $L^2_{\rho_0}(-\infty, \infty)$ are such that $F(t) = F_c(t) + F_d(t)$, where (i) $F_c(t)$ is a continuously differentiable function which vanishes outside a compact subset of E_2, and (ii) $F_d(t)$ is zero except possibly at a finite number of the t_k. We note that S is a linear manifold which is dense in $\mathfrak{H}$. We note also that $F_c(t)$ is zero outside a finite number of the J_m, say $J_{m_1}, \cdots, J_{m_n}$, and that these intervals contain closed bounded intervals $[a_{m_1}, b_{m_1}], \cdots, [a_{m_n}, b_{m_n}]$ such that $F_c(t)$ is zero outside these intervals.

Suppose that $R_0(\lambda)$ is the resolvent of A_0. Let $g(\lambda) = g_0 + (\lambda - \lambda_0)R_0(\lambda)g_0$, and let $Q(\lambda) = i \operatorname{Im} \lambda_0 + (\lambda - \lambda_0)(g_0, g(\bar{\lambda}))$. As is indicated in [2], for $\operatorname{Im} \lambda \neq 0$ the resolvent or generalized resolvent $R(\lambda)$ of A corresponding to a selfadjoint extension or dilation A^+ of A has the form

$$R(\lambda) = R_0(\lambda) - [\theta(\lambda) + Q(\lambda)]^{-1}(\cdot, g(\bar{\lambda}))g(\lambda), \tag{9}$$

where $\theta(\lambda)$ is an analytic function for $\operatorname{Im}\lambda \neq 0$ which has non-negative imaginary part in the upper half-plane, and $\theta(\bar{\lambda}) = [\theta(\lambda)]^-$. $Q(\lambda)$ is analytic for $\operatorname{Im}\lambda \neq 0$, has positive imaginary part in the upper half-plane, and $Q(\bar{\lambda}) = [Q(\lambda)]^-$. We note that $\operatorname{Im} Q(\bar{\lambda}) = -\operatorname{Im} Q(\lambda)$.

For $f \in S$ and $\operatorname{Im} \lambda \neq 0$ we define $C_1(f; \lambda)$ and $C_2(f; \lambda)$ by means of the following equations:

$$C_1(f; \lambda) = (f, g_0) + (\lambda - \bar{\lambda}_0) \int_{-\infty}^{\infty} \operatorname{Re}[(t - \lambda)^{-1}] F(t) d\rho_0(t), \tag{10}$$

$$C_2(f; \lambda) = (\lambda - \bar{\lambda}_0) \int_{-\infty}^{\infty} \operatorname{Im}[(t - \lambda)^{-1}] F(t) d\rho_0(t). \tag{11}$$

We note that under the above definitions we have that

$$(f, g(\bar{\lambda})) = C_1(f; \lambda) + iC_2(f; \lambda), \tag{12}$$

$$(g(\lambda), h) = [C_1(h; \bar{\lambda})]^- - i[C_2(h; \bar{\lambda})]^- \quad \text{for } f, h \in S. \tag{13}$$

The following lemmas are immediate consequences of Lemma 1.

LEMMA 3. *If $f, h \in S$,*

$$(R_0(\lambda)f,\ h) = \int_{-\infty}^{\infty} (t-\lambda)^{-1} F_c(t)[H_c(t)]^{-} d\rho_0(t) + \sum_k (t_k - \lambda)^{-1} F_d(t_k)[H_d(t_k)]^{-} \rho_0[t_k]\ . \tag{14}$$

The first term on the right is analytic across any open interval on which $F_c(t)[H_c(t)]^{-} \equiv 0$ (in particular, in a neighborhood of each t_k), it can be extended continuously down (up) to the real axis everywhere, and

$$\lim_{\eta\to 0\pm} \int_{-\infty}^{\infty} (t-\lambda)^{-1} F_c(t)[H_c(t)]^{-} d\rho_0(t) = \mathrm{P} \int_{-\infty}^{\infty} (t-\xi)^{-1} F_c(t)[H_c(t)]^{-} d\rho_0(t) \pm i\pi F_c(\xi)[H_c(\xi)]^{-} \rho_0{}'(\xi)\,, \tag{15}$$

where we interpret $F_c(t_k)[H_c(t_k)]^{-}\rho_0{}'(t_k)$ to be zero. The order of approach on any bounded interval is $O(|\eta| \log |\eta|^{-1})$ uniformly in ξ. $(R_0(\lambda)f,\ h)$ can be extended continuously down (up) to the real axis everywhere except at the t_k, and

$$\lim_{\eta\to 0\pm} (R_0(\lambda)f, h) = \mathrm{P} \int_{-\infty}^{\infty} (t-\xi)^{-1} F_c(t)[H_c(t)]^{-} d\rho_0(t) \pm i\pi F_c(\xi)[H_c(\xi)]^{-} \rho_0{}'(\xi) + \sum_k (t_k - \xi)^{-1} F_d(t_k)[H_d(t_k)]^{-} \rho_0[t_k]\ . \tag{16}$$

The order of approach on any bounded closed interval not containing a t_k is $O(|\eta| \log |\eta|^{-1})$ uniformly in ξ.

LEMMA 4. *If $f \in S$,*

$$C_1(f; \lambda) = \left[(f, g_0) + (\lambda - \bar{\lambda}_0) \int_{-\infty}^{\infty} \mathrm{Re}[(t-\lambda)^{-1}]\ F_c(t) d\rho_0(t)\right] + (\lambda - \bar{\lambda}_0) \sum_k \mathrm{Re}[(t_k - \lambda)^{-1}]\, F_d(t_k)\rho_0[t_k]\ . \tag{17}$$

The first term on the right can be extended continuously across the real axis everywhere, and

$$\lim_{\eta\to 0\pm}\left[(f, g_0) + (\lambda - \bar{\lambda}_0)\int_{-\infty}^{\infty} \mathrm{Re}[(t-\lambda)^{-1}]\ F_c(t)d\rho_0(t)\right] = (f, g_0) + (\xi - \bar{\lambda}_0)\mathrm{P}\int_{-\infty}^{\infty}(t-\xi)^{-1}\ F_c(t)d\rho_0(t)\,. \tag{18}$$

The order of approach on any bounded interval is $O(|\eta|)$ uniformly in ξ. $C_1(f;\lambda)$ can be extended continuously across the real axis everywhere except at the t_k, and

$$\lim_{\eta\to 0\pm} C_1(f;\lambda) = (f, g_0) + (\xi - \bar{\lambda}_0)\mathrm{P}\int_{-\infty}^{\infty}(t-\xi)^{-1}F_c(t)d\rho_0(t) + (\xi - \bar{\lambda}_0)\sum_k (t_k - \xi)^{-1}F_d(t_k)\rho_0[t_k]\,. \tag{19}$$

The order of approach on any bounded closed interval not containing a t_k is $O(|\eta|)$ uniformly in ξ.

In the remainder of the paper we let

$$C_1(f;\xi) = (f, g_0) + (\xi - \bar{\lambda}_0)\mathrm{P}\int_{-\infty}^{\infty}(t-\xi)^{-1}F_c(t)d\rho_0(t) + (\xi - \bar{\lambda}_0)\sum_k (t_k - \xi)^{-1}F_d(t_k)\rho_0[t_k]\,.$$

Then, equation (19) becomes

$$\lim_{\eta\to 0\pm} C_1(f;\lambda) = C_1(f;\xi)\,. \tag{20}$$

LEMMA 5. *If $f \in S$,*

$$C_2(f;\lambda) = (\lambda - \bar{\lambda}_0)\int_{-\infty}^{\infty}\mathrm{Im}\,[(t-\lambda)^{-1}]\,F_c(t)d\rho_0(t) + (\lambda - \bar{\lambda}_0)\sum_k \mathrm{Im}\,[(t_k-\lambda)^{-1}]\,F_d(t_k)\rho_0[t_k]\,. \tag{21}$$

The first term on the right can be extended continuously down (up) to the real axis everywhere, and

$$\lim_{\eta\to 0\pm}(\lambda - \bar{\lambda}_0)\int_{-\infty}^{\infty}\mathrm{Im}[(t-\lambda)^{-1}]\,F_c(t)d\rho_0(t) = \pm(\xi - \bar{\lambda}_0)\pi F_c(\xi)\rho_0'(\xi)\,, \tag{22}$$

where $F_c(t_k)\rho_0'(t_k)$ is interpreted to be zero. The order of approach on any bounded interval is $O(|\eta|\log|\eta|^{-1})$ uniformly in ξ. $C_2(f;\lambda)$ can be extended continuously down (up) to the real axis everywhere except at the t_k, and

$$\lim_{\eta \to 0\pm} C_2(f; \lambda) = \pm(\xi - \bar{\lambda}_0)\pi F_c(\xi)\rho_0{}'(\xi)\,. \tag{23}$$

The order of approach on any bounded closed interval not containing a t_k *is* $O(|\eta| \log |\eta|^{-1})$ *uniformly in* ξ.

LEMMA 6. *If* $f \in S$,

$$(f, g(\bar{\lambda})) = C_1(f; \lambda) + iC_2(f; \lambda)$$

$$= \left[(f, g_0) + (\lambda - \bar{\lambda}_0)\int_{-\infty}^{\infty}(t - \lambda)^{-1}F_c(t)d\rho_0(t)\right] + (\lambda - \bar{\lambda}_0)\sum_k (t_k - \lambda)^{-1}F_d(t_k)\rho_0[t_k]\,. \tag{24}$$

The first term on the right is analytic across any open interval on which $F_c(t) \equiv 0$ *(in particular, in a neighborhood of each* t_k*), it can be extended continuously down (up) to the real axis everywhere, and*

$$\lim_{\eta \to 0\pm}\left[(f, g_0) + (\lambda - \bar{\lambda}_0)\int_{-\infty}^{\infty}(t - \lambda)^{-1}F_c(t)d\rho_0(t)\right]$$

$$= (f, g_0) + (\xi - \bar{\lambda}_0)\mathrm{P}\int_{-\infty}^{\infty}(t - \xi)^{-1}F_c(t)d\rho_0(t) \pm i(\xi - \bar{\lambda}_0)\pi F_c(\xi)\rho_0{}'(\xi)\,, \tag{25}$$

where $F_c(t_k)\rho'(t_k)$ *is interpreted to be zero. The order of approach on any bounded interval is* $O(|\eta| \log |\eta|^{-1})$ *uniformly in* ξ. $(f, g(\bar{\lambda}))$ *is continuous across any interval not containing a* t_k *and in which* $F_c(t) \equiv 0$*, and*

$$\lim_{\eta \to 0\pm}(f, g(\bar{\lambda})) = C_1(f; \xi)\,. \tag{26}$$

The order of approach on any bounded closed interval not containing a t_k *and in which* $F_c(t) \equiv 0$ *is* $O(|\eta| \log |\eta|^{-1})$ *uniformly in* ξ.

LEMMA 7. *If* $h \in S$,

$$(g(\lambda), h) = [C_1(h; \bar{\lambda})]^- - i[C_2(h; \bar{\lambda})]^-$$

$$= \left[(g_0, h) + (\lambda - \lambda_0)\int_{-\infty}^{\infty}(t - \lambda)^{-1}[H_c(t)]^-d\rho_0(t)\right] + (\lambda - \lambda_0)\sum_k (t_k - \lambda)^{-1}[H_d(t_k)]^-\rho_0[t_k]\,. \tag{27}$$

The first term on the right is analytic across any open interval on which $H_c(t) \equiv 0$ *(in particular, in a neighborhood of each* t_k*), it can be extended continuously down (up) to the real axis everywhere, and*

$$\lim_{\eta\to 0\pm}\left[\ (g_0, h) + (\lambda - \lambda_0)\int_{-\infty}^{\infty}(t-\lambda)^{-1}[H_c(t)]^{-}d\rho_0(t)\ \right]$$

$$\begin{aligned}(28)\qquad &= (g_0, h) + (\xi - \lambda_0)\mathrm{P}\int_{-\infty}^{\infty}(t-\xi)^{-1}[H_c(t)]^{-}d\rho_0(t)\\ &\pm i(\xi-\lambda_0)\pi[H_c(\xi)]^{-}\rho_0{}'(\xi)\,,\end{aligned}$$

where $[H_c(t_k)]^{-}\rho_0{}'(t_k)$ *is interpreted to be zero. The order of approach on any bounded interval is* $O(|\eta|\log|\eta|^{-1})$ *uniformly in* ξ. $(g(\lambda), h)$ *is continuous across any interval not containing a* t_k *and in which* $H_c(t)\equiv 0$, *and*

$$(29)\qquad \lim_{\eta\to 0\pm}(g(\lambda), h) = [C_1(h;\xi)]^{-}\,.$$

The order of approach on any bounded closed interval not containing a t_k *and in which* $H_c(t)\equiv 0$ *is* $O(|\eta|\log|\eta|^{-1})$ *uniformly in* ξ.

LEMMA 8. *Let* $\rho_0(t) = \rho_{0c}(t) + \rho_{0d}(t)$ *be the standard decomposition of* $\rho_0(t)$ *into continuous and discrete parts. Then,* $Q(\lambda)$ *can be written in the form*

$$(30)\qquad Q(\lambda) = Q_c(\lambda) + Q_d(\lambda),$$

where

$$(31)\qquad \begin{aligned}Q_c(\lambda) &= [i\eta_0 + (\lambda-\lambda_0)]\int_{-\infty}^{\infty}1\,d\rho_{0c}(t)\\ &+ (\lambda-\lambda_0)(\lambda-\bar\lambda_0)\int_{-\infty}^{\infty}(t-\lambda)^{-1}d\rho_{0c}(t)\,,\end{aligned}$$

$$(32)\qquad \begin{aligned}Q_d(\lambda) &= [i\eta_0 + (\lambda-\lambda_0)]\sum_k\rho_0[t_k]\\ &+ (\lambda-\lambda_0)(\lambda-\bar\lambda_0)\sum_k(t_k-\lambda)^{-1}\rho_0[t_k]\,.\end{aligned}$$

$Q_c(\lambda)$ *and* $Q_d(\lambda)$ *are analytic in the upper and lower half-planes, both have positive imaginary part in the upper half-plane (unless one is identically zero), and satisfy the equations* $Q_c(\bar\lambda) = [Q_c(\lambda)]^{-}$, $Q_d(\bar\lambda) = [Q_d(\lambda)]^{-}$. Im $Q_c(\lambda)$ *and* Im $Q(\lambda)$ *can be extended continuously down (up) to the real axis except at the* t_k, *and*

$$(33)\qquad \lim_{\eta\to 0\pm}\mathrm{Im}\,Q(\lambda) = \lim_{\eta\to 0\pm}Q_c(\lambda) = \pm|\xi-\lambda_0|^2\pi\rho_0{}'(\xi)\,.$$

The order of approach on any closed bounded interval not containing a t_k *is* $O(|\eta|\log|\eta|^{-1})$ *uniformly in* ξ.

LEMMA 9. *If* $f \in \mathrm{S}$, $C_2(f; \lambda)/\mathrm{Im}\, Q(\lambda)$ *can be extended continuously across the real axis on* E_2, *and*

$$\lim_{\eta \to 0\pm} C_2(f; \lambda)/\mathrm{Im}\, Q(\lambda) = F_c(\xi)(\xi - \lambda_0)^{-1}. \tag{34}$$

On any closed bounded subinterval of E_2, *the order of approach is* $O(|\eta| \log |\eta|^{-1})$ *uniformly in* ξ.

PROOF. Use Lemmas 5 and 8.

In the remainder of the paper we shall use the notation $D_1(f; \lambda) = C_1(f; \lambda)$, $D_1(f; \xi) = C_1(f; \xi)$, $D_2(f; \lambda) = C_2(f; \lambda)/\mathrm{Im}\, Q(\lambda)$, $D_2(f; \xi) = F_c(\xi)(\xi - \lambda_0)^{-1}$, where $f \in \mathrm{S}$. Then equations (19) and (34) become

$$\lim_{\eta \to 0\pm} D_1(f; \lambda) = D_1(f; \xi), \tag{35}$$

except at the t_k,

$$\lim_{\eta \to 0\pm} D_2(f; \lambda) = D_2(f; \xi) \quad \text{on } E_2. \tag{36}$$

We note that $D_1(f; \xi)$ and $D_2(f; \xi)$ are linear functionals over S.

LEMMA 10. *Let* $\theta(\lambda)$ *be a function which is analytic in the upper half-plane with nonnegative imaginary part. Suppose that the matrix* $\Phi(\lambda) = (\Phi_{rs}(\lambda))$ *and the function* $\Psi_{11}(\lambda)$ *are defined in the upper half-plane as follows*:

$$\Phi_{11}(\lambda) = -[\theta(\lambda) + Q(\lambda)]^{-1}; \tag{37}$$

$$\Phi_{12}(\lambda) = \Phi_{21}(\lambda) = i\Phi_{11}(\lambda)\, \mathrm{Im}\, Q(\lambda); \tag{38}$$

$$\Phi_{22}(\lambda) = -\Phi_{11}(\lambda)[\mathrm{Im}\, Q(\lambda)]^2 + i\, \mathrm{Im}\, Q(\lambda); \tag{39}$$

$$\Psi_{11}(\lambda) = -\{Q_d(\lambda)[\theta(\lambda) + Q_c(\lambda)] - 1\}\Phi_{11}(\lambda). \tag{40}$$

Then the following statements (in which we take $\lambda = \sigma + i\eta$*) are true*:

(I) $\Phi_{11}(\lambda)$ *is analytic and has positive imaginary part in the upper half-plane.* $\rho_{11}(\xi) = \lim_{\eta \to 0+}(1/\pi) \int_0^\xi \mathrm{Im}\, \Phi_{11}(\lambda) d\sigma$ *exists for all* ξ. *For any interval* $[a, b]$, $\int_a^b |\Phi_{11}(\lambda)| d\sigma = O(\log n^{-1})$ *as* $\eta \to 0+$.

(II) $\Psi_{11}(\lambda)$ *is analytic and has positive imaginary part in the upper half-plane.* $\tau_{11}(\xi) = \lim_{\eta \to 0+} (1/\pi) \int_0^\xi \mathrm{Im}\, \Psi_{11}(\lambda) d\sigma$ *exists for all* ξ. *For any interval* $[a, b]$, $\int_a^b |\Psi_{11}(\lambda)| d\sigma = O(\log \eta^{-1})$ *as* $\eta \to 0+$.

(III) $\Phi_{22}(\lambda)$ *is continuous and* $\mathrm{Im}\, \Phi_{22}(\lambda) \geqq 0$ *in the upper half-plane. If* $[a, b] \subset I_k$ *for some* k, $\int_a^b |\Phi_{22}(\lambda)| d\sigma = O(\log \eta^{-1})$ *as* $\eta \to 0+$. *If we put* $\rho_{22}^k(\xi, \eta) = (1/\pi) \int_{a_k}^\xi \mathrm{Im}\, \Phi_{22}(\sigma + i\eta) d\sigma$, *where* a_k *is an arbitrary point in* I_k, *then* $\rho_{22}^k(\xi) = \lim_{\eta \to 0+} \rho_{22}^k(\xi, \eta)$ *exists for all* ξ *in* I_k.

(IV) *$\Phi_{12}(\lambda)$ is continuous and* $\operatorname{Im}\Phi(\lambda) \geqq 0$ *in the upper half-plane. If* $[a, b] \subset I_k$ *for some* k, $\int_a^b |\Phi_{12}(\lambda)| d\sigma = O(\log \eta^{-1})$ *as* $\eta \to 0+$. *If we put* $\rho_{12}^k(\xi, \eta) = (1/\pi) \int_{a_k}^{\xi} \operatorname{Im}\Phi_{12}(\sigma + i\eta) d\sigma$, *where* a_k *is an arbitrary point in* I_k *and if* $[a, b] \subset I_k$, *then there exists an* η_0 *such that* $\rho_{12}^k(\xi, \eta)$ *is of uniformly bounded variation in* ξ *for* ξ *in* $[a, b]$ *and for* $0 < \eta \leqq \eta_0$. *There exists a decreasing sequence* $\{\eta_n\}$ *approaching zero such that* $\rho_{12}^k(\xi) = \rho_{21}^k(\xi) = \lim_{n \to \infty} \rho_{12}^k(\xi, \eta_n)$ *exists for each* ξ *in* I_k.

(V) *The matrix* $\rho^k(\xi)$ *with elements* $\rho_{11}^k(\xi) = \rho_{11}(\xi)$, $\rho_{21}^k(\xi) = \rho_{12}^k(\xi)$, *and* $\rho_{22}^k(\xi)$ *is a nondecreasing function of* ξ *for* ξ *in* I_k. *Its elements are of bounded variation in each closed bounded subinterval of* I_k.

Proof. The analyticity of Φ_{11} and Ψ_{11} in the upper half-plane as stated in (I) and (II) follows from the fact that $\operatorname{Im} Q > 0$ in the upper half-plane. The positivity of $\operatorname{Im}\Phi_{11}$ and $\operatorname{Im}\Psi_{11}$ follows from the equations

$$\operatorname{Im}\Phi_{11} = [\operatorname{Im}\theta + \operatorname{Im} Q]\, |\theta + Q|^{-2},$$

$$\operatorname{Im}\Psi_{11} = [(|\theta + Q_c|^2 + 1)\operatorname{Im} Q_d + (|Q_d|^2 + 1)\operatorname{Im}(\theta + Q_c)]\, |\theta + Q|^{-2}.$$

The remaining statements of (I) and (II) then follow from Štraus [7, Lemmas 3 and 4].

Let us now prove (III). The continuity of Φ_{22} in the upper half-plane is obvious. If $[a, b] \subset I_k$ for some k, $\operatorname{Im} Q(\lambda)$ is continuous down to the real axis on $[a, b]$, by Lemma 8, and therefore

$$\begin{aligned}\int_a^b |\Phi_{22}(\lambda)|\, d\sigma &\leqq K_1 \int_a^b |\Phi_{11}(\lambda)| d\sigma + K_2 \\ &= O(\log \eta^{-1}) \quad \text{as } \eta \to 0+, \quad \text{by (I)}.\end{aligned}$$

To prove that $\rho_{22}^k(\xi) = \lim_{\eta \to 0+} \rho_{22}^k(\xi, \eta)$ exists for all ξ in I_k, we note that

$$\begin{aligned}\rho_{22}^k(\xi, \eta) &= (1/\pi) \int_{a_k}^{\xi} \{-\operatorname{Im}\Phi_{11}(\lambda)[\operatorname{Im} Q(\lambda)]^2 + \operatorname{Im} Q(\lambda)\} d\sigma \\ &= (-1/2\pi i) \int_{a_k}^{\xi} \{\Phi_{11}(\lambda)[\operatorname{Im} Q(\lambda)]^2 - [\Phi_{11}(\lambda)]^- [\operatorname{Im} Q(\bar{\lambda})]^2\} d\sigma \\ &\quad + (1/\pi) \int_{a_k}^{\xi} \operatorname{Im} Q(\lambda) d\sigma.\end{aligned}$$

From (I) and Lemma 8 it follows that we can use Lemma 2 to show that the limit of the first integral on the right in the above equation exists as $\eta \to 0+$. The limit of the second integral exists by the continuity of $\operatorname{Im} Q(\lambda)$ down to the real axis. Thus, we obtain that $\rho_{22}^k(\xi) = \lim_{\eta\to 0+} \rho_{22}^k(\xi, \eta)$ exists for all ξ in I_k, and, in fact, $\rho_{22}^k(\xi) = \int_{a_k}^{\xi} [\operatorname{Im} Q(\sigma)]^2 d\rho_{11}(\sigma) + (1/\pi) \int_{a_k}^{\xi} \operatorname{Im} Q(\sigma) d\sigma$.

Let us now prove (IV). The continuity of $\Phi_{12}(\lambda)$ in the upper half-plane is clear. To prove that $\operatorname{Im}\Phi \geqq 0$, we denote the components of $\operatorname{Im}\Phi$ by b_{uv}, and observe that $b_{11} = \operatorname{Im}\Phi_{11}$, $b_{12} = b_{21} = (\operatorname{Im} Q)\cdot(\operatorname{Re}\Phi_{11})$, $b_{22} = -(\operatorname{Im}\Phi_{11})(\operatorname{Im} Q)^2 + \operatorname{Im} Q$. Then by completing the square, we obtain that

$$\sum_{u,v=1}^{2} b_{uv}x_u\bar{x}_v = (\operatorname{Im}\Phi_{11})|x_1 + (\operatorname{Im} Q)(\operatorname{Re}\Phi_{11})[\operatorname{Im}\Phi_{11}]^{-1}x_2|^2$$

$$+ (\operatorname{Im} Q)(\operatorname{Im}\theta)[\operatorname{Im}\Phi_{11}]^{-1}|\theta + Q|^{-2}|x_2|^2 \geqq 0$$

for all complex numbers x_1, x_2. Hence, $\operatorname{Im}\Phi \geqq 0$.

If $[a, b] \subset I_k$ for some k, we have by (I) and Lemma 8 that

$$\int_a^b |\Phi_{12}(\lambda)|d\sigma \leqq K \int_a^b |\Phi_{11}(\lambda)|d\sigma = O(\log\eta^{-1}) \quad \text{as } \eta \to 0+.$$

To prove the remainder of (IV), we consider the matrix $\rho^k(\xi, \eta) = (1/\pi) \int_{a_k}^{\xi} \operatorname{Im}\Phi(\lambda)d\sigma$ with elements denoted by $\rho_{uv}^k(\xi, \eta)$. Since $\operatorname{Im}\Phi(\lambda) \geqq 0$, the symmetric matrix $\rho^k(\xi, \eta)$ is a nondecreasing function of ξ for fixed η. Hence, for $\xi_1 < \xi_2$,

$$|\rho_{12}^k(\xi_2, \eta) - \rho_{12}^k(\xi_1, \eta)|$$

$$\leqq (1/2)\{[\rho_{11}^k(\xi_2, \eta) - \rho_{11}^k(\xi_1, \eta)] + [\rho_{22}^k(\xi_2, \eta) - \rho_{22}^k(\xi_1, \eta)]\}.$$

Since we already know by (I) and (III) that the limit as $\eta \to 0+$ of the right side of this inequality exists, it follows that for an arbitrary closed bounded subinterval $[a, b]$ of I_k and for some η_0, $\rho_{12}^k(\xi, \eta)$ is of uniformly bounded variation in ξ for ξ in $[a, b]$ and for $0 < \eta \leqq \eta_0$. Further

$$|\rho_{12}^k(a, \eta)| \leqq (1/2)\{|\rho_{11}^k(a, \eta) - \rho_{11}^k(a_k, \eta)| + |\rho_{22}^k(a, \eta) - \rho_{12}^k(a_k, \eta)|\},$$

so that η_0 can be chosen such that $|\rho_{12}^k(a, \eta)|$ is bounded by a constant for $0 < \eta \leqq \eta_0$. By Helly's selection theorem, then, there exists a nondecreasing sequence $\{\eta_n\}$ approaching zero such that $\rho_{12}^k(\xi) = \lim_{n\to\infty} \rho^k(\xi, \eta_n)$ exists for ξ in $[a, b]$. (See, for example, Widder [3, Chapter I, Theorem 16.3].) By means of a diagonal process, we can now show that there exists a decreasing sequence $\{\eta_n\}$ approaching zero such that $\rho_{12}^k(\xi) = \lim_{n\to\infty}\rho^k(\xi, \eta_n)$ exists for all ξ in I_k. This proves (IV).

Statement (V) follows from the fact that

$$\rho^k(\xi_2) - \rho^k(\xi_1) = \lim_{n\to\infty} [\rho^k(\xi_2, \eta_n) - \rho^k(\xi_1, \eta_n)]$$

and the fact that $\rho^k(\xi, \eta_n)$ is a nondecreasing function of ξ for each fixed η_n. The elements of $\rho^k(\xi)$ are of bounded variation in each closed bounded subinterval of I_k because $\rho^k(\xi)$ is nondecreasing.

This completes the proof of Lemma 10.

3. Spectral representation.

Theorem 1 (Expansion Theorem). *Let A be a closed symmetric operator with deficiency index $(1, 1)$ in the Hilbert space $\mathfrak{H}$. Suppose that $\rho_0(t) = (E_0(t)g_0, g_0)$ is twice continuously differentiable everywhere except possibly at a countable set $\{t_k\}$ with no finite limit points and with $\rho_0[t_k] \neq 0$ for each k, where $E_0(t)$ is the spectral function of a selfadjoint extension of A in $\mathfrak{H}$, and g_0 is an element of norm 1 in the deficiency subspace of A corresponding to the complex number $\bar{\lambda}_0$ with $\operatorname{Im}\lambda_0 > 0$. Let $E^+(t)$ be the spectral function of a selfadjoint extension or dilation A^+ of A. Then, for an arbitrary interval $[\alpha, \beta)$ and for arbitrary $f, h \in S$,*

$$\begin{aligned}([E^+(\beta) - E^+(\alpha)]f, h) &= \sum_{t_k\in[\alpha,\beta)} F_d(t_k)(t_k-\lambda_0)^{-1}[H_d(t_k)(t_k-\lambda_0)^{-1}]^-\tau_{11}[t_k] \\ &\quad + \sum_k \int_{[\alpha,\beta)\cap I_k} \sum_{u,v=1}^{2} D_u(f;\xi)[D_v(h;\xi)]^- d\rho^k_{uv}(\xi),\end{aligned} \tag{41}$$

where $\rho^k_{11}(\xi) = \rho_{11}(\xi)$ for each k, and the remaining $\rho^k_{uv}(\xi)$ are defined as in Lemma 10. The integral

$$\int_{[\alpha,\beta)\cap I_k} \sum_{u,v=1}^{2} D_u(f;\xi)[D_v(h;\xi)]^- d\rho^k_{uv}(\xi)$$

is to be interpreted as the Lebesgue-Stieltjes integral

$$\int_{[\alpha,\beta)\cap I_k} \left\{ \sum_{u,v=1}^{2} D_u(f;\xi)[D_v(h;\xi)]^- \delta^k_{uv}(\xi) \right\} d\sigma^k(\xi),$$

where $\sigma^k(\xi) = \rho^k_{11}(\xi) + \rho^k_{22}(\xi)$, and $\delta^k_{uv}(\xi) = d\rho^k_{uv}(\xi)/d\sigma^k(\xi)$.

(Remark. Lebesgue-Stieltjes integrals of the above type are discussed in Dunford and Schwartz [1, XIII.5.9] and in Kac [5].)

Proof. If α, β are two arbitrary real numbers, $\alpha < \beta$, and if f, h

are two arbitrary elements in $\mathfrak{H}$, the Stieltjes inversion formula states that

$$
\begin{aligned}
&([(1/2)\{E^+(\beta) + E^+(\beta+0)\} - (1/2)\{E^+(\alpha) + E^+(\alpha+0)\}]f, h) \\
(42)\qquad & \qquad = (2\pi i)^{-1} \lim_{\eta\to 0+} \int_\alpha^\beta [(R(\lambda)f, h) - (R(\bar{\lambda})f, h)]\, d\xi,
\end{aligned}
$$

where $\lambda = \xi + i\eta$.

For fixed $f, h \in S$, we shall first of all evaluate the limit on the right of equation (42) for the following types of intervals $[\alpha, \beta]$:

Type 1. $\rho_0 \in C^2$ and $\rho_0{}' > 0$ in a neighborhood of $[\alpha, \beta]$, (i.e., $[\alpha, \beta] \subset E_2$).

Type 2. $\rho_0 \in C^2$ in a neighborhood of $[\alpha, \beta]$, and $F_c(\xi) = H_c(\xi) = 0$ for all ξ in $[\alpha, \beta]$.

Type 3. $[\alpha, \beta]$ contains one and only one t_k, which is in the interior of $[\alpha, \beta]$, and $F_c(\xi) = H_c(\xi) = 0$ in a neighborhood of $[\alpha, \beta]$.

The rationale behind the choice of these types is this: If $f \in S$, then, as has already been noted, $F_c(\xi)$ is zero outside a finite number of closed bounded intervals contained in E_2 (the set on which $\rho_0 \in C^2$ and $\rho_0{}' > 0$). Hence, if $[\alpha, \beta]$ is any interval not having a t_k for an endpoint, it can be partitioned into a finite number of intervals of types 1, 2, 3.

Suppose now that $[\alpha, \beta]$ is of type 1. Using equations (9), (12), (13), (37), (38), (39), we can write

$$
\begin{aligned}
(R(\lambda)f, h) = (R_0(\lambda)f, h) &- D_2(f; \lambda)[D_2(h; \bar{\lambda})]^- i \operatorname{Im} Q(\lambda) \\
&+ \sum_{u,v=1}^{2} D_u(f; \lambda)[D_v(h; \lambda)]^- \Phi_{uv}(\lambda),
\end{aligned}
$$

and similarly for $(R(\bar{\lambda})f, h)$. Hence,

$$
\begin{aligned}
&\int_\alpha^\beta [(R(\lambda)f, h) - (R(\bar{\lambda})f, h)]\, d\xi \\
&\quad = \int_\alpha^\beta \{(R_0(\lambda)f, h) - (R_0(\bar{\lambda})f, h) - D_2(f; \lambda)[D_2(h; \bar{\lambda})]^- i \operatorname{Im} Q(\lambda) \\
&\qquad\qquad\qquad - D_2(f; \bar{\lambda})[D_2(h; \lambda)]^- i \operatorname{Im} Q(\lambda)\}\, d\xi \\
&\qquad + \sum_{u,v=1}^{2} \int_\alpha^\beta \{D_u(f; \lambda)[D_v(h; \bar{\lambda})]^- \Phi_{uv}(\lambda) \\
&\qquad\qquad\qquad - D_u(f; \bar{\lambda})[D_v(h; \lambda)]^- [\Phi_{uv}(\lambda)]^-\}\, d\xi .
\end{aligned}
$$

By Lemmas 3 and 9, we see that the integrand in the first integral on the right in the above equation is continuous down to the real axis on E_2, and

$$\lim_{\eta\to 0+} \{(R_0(\lambda)f, h) - (R_0(\bar{\lambda})f, h) - D_2(f; \lambda)[D_2(h; \bar{\lambda})]^- i \operatorname{Im} Q(\lambda)$$
$$- D_2(f; \bar{\lambda})[D_2(h; \lambda)]^- i \operatorname{Im} Q(\lambda)\} = 0 .$$

Hence, the limit of this integral is zero.

From Lemmas 4, 9 and 10 it follows that we can use Lemma 2 (and the remarks following Lemma 2) in order to evaluate the limits of the remaining integrals. We obtain, then, from the Stieltjes inversion formula that if $[\alpha, \beta]$ is of type 1,

$$\begin{aligned}&([(1/2)\{E^+(\beta) + E^+(\beta + 0)\} - (1/2)\{E^+(\alpha) + E^+(\alpha + 0)\}]f, h)\\ &= \sum_{u,v=1}^{2} \int_\alpha^\beta D_u(f; \xi)[D_v(h; \xi)]^- d\rho_{uv}^k(\xi), \quad \text{where } [\alpha, \beta] \subset I_k .\end{aligned} \tag{43}$$

We note that in the above derivation the bilinear functional $D_2(f; \lambda)[D_2(h; \bar{\lambda})]^- i \operatorname{Im} Q(\lambda)$ takes the place of the fundamental solution in the derivation of an expansion theorem for a Sturm-Liouville operator by Štraus [7], and the linear functional $D_1(f; \lambda)$ and $D_2(f; \lambda)$ take the place of the basis for the solutions of the equation $Af = \lambda f$.

Suppose next that $[\alpha, \beta]$ is of type 2. Using equation (9), we write

$$\int_\alpha^\beta [(R(\lambda)f, h) - (R(\bar{\lambda})f, h)]\, d\xi$$
$$= \int_\alpha^\beta [(R_0(\lambda)f, h) - (R_0(\bar{\lambda})f, h)]\, d\xi$$
$$+ \int_\alpha^\beta \{(f, g(\bar{\lambda}))(g(\lambda), h)\Phi_{11}(\lambda) - (f, g(\lambda))(g(\bar{\lambda}), h)[\Phi_{11}(\lambda)]^-\} d\xi .$$

By Lemma 3, the limit as $\eta \to 0+$ of the first integral on the right in the above equation is zero. From Lemmas 6, 7, and 10 it follows that we can use Lemma 2 to evaluate the limit of the second integral. We obtain that

$$\lim_{\eta\to 0+} (2\pi i)^{-1} \int_\alpha^\beta \{(f, g(\bar{\lambda}))(g(\lambda), h)\Phi_{11}(\lambda) - (f, g(\lambda))(g(\bar{\lambda}), h)[\Phi_{11}(\lambda)]^-\} d\xi$$
$$= \int_\alpha^\beta D_1(f; \xi)[D_1(h; \xi)]^- d\rho_{11}(\xi) .$$

Since $F_c(\xi) = H_c(\xi) = 0$ and therefore $D_2(f;\ \xi) = D_2(h;\ \xi) = 0$ for ξ in $[\alpha, \beta]$, we can write the right side of this equation in the form of the right side of equation (43). Thus, we conclude that if $[\alpha, \beta]$ is of type 2, equation (43) is again true.

Suppose, finally, that $[\alpha, \beta]$ is of type 3. In this case we write $(R(\lambda)f, h) = A_1(\lambda) + A_2(\lambda)\Psi_{11}(\lambda)$, where

$$A_1(\lambda) = \{(R_0(\lambda)f, h)[Q_d^2(\lambda) + 1] - (f, g(\bar{\lambda}))(g(\lambda), h)Q_d(\lambda)\}[Q_d^2(\lambda) + 1]^{-1},$$

$$A_2(\lambda) = (f, g(\bar{\lambda}))(g(\lambda), h)[Q_d^2(\lambda) + 1]^{-1}.$$

Using Lemmas 3, 6, 7, and 8, it can be checked that $A_1(\lambda)$ and $A_2(\lambda)$ are both analytic in a neighborhood of $[\alpha, \beta]$. From Lemmas 2 and 10 and the Stieltjes inversion formula it therefore follows that if $[\alpha, \beta]$ is of type 3, then

$$([(1/2)\{E^+(\beta) + E^+(\beta + 0)\} - (1/2)\{E^+(\alpha) + E^+(\alpha + 0)\}]f, h)$$

$$= \int_\alpha^\beta A_2(\xi)d\tau_{11}(\xi) = \int_\alpha^{t_k-} A_2(\xi)d\tau(\xi) + \int_{t_k+}^\beta A_2(\xi)d\tau_{11}(\xi) + A_2(t_k)\tau_{11}[t_k].$$

Since

$$\rho_{11}(\xi) = \int_0^\xi [Q_d^2(\sigma) + 1]^{-1}d\tau_{11}(\sigma)$$

for all ξ, where $[Q_d^2(\sigma + 1)]^{-1}$ is defined by continuity at the points t_k, and since $A_2(\xi) = D_1(f;\xi)[D_1(h;\xi)]^-[Q_d^2(\xi) + 1]^{-1}$ for all ξ, where the right side is defined by continuity at the t_k,

$$\int_\alpha^{t_k-} A_2(\xi)d\tau_{11}(\xi) = \int_\alpha^{t_k-} D_1(f;\xi)[D_1(h;\xi)]^- d\rho_{11}(\xi).$$

This last integral is equal to

$$\sum_{u,v=1}^{2} \int_\alpha^{t_k-} D_u(f;\xi)[D_v(h;\xi)]^- d\rho_{uv}^{k-1}(\xi),$$

because $F_c(\xi) = H_c(\xi) = 0$ in a neighborhood of $[\alpha, \beta]$. Similar equations are valid for $\int_{t_k+}^\beta A_2(\xi)d\tau_{11}(\xi)$. It can be checked that $A_2(t_k) = F_d(t_k)(t_k - \lambda_0)^{-1}[H_d(t_k)(t_k - \lambda_0)^{-1}]^-$. Hence, if $[\alpha, \beta]$ is of type 3, then

$$([(1/2)\{E^+(\beta) + E^+(\beta+0)\} - (1/2)\{E^+(\alpha) + E^+(\alpha+0)\}]f, h)$$

$$\begin{aligned} &= \sum_{u,v=1}^{2} \int_{\alpha}^{t_k-} D_u(f;\xi)[D_v(h;\xi)]^- d\rho_{uv}^{k-1}(\xi) \\ (44) \qquad &+ \sum_{u,v=1}^{2} \int_{t_k+}^{\beta} D_u(f;\xi)[D_v(h;\xi)]^- d\rho_{uv}^{k}(\xi) \\ &+ F_d(t_k)(t_k-\lambda_0)^{-1}[H_d(t_k)(t_k-\lambda_0)^{-1}]^- \tau_{11}[t_k]\,. \end{aligned}$$

Now let $[\alpha, \beta]$ be an arbitrary interval not having a t_k for an end-point. By partitioning $[\alpha, \beta]$ into intervals of types 1, 2, and 3 and using equations (43) and (44), we see that

$$([(1/2)\{E^+(\beta) + E^+(\beta+0)\} - (1/2)\{E^+(\alpha) + E^+(\alpha+0)\}]f, h)$$

$$\begin{aligned} &= \sum_{t_k \in [\alpha,\beta]} F_d(t_k)(t_k-\lambda_0)^{-1}[H_d(t_k)(t_k-\lambda_0)^{-1}]^- \tau_{11}[t_k] \\ (45) \qquad &+ \sum_k \left\{ \sum_{u,v=1}^{2} \int_{[\alpha,\beta]\cap I_k} D_u(f;\xi)[D_v(h;\xi)]^- d\rho_{uv}^k(\xi) \right\}, \end{aligned}$$

where the integrals are to be interpreted as Stieltjes integrals. As is indicated by Kac [5], $\rho_{uv}^k(\xi)$ is absolutely continuous with respect to $\sigma^k(\xi) = \rho_{11}^k(\xi) + \rho_{22}^k(\xi)$. Let $\delta_{uv}^k(\xi) = d\rho_{uv}^k(\xi)/d\sigma^k(\xi)$. Then, if α, β are continuity points of $\sigma^k(\xi)$.

$$\sum_{u,v=1}^{2} \int_{[\alpha,\beta]\cap I_k} D_u(f;\xi)[D_v(h;\xi)]^- d\rho_{uv}^k(\xi)$$

$$= \int_{[\alpha,\beta]\cap I_k} \left\{ \sum_{u,v=1}^{2} D_u(f;\xi)[D_v(h;\xi)]^- \delta_{uv}^k(\xi) \right\} d\sigma^k(\xi)\,,$$

where this last integral is a Lebesgue-Stieltjes integral. It is also denoted by

$$\int_{[\alpha,\beta]\cap I_k} \sum_{u,v=1}^{2} D_u(f;\xi)[D_v(h;\xi)]^- d\rho_{uv}^k(\xi)\,.$$

Thus, if α, β are continuity points of $\sigma^k(\xi)$, (45) can be written in the form

$$([(1/2)\{E^+(\beta) + E^+(\beta+0)\} - (1/2)\{E^+(\alpha) + E^+(\alpha+0)\}]f, h)$$

$$\begin{aligned} &= \sum_{t_k \in [\alpha,\beta]} F_d(t_k)(t_k-\lambda_0)^{-1}[H_d(t_k)(t_k-\lambda_0)^{-1}]^- \tau_{11}[t_k] \\ (46) \qquad &+ \sum_k \int_{[\alpha,\beta]\cap I_k} \sum_{u,v=1}^{2} D_u(f;\xi)[D_v(h;\xi)]^- d\rho_{uv}^k(\xi). \end{aligned}$$

Now let α, β be finite real numbers with $\alpha < \beta$. We can choose increasing sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with the following properties: $\{\alpha_n\}$ and $\{\beta_n\}$ approach α and β respectively; there is no t_k in $[\alpha_1, \alpha)$ nor in $[\beta_1, \beta)$; if $\alpha \in I_k$, then α_n is a continuity point of σ^k for each n, and if $\beta \in I_k$, then β_n is a continuity point of σ^k for each n. Then by equation (46),

$$\begin{aligned}
&([E^+(\beta) - E^+(\alpha)]f, h)\\
&\quad = \lim_{n\to\infty}([(1/2)\{E^+(\beta_n) + E^+(\beta_n + 0)\}\\
&\qquad\qquad - (1/2)\{E^+(\alpha_n) + E^+(\alpha_n + 0)\}]f, h)\\
&\quad = \lim_{n\to\infty}\Big\{ \sum_{t_k \in [\alpha_n, \beta_n]} F_d(t_k)(t_k - \lambda_0)^{-1}[H_d(t_k)(t_k - \lambda_0)^{-1}]^- \tau_{11}[t_k]\\
&\qquad\qquad + \sum_k \int_{[\alpha_n, \beta_n] \cap I_k} \sum_{u,v=1}^{2} D_u(f; \xi)[D_v(h; \xi)]^- d\rho_{uv}^k(\xi) \Big\}\\
&\quad = \sum_{t_k \in [\alpha, \beta)} F_d(t_k)(t_k - \lambda_0)^{-1}[H_d(t_k)(t_k - \lambda_0)^{-1}]^- \tau_{11}[t_k]\\
&\qquad\qquad + \sum_k \int_{[\alpha, \beta) \cap I_k} \sum_{u,v=1}^{2} D_u(f; \xi)[D_v(h; \xi)]^- d\rho_{uv}^k(\xi),
\end{aligned}$$

where $\int_{[\alpha,\beta) \cap I_k} \sum_{u,v=1}^{2} D_u(f; \xi)[D_v(h; \xi)]^- d\rho_{uv}^k(\xi)$ is to be interpreted as the Lebesgue-Stieltjes integral

$$\int_{[\alpha,\beta) \cap I_k} \Big\{ \sum_{u,v=1}^{2} D_u(f; \xi)[D_v(h; \xi)]^- \delta_{uv}^k(\xi) \Big\} d\sigma^k(\xi).$$

This completes the proof of Theorem 1.

In what follows Δ will always denote a bounded interval of the form $[\alpha, \beta)$, and Δ^c will denote $[\alpha, \beta]$. $E(\Delta)$ will denote $E(\beta) - E(\alpha)$.

We note that in view of the fact that $D_2(f; \xi) = F_c(\xi)(\xi - \lambda_0)^{-1} = 0$ outside E_2, equation (41) can be written in the form

$$\begin{aligned}
(E^+(\Delta)f, h) = &\sum_{t_k \in \Delta} F_d(t_k)(t_k - \lambda_0)^{-1}[H_d(t_k)(t_k - \lambda_0)^{-1}]^- \tau_{11}[t_k]\\
&+ \sum_k \int_{\Delta \cap I_k \cap \bar{E}_2} \sum_{u,v=1}^{2} D_u(f; \xi)[D_v(h; \xi)]^- d\rho_{uv}^k(\xi)\\
&+ \int_{\Delta \cap E_3} D_1(f; \xi)[D_1(h; \xi)]^- d\rho_{11}(\xi).
\end{aligned} \tag{47}$$

Let $\mathfrak{H}_1 = \mathfrak{L}_{\tau_{11}}^2(E_1)$, i.e., let $\mathfrak{H}_1$ consist of all sequences $\{a(t_k)\}_k$ for which $\sum_k |a(t_k)|^2 \tau_{11}[t_k] < \infty$. Let $\mathfrak{H}_2 = \sum_k L^2_{\rho^k}(E_2 \cap I_k)$, where $L^2_{\rho^k}(E_2 \cap I_k)$ consists of all vector functions $[F_1(\xi), F_2(\xi)]$ whose components are measurable with respect to $\sigma^k(\xi)$ on $E_2 \cap I_k$ and such that

$$\int_{E_2 \cap I_k} \sum_{u,v=1}^{2} F_u(\xi)[F_v(\xi)]^- \delta^k_{uv}(\xi) d\sigma^k(\xi) < \infty .$$

(Integrals of this last type will also be written in the form

$$\int_{E_2 \cap I_k} \sum_{u,v=1}^{2} F_u(\xi)[F_v(\xi)]^- d\rho^k_{uv}(\xi) .$$

See Dunford and Schwartz [1] or Kac [5].) Let $\mathfrak{H}_3 = L^2_{\rho_{11}}(E_3)$. Then, $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$ are Hilbert spaces under the usual definitions of addition, scalar multiplication and inner product.

THEOREM 2 (SPECTRAL REPRESENTATION). *Under the hypotheses of Theorem 1, if A^+ is a selfadjoint extension of A or a minimal self-adjoint dilation of A, then A^+ is unitarily equivalent to the multiplication operator in $\mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \mathfrak{H}_3$.*

PROOF. If $f \in S$, let

$$\varphi_1(f; t_k) = F_d(t_k)(t_k - \lambda_0)^{-1} \quad \text{for } t_k \in E_1;$$

$$\varphi_2(f; \xi) = [D_1(f; \xi), D_2(f; \xi)] \quad \text{for } \xi \in E_2;$$

$$\varphi_3(f; \xi) = D_1(f; \xi) \quad \text{for } \xi \in E_3 .$$

By taking $f = h$ in equation (47) and letting $\beta \to +\infty$, $\alpha \to -\infty$ (recalling that $\Delta = [\alpha, \beta)$), we see that $\{\varphi_1(f; t_k)\}_k \in \mathfrak{H}_1$, $\varphi_2(f; \xi) \in \mathfrak{H}_2$, $\varphi_3(f; \xi) \in \mathfrak{H}_3$.

If $f \in S$, we define the transformation V on elements $E^+(\Delta)f$ by the equation

$$(48) \quad VE^+(\Delta)f = [\{\chi_\Delta(t_k)\varphi_1(f; t_k)\}_k, \chi_\Delta(\xi)\varphi_2(f; \xi), \chi_\Delta(\xi)\varphi_3(f; \xi)] ,$$

where $\chi_\Delta(\xi)$ is the characteristic function of Δ. Then, $VE^+(\Delta)f \in \mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \mathfrak{H}_3$, and from equation (47) it follows that $\|VE^+(\Delta)f\| = \|E^+(\Delta)f\|$.

Let Z_1 consist of all elements of the form $E^+(\Delta)f$, where Δ is an arbitrary interval, and f is an arbitrary element in S. Since A^+ is assumed to be minimal, the Hilbert space $\mathfrak{H}^+$ in which A^+ acts is the closed linear hull of Z_1. (See Naimark [6].) If f is in the closed linear hull of Z_1, f can be written in the form $f = \sum_{r=1}^m E^+(\Delta_r)f_r'$, where $f_r' \in S$, and the Δ_r are disjoint intervals of the form $[\alpha, \beta)$. We define

Vf by means of equation (48) and the equation $Vf = \sum_{r=1}^{m} VE^{+}(\Delta_r)f_r'$. Using equation (47), it can be shown that V is defined uniquely on the linear hull of Z_1 into $\mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \mathfrak{H}_3$, that V is linear, and that V is norm-preserving, i.e., that $\|Vf\| = \|f\|$. By continuity, then, V can be extended to all of $\mathfrak{H}^{+}$. V will be a linear, norm-preserving transformation on $\mathfrak{H}^{+}$ into $\mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \mathfrak{H}_3$.

V is, in fact, on $\mathfrak{H}^{+}$ onto $\mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \mathfrak{H}_3$. To prove this, suppose that $q(\xi) = [\{q_1(t_k)\}_k, q_2(\xi), q_3(\xi)] \in \mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \mathfrak{H}_3$. We shall prove that if q is perpendicular to the range of V, then $q = 0$ in the norm of $\mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \mathfrak{H}_3$; indeed, we shall show that if q is perpendicular to VZ_1, then $q = 0$.

Suppose, then, that q is perpendicular to VZ_1, i.e.,

$$\begin{aligned} 0 = & \sum_{t_k \in \Delta} F_d(t_k)(t_k - \lambda_0)^{-1}[q_1(t_k)]^{-}\tau_{11}[t_k] \\ & + \sum_k \int_{I_k \cap E_2 \cap \Delta} \sum_{u,v=1}^{2} D_u(f;\xi)[q_{2v}(\xi)]^{-} d\rho_{uv}^{k}(\xi) \\ & + \int_{E_3 \cap \Delta} D_1(f;\xi)[q_3(\xi)]^{-} d\rho_{11}(\xi) \end{aligned} \tag{49}$$

for all $f \in S$ and for all Δ. We shall prove that

$$\begin{aligned} 0 = & \sum_k |q_1(t_k)|^2 \tau_{11}[t_k] \\ & + \sum_k \int_{I_k \cap E_2} \sum_{u,v=1}^{2} q_{2u}(\xi)[q_{2v}(\xi)]^{-} d\rho_{uv}^{k}(\xi) \\ & + \int_{E_3} |q_3(\xi)|^2 d\rho_{11}(\xi). \end{aligned} \tag{50}$$

Let us note that since A_0 is unbounded and therefore either E_1 or E_2 is unbounded, we can choose a sequence $\{f_n\}$, $f_n \in S$, such that $F_n(t) = 0$ for $-n \leqq t \leqq n$, $(f_n, g_0) = \int_{-\infty}^{\infty} F_{nc}(t)d\rho_0(t) + \sum_k F_{nd}(t_k)\rho_0[t_k] = 1$. Then for any interval $[\alpha, \beta]$, $D_1(f_n;\xi) \to 1$ as $n \to \infty$, uniformly for $\xi \in [\alpha, \beta]$, and $D_2(f_n;\xi) = 0$ for $\xi \in [\alpha, \beta]$ and n sufficiently large.

We shall work first of all with the second integral in (49) by taking $\Delta \subset E_2$, and we shall show that the second integral in (50) is zero. Now, $E_2 = \bigcup_m J_m$, where the J_m are open intervals. Let us consider an arbitrary but fixed J_m, and suppose that $J_m \subset I_k$. From (49) we obtain that

$$0 = \int_{\Delta} \sum_{u,v=1}^{2} D_u(f;\xi)[q_{2v}(\xi)]^{-} d\rho_{uv}^{k}(\xi) \tag{51}$$

for an arbitrary $f \in S$ and for an arbitrary interval Δ, where $\Delta^c \subset J_m$.

Taking $f = f_n$ in (51), we obtain

$$(52)\quad 0 = \int_{\Delta} D_1(f_n; \xi)\{[q_{21}(\xi)]^{-}\delta_{11}^{k}(\xi) + [q_{22}(\xi)]^{-}\delta_{12}^{k}(\xi)\}d\sigma^k(\xi)$$

for arbitrary Δ, $\Delta^c \subset J_m$. Since $[\chi_{\Delta}(\xi), 0] \in L_{\rho k}^2(J_m)$ and since $[q_{21}(\xi), q_{22}(\xi)] \in L_{\rho k}^2(J_m)$, we see by taking the inner product of these two elements in $L_{\rho k}^2(J_m)$ that $\bar{q}_{21}\,\delta_{11}^{k} + \bar{q}_{22}\,\delta_{12}^{k}$ is integrable with respect to σ^k over Δ. Letting $n \to \infty$ in (52) we then obtain that

$$(53)\qquad 0 = \int_{\Delta} \{[q_{21}(\xi)]^{-}\delta_{11}^{k}(\xi) + [q_{22}(\xi)]^{-}\delta_{12}^{k}(\xi)\}d\sigma^k(\xi)$$

for arbitrary Δ, $\Delta^c \subset J_m$. From this equation it follows that

$$(54)\qquad 0 = \int_{\Delta} F(\xi)\{[q_{21}(\xi)]^{-}\delta_{11}^{k}(\xi) + [q_{22}(\xi)]^{-}\delta_{12}^{k}(\xi)\}d\sigma^k(\xi)$$

for an arbitrary continuous function F on J_m and for arbitrary Δ, $\Delta_c \subset J_m$.

Now for an arbitrary interval $\Delta = [\alpha, \beta)$, $\Delta^c \subset J_m$, let $F \in C'$ and such that $F(t) \equiv 1$ on $[\alpha, \beta]$, $F(t) \equiv 0$ outside $[\alpha_1, \beta_1]$, where $[\alpha_1, \beta_1] \subset J_m$ and $\alpha_1 < \alpha < \beta < \beta_1$. Let $H(t) = F(t)(t - \lambda_0)$. If h is the element in $\mathfrak{H}$ whose transform in $L_{\rho_0}^2(-\infty, \infty)$ is H, then $h \in S$, and $D_1(h; \xi)$ is continuous for all ξ, and $D_2(h; \xi) = F(\xi)$ for all ξ. From (51) and (54) we obtain

$$(55)\qquad 0 = \int_{\Delta} ([q_{21}(\xi)]^{-}\delta_{21}^{k}(\xi) + [q_{22}(\xi)]^{-}\delta_{22}^{k}(\xi))d\sigma^k(\xi)$$

for arbitrary Δ, $\Delta^c \subset J_m$.

From (53) and (55) it follows that $[q_{21}(\xi), q_{22}(\xi)]$ is orthogonal in $L_{\rho k}^2(J_m)$ to all functions $[\chi_{\Delta_1}(\xi), \chi_{\Delta_2}(\xi)]$, where $\Delta_i{}^c \subset J_m$, and since the set of all linear combinations of such functions is dense in $L_{\rho k}^2(J_m)$, we have that $[q_{21}(\xi), q_{22}(\xi)] = 0$ in $L_{\rho k}^2(J_m)$, i.e.,

$$0 = \int_{J_m} \sum_{u,v=1}^{2} q_{2u}(\xi)[q_{2v}(\xi)]^{-}d\rho_{uv}^{k}(\xi)\,.$$

Summing up over all J_m, we obtain

$$(56)\qquad 0 = \sum_{k} \int_{I_k \cap E_2} \sum_{u,v=1}^{2} q_{2u}(\xi)[q_{2v}(\xi)]^{-}d\rho_{uv}^{k}(\xi)\,.$$

Equation (49) now becomes

$$(57)\qquad \begin{aligned} 0 = &\sum_{t_k \in \Delta} F_d(t_k)(t_k - \lambda_0)^{-1}[q_1(t_k)]^{-}\tau_{11}[t_k] \\ &+ \int_{E_3 \cap \Delta} D_1(f; \xi)[q(\xi)]^{-}d\rho_{11}(\xi) \end{aligned}$$

for all $f \in S$ and all Δ.

Let I_k be arbitrary but fixed. Taking $f = f_n$ and $\Delta \subset I_k$, we obtain from (57) that

$$0 = \int_{E_3 \cap \Delta} D_1(f_n; \xi)[q_3(\xi)]^- d\rho_{11}(\xi) \quad \text{for all } \Delta \subset I_k . \tag{58}$$

Since $\int_{E_3} |q_3(\xi)|^2 d\rho_{11}(\xi) < \infty$ and since Δ is assumed to be bounded, it follows that $\int_{E_3 \cap \Delta} |q_3(\xi)| d\rho_{11}(\xi) < \infty$. Hence, we can let $n \to \infty$ in (58) and obtain

$$0 = \int_{E_3 \cap \Delta} [q_3(\xi)]^- d\rho_{11}(\xi) \quad \text{for all } \Delta \subset I_k .$$

This means that q_3 is perpendicular in the space $L^2_{\rho_{11}}(I_k \cap E_3)$ to functions χ_Δ, where $\Delta \subset I_k$. Since the set of linear combinations of such functions is dense in $L^2_{\rho_{11}}(I_k \cap E_3)$, it follows that $0 = \int_{E_3 \cap I_k} |q_3(\xi)|^2 d\rho_{11}(\xi)$. Summing over k, we obtain

$$0 = \int_{E_3} |q_3(\xi)|^2 d\rho_{11}(\xi) . \tag{59}$$

Equation (49) can now be written

$$0 = \sum_{t_k \in \Delta} F_d(t_k)(t_k - \lambda_0)^{-1}[q_1(t_k)]^- \tau_{11}[t_k] \tag{60}$$

for all $f \in S$ and all Δ. Let t_k be arbitrary but fixed. Let $F(t_k) = 1$, $F(t) = 0$ elsewhere. Let f be the element in $\mathfrak{H}$ whose transform in $L^2_{\rho_0}(-\infty, \infty)$ is f. Let Δ be an interval containing t_k. By (60), $0 = (t_k - \lambda_0)^{-1}[q_1(t_k)]^- \tau_{11}[t_k]$. Hence, $0 = |q_1(t_k)|^2 \tau_{11}[t_k]$. Summing up over the t_k, we obtain

$$0 = \sum_k |q_1(t_k)|^2 \tau_{11}[t_k] . \tag{61}$$

Equation (50) now follows from equations (56), (59), and (61). This completes the proof that V is on $\mathfrak{H}^+$ onto $\mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \mathfrak{H}_3$.

It is not difficult, finally, to verify that V takes the spectral function $E^+(\Delta)$ of A^+ into the spectral function of the multiplication operator in $\mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \mathfrak{H}_3$.

This completes the proof of Theorem 2.

References

1. N. Dunford and Jacob T. Schwartz, *Linear operators. II: Spectral theory. Selfadjoint operators in Hilbert space*, Interscience, New York, 1963. MR 32 #6181.

2. R. C. Gilbert, *Spectral multiplicity of selfadjoint dilations*, Proc. Amer. Math. Soc. **19** (1968), 477–482. MR **36** #6964.

3. ——, *Symmetric operators with singular spectral functions,* Proc. Amer. Math. Soc. **21** (1969), 43–49. MR **38** #5030.

4. ——, *Symmetric operators with twice continuously differentiable spectral functions,* Proc. Amer. Math. Soc. **23** (1969), 347–355.

5. I. S. Kac, *Spectral multiplicity of a second-order differential operator and expansion in eigenfunction,* Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 1081–1112. (Russian) MR **28** #3196.

6. M. A. Naĭmark, *Spectral functions of a symmetric operator,* Izv. Akad. Nauk SSSR Ser. Mat. **4** (1940), 277–318. (Russian) MR **2**, 105.

7. A. V. Štraus, *On spectral functions of differential operators,* Izv. Akad. Nauk SSSR Ser. Mat. **19** (1955), 201–220. (Russian) MR **17**, 620.

8. D. V. Widder, *The Laplace transform,* Princeton Math. Series, vol. 6, Princeton Univ. Press, Princeton, N.J., 1941. MR **3**, 232.

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