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PRIMARY COHOMOLOGY OPERATIONS IN *BSJ*

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I. **Introduction.** The study of fiber spaces and fiber bundles has led to several different definitions of equivalence. Two of the most important are "fiber homotopy equivalence" for Hurewicz spherical fiber spaces [8, p. 100] and "bundle equivalence" for spherical fiber bundles [8, p. 92]. If X is a reasonable space, then the set of classes of stable oriented spherical Hurewicz fiber spaces over X is a group, called $\tilde{K}SF(X)$; also the set of classes of stable oriented spherical fiber bundles is a group, called $\tilde{K}SO(X)$.

The contravariant functors $\tilde{K}SF$ and $\tilde{K}SO$ are representable. This means that there are spaces BSF and BSO such that there exist natural isomorphisms $[\ ;BSF] = \tilde{K}SF$ and $[\ ;BSO] = \tilde{K}SO$ when the functors are restricted to a reasonable class of spaces.

For the rest of this introduction, we shall use slightly nonstandard notation. This will serve two purposes. First, it will help to distinguish between the J homomorphism and the contravariant functor which J induces. Second, it will allow our notation to be consistent.

There is a natural transformation $J:\tilde{K}SO \rightarrow \tilde{K}SF$, namely the map that associates to each class of bundles over X the class of fiber spaces which includes it. This transformation is the stable J-homomorphism. We use the symbol $\tilde{K}SJ(X)$ to denote the group $J[\tilde{K}SO(X)]$. Thus $\tilde{K}SJ$ is a contravariant functor (usually denoted by J). Adams has shown that $\tilde{K}SJ$ is not a representable functor [1].

Recently there has been some interest in spaces which are "approximately" classifying spaces for the functor $\tilde{K}SJ$. No space does the job perfectly, the argument goes, but some spaces do a better job than others. A good approximation should behave as much as possible as BSJ would behave if it existed. For example, because of the commutative diagram

$$\begin{array}{ccc} \tilde{K}SO & \xrightarrow{J} & \tilde{K}SF \\ & {\scriptstyle J}\searrow \quad \nearrow & \\ & \tilde{K}SJ & \end{array}$$

there should exist a corresponding homotopy commutative diagram

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$$\begin{array}{ccc} BSO & \longrightarrow & BSF \\ & \searrow \quad \nearrow & \\ & BSJ & \end{array}$$

by [3]. The arrow $BSJ \dashrightarrow BSF$ is dotted for reasons explained below. Also, $[S^n; BSJ]$ should look as much as possible like $\tilde{K}SJ(S^n)$. If the diagram exists, then $[S^n; BSJ]$ cannot look exactly like $\tilde{K}SJ(S^n)$ for reasons explained in [5].

If we convert the map $(\psi^3 - 1): BSO \rightarrow BSO$ (where ψ^3 is the Adams operation [2]) into a fiber map and define SJ to be the 2-primary component of the fiber, then SJ is an H-space that has a classifying space which we agree to call BSJ. We ignore all odd torsion for reasons explained in [5]. The homotopy groups $[S^n, BSJ]$ are exactly as required by the table in [5]. The search for a map $BSJ \rightarrow BSF$ continues, although Stasheff has checked that BSF actually splits into $BSJ \times X$ in low dimensions. This explains why we have used a dotted arrow in our diagram above.

The author's calculation of $H^*(BSJ; Z_2)$ as an algebra [5] and more recent calculations by Milgram and May [6] of $H^*(BSF; Z_2)$ show that the first obvious class of obstructions to a map $BSJ \rightarrow BSF$ vanishes. In the present work, we prove that the second obvious class of obstructions vanishes: Namely we show that Stiefel-Whitney classes can be chosen in $H^*(BSJ; Z_2)$ so as to satisfy the Wu formula. It is known that the Stiefel-Whitney classes in $H^*(BSO; Z_2)$ and $H^*(BSF; Z_2)$ satisfy the Wu formulae. Hence, they must also in $H^*(BSJ; Z_2)$ if the map exists. Indeed, we shall prove the following theorem:

THEOREM 1. *There exist isomorphisms*

$$H^*(BSJ; Z_2) = H^*(BSO \times BBSO; Z_2);$$

$$H^*(SJ; Z_2) = H^*(SO \times BSO; Z_2).$$

Both are simultaneously isomorphisms of Hopf algebras and isomorphisms of modules over the Steenrod algebra.

Of course, the Hopf algebra and Steenrod algebra module structure of both objects on the right-hand side of the equalities are known. $H^*(BBSO; Z_2)$ is the least well known [5].

II. **Details.** We first agree on certain conventions. All homology and cohomology groups will have Z_2 coefficients. Hence $H_*(X)$ and $H^*(X)$ will mean $H_*(X; Z_2)$ and $H^*(X; Z_2)$. Let $Y = \{y_{\alpha(1)}, y_{\alpha(2)}, \cdots\}$ be a set of indeterminates over Z_2, where α is an increasing function with values in the set of positive integers. Then $P[Y] = P[y_{\alpha(1)}, y_{\alpha(2)}, \cdots]$ is the graded Z_2 polynomial algebra on the $y_{\alpha(i)}$, where degree $(y_{\alpha(i)}) = \alpha(i)$. As an example, $H^*(BO) = P[w_1, w_2, \cdots]$ is the polynomial algebra on the Stiefel-Whitney classes. Also, $E[Y] = E[y_{\alpha(1)}, y_{\alpha(2)}, \cdots]$ is the graded Z_2 exterior algebra on the $y_{\alpha(i)}$. As an example, $H^*(BBO) = E[e_2, e_3, e_4, \cdots]$. Finally $\mathcal{A}(2)$ is the Z_2 Steenrod algebra.

Let us briefly review the definition of BSJ. Let BOQ_2 denote the classifying space for the representable functor $\tilde{K}O \otimes Q_2$, where Q_2 is the ring of rational numbers which, when reduced to lowest terms, have odd denominators. Let $BSOQ_2$ denote the classifying space for $\tilde{K}SO \otimes Q_2$. Then we have homotopy commutative diagrams

$$\begin{array}{ccc} BO & \xrightarrow{\psi^3 - 1} & BO \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ BOQ_2 & \xrightarrow{\chi} & BOQ_2 \end{array} \qquad \begin{array}{ccc} BSO & \xrightarrow{\psi^3 - 1} & BSO \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ BSOQ_2 & \xrightarrow{\chi} & BSOQ_2 \end{array}$$

where the vertical maps are 2-primary homotopy equivalences and infinite loop maps. The maps χ are the 8-fold loops of maps $BOQ_2[9] \to BOQ_2[9]$ and $BSOQ_2[10] \to BSOQ_2[10]$. Here $X[n]$ denotes the $(n-1)$-connected covering of X. We define BJ and BSJ by insisting that the following sequences be fibrations:

$$BJ \to BBOQ_2 \xrightarrow{B\chi} BBSOQ_2; \qquad BSJ \to BBSOQ_2 \xrightarrow{B\chi} BBSOQ_2 .$$

The upper map is a lifting of the map $BBOQ_2 \to BBOQ_2$ into $BBSOQ_2$. Let $J = \Omega BJ$, $SJ = \Omega BSJ$, and $SOQ_2 = \Omega BSOQ_2$. Then we have fibrations

$$SOQ_2 \to SJ \to BSOQ_2; \qquad BSOQ_2 \to BSJ \to BBSOQ_2;$$

$$SOQ_2 \to J \to BOQ_2; \qquad BSOQ_2 \to BJ \to BBOQ_2.$$

In [5] it is shown that all differentials (starting with d_2) in the Z_2 Serre homology and cohomology spectral sequences are 0. Thus the isomorphisms of Theorem 1 are algebra isomorphisms.

It will be convenient to do our work first in $H^*(J)$, then in $H^*(BJ)$, and finally in $H^*(SJ)$ and $H^*(BSJ)$. This is because there is a map $RP \to BO$ which sends the Stiefel-Whitney classes to the symmetric functions of $H^*(RP \times \cdots \times RP)$, where RP denotes real infinite dimensional projective space.

THEOREM 2. *There exists an isomorphism*

$$H^*(J; Z_2) = H^*(SO \times BO; Z_2)$$

which is simultaneously an isomorphism of Hopf algebras and of modules over the Steenrod algebra.

PROOF OF THEOREM 2.

Step 1. *The Hopf algebra structure of* $H^*(J)$. As an algebra, $H_*(J)$ is isomorphic to $E[x_1, x_2, \cdots] \otimes P[y_1, y_2, \cdots]$. Since the space of primitive elements of $H^*(J)$ is the dual of the space of indecomposable elements of $H_*(J)$, we know that the primitive subspace of $H^{2n+1}(J)$ is isomorphic to $Z_2 \oplus Z_2$. Then there exists a primitive element $p_{2n+1} \in H^{2n+1}(J)$ such that p_{2n+1} is not in the image of $H^*(BO) \to H^*(J)$. We claim that p_{2n+1} is indecomposable. Obviously p_1 is indecomposable. Suppose p_{2k+1} is indecomposable for $k < n$. Since $P[p_{2k+1} \mid k < n] \otimes P[w_1, w_2, \cdots]$ has only one primitive of degree $2n + 1$, namely the one in the image of $H^*(B) \to H^*(BJ)$, and since p_{2n+1} is not in this image, we know that

$$p_{2n+1} \notin P[p_{2k+1} \mid k < n] \otimes P[w_1, w_2, \cdots] .$$

Hence, p_{2n+1} is indecomposable. Thus indeed, $H^*(J) = P[p_n \mid n \text{ is odd}] \otimes P[w_1, w_2, \cdots] = H^*(SO \times BO)$, where the equalities are all Hopf algebra isomorphisms.

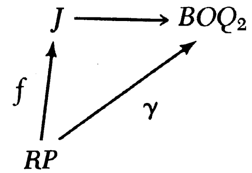
Step 2. *The behavior of* Sq^1 *in* $H^*(J)$. Since the w_i of $H(J)$ satisfy the Wu relations, we know that Sq^1 annihilates each primitive element p'_{2n} of even degree in $P[w_1, w_2, \cdots] \subset H^*(J)$ and sends each primitive element p'_{2n-1} of odd degree to p'_{2n}. Now Sq^1p_{2n-1} is primitive and maps nontrivially into $H^*(SO)$. Thus Sq^1p_{2n-1} is either $p_1{}^{2n}$ or $p_1{}^{2n} + w_1{}^{2n}$ when n is a power of two. In the latter case, $Sq^1(p_{2n-1} + p'_{2n-1}) = p_1{}^{2n}$. We now officially redefine p_{2n-1} to be that primitive of degree $2n - 1$ such that $Sq^1p_{2n-1} = p_1{}^{2n}$ whenever n is a power of two. We redefine p_{2n-1} in general to be Sq^jp_{2k-1} where k is a power of two, $j + 2k - 1 = 2n - 1$ and $0 \leqq j < 2k$. We recall that

$$H^*(SO) = P[p_1, p_2, p_3, \cdots]/\text{Ideal } [p_i{}^2 + p_{2i} \mid i = 1, 2, \cdots]$$
$$= P[p_i \mid i \text{ is odd}]$$

and that $Sq^ip_j = \binom{j}{i}p_{i+j}$ where the p_i are all primitive. Our alterations of the p_i in $H^*(J)$ have not altered the indecomposability of the p_i, nor have they altered the fact that the p_i of $H^*(J)$ map to the p_i of $H^*(SO)$. Since the Sq^i preserve primitivity, the new p_i are still primitive. Notice that our choice of p_i depend on our choice of p_1. If we had taken $p_1 + w_1$ as the primitive not in $H^1(BO) \to H^1(J)$, we would have had to take $p_{2n-1} + p'_{2n-1}$ as our primitive of $H^{2n-1}(J)$ for n a power of two. We have proved the following lemma:

LEMMA 1. *For each choice of $p_1 \in H^1(J)$ there is exactly one choice of primitive indecomposable elements $p_i \in H^i(J)$ such that $H^*(J)$ and $H^*(SO \times BO)$ are isomorphic simultaneously as Hopf algebras and as differential algebras with differential* Sq^1. ∎

Step 3. *$H^*(J)$ as an $\mathcal{A}(2)$ module.* The fact that $\chi : BOQ_2 \to BOQ_2$ lifts into $BSOQ_2$ is a consequence of the fact that ψ^3 is the identity map on line bundles. Another consequence is that there is a homotopy commutative diagram



where γ is the canonical line bundle. We now pick $p_1 \in H^*(J)$ once and for all to be the primitive in $H^1(J)$ such that $f^* p_1 = 0$. Then we make the choice of p_i which Lemma 1 permits.

We remark that $f^* p_{2n-1} = 0$ for each n. For suppose n is a power of two and $f^* p_{2n-1} = \alpha^{2n-1}$ where $\alpha \in H^1(RP)$ is the generator. Then

$$\begin{aligned} 0 &= (f^* p_1)^{2n} = f^*(p_1{}^{2n}) = f^* Sq^1 p_{2n-1} \\ &= Sq^1 f^* p_{2n-1} = Sq^1 \alpha^{2n-1} = \alpha^{2n} \neq 0 . \end{aligned}$$

This contradiction proves that $f^* p_{2n-1} = 0$.

LEMMA 2. *Let G be a homotopy commutative H-space and let A and B be sub-Hopf algebras of G such that $H^*(G)$ and $A \otimes B$ are isomorphic as Hopf algebras. Let X and Y be connected pointed spaces and $f: X \to G$, $g : Y \to G$ be point preserving maps such that $f^*[A] = Z_2 = H^0(X)$ and $g^*[A] = Z_2 = H^0(Y)$. Define $h : X \times Y \to G$ by $h(x, y) = f(x)g(y)$. Then $h^*[A] = Z_2 = H^0(X \times Y)$.*

PROOF. Let $\{a_1, a_2, \cdots\}$ be a basis of A and B where $a_1 = 1$ and subscripts do not denote degree. Let $\mu : G \times G \to G$ be multiplication. Then $h^* a_k = (f^* \otimes g^*) \circ \mu^* a_k = f^* \otimes g^* \sum \epsilon(i, j) a_i \otimes a_j = \sum \epsilon(i, j) f^* a_i \otimes g^* a_j = 0$ unless $i = j = 1$ and $a_i = a_j = 1$. ∎

By Lemma 2, if we take the map $RP \times \cdots \times RP \to J$, where there are n copies of RP, we see that all of the p_i are sent to 0. The w_i for $i \leqq n$ map to the symmetric functions on the $\alpha(i)$ in the usual fashion, where $H^*(RP \times \cdots \times RP) = P[\alpha(1), \cdots, \alpha(n)]$. Here each $\alpha(k)$ has degree 1.

LEMMA 3. *Let G be a homotopy commutative H-space such that $H^*(G) = A \otimes B$ as Hopf algebras. Let $f: X \to G$ and $g: Y \to G$ be continuous base point-preserving functions where X and Y are spaces with base points. Suppose $f^*[B] = Z_2 = H^0(X)$ and $g^*[A] = Z_2 = H^0(Y)$. Let $h: X \times Y \to G$ be defined by $h(x, y) = f(x) \cdot g(y)$. Then $h^*[A] \subset H^*(X)$ and $h^*[B] \subset H^*(Y)$, where $H^*(X)$ and $H^*(Y)$ are naturally included in $H^*(X \times Y)$.*

PROOF. Let $\{a_1, a_2, \cdots\}$ and $\{b_1, b_2, \cdots\}$ be bases for A and B; here subscripts do not denote degree, but $a_1 = 1 = b_1$. Then $h^*a_k = (f^* \otimes g^*) \circ \mu^*a = f^* \otimes g^* \sum \epsilon(i,j) a_i \otimes a_j = \sum \epsilon(i,j) f^* a_i \otimes g^* a_j = \sum \epsilon(i, 1) f^* a_i \otimes 1 \in H^*(X)$. The rest of the proof is similar. ■

By Lemma 3, if we now combine $SO \to J$ with $RP \times \cdots \times RP \to J$ to obtain $SO \times RP \times \cdots \times RP \to J$, we have a monomorphism $H^*(J) \to H^*(SO \times RP \times \cdots \times RP)$ in degrees $\leqq n$. Furthermore, the w_i are sent into $H^*(RP \times \cdots \times RP)$ and the p_i are sent into $H^*(SO)$. This proves Theorem 2.

PROOF OF THEOREM 1. Theorem 1 for SJ follows immediately from Theorem 2.

By selecting primitive indecomposables in the base of the homology Serre spectral sequence of $BSO \to BSJ \to BBSO$ we show that $H^*(BSJ) = H^*(BSO \times BBSO)$ as Hopf algebras. Because $H^*(BBSO)$ has odd dimensional primitive indecomposables and no even dimensional primitives, there is exactly one choice of $w_n \in H^*(BSJ)$ such that

$$\iota^* w_n = w_n \in H^*(BSO), \qquad \Delta w_n = \sum \{w_i \oplus w_j \mid i + j = n\},$$

and $\sigma w_k \in H^*(SO) \subset H^*(SJ)$ for $k = 2$, 4, or odd. We wish to show that $Sq^i w_j \in P[w_n \mid n \geqq 2]$ for all i and j. This turns out to be false in BJ, a fact due to the absence of w_4 suspending into $H^*(SO) \subset H^*(J)$.

Let n be the least k such that for some i, $Sq^i w_k \notin P[w_j \mid j \geqq 2]$. Let m be the least i such that $Sq^i w_n \notin P[w_j \mid j \geqq 2]$. Now

$$\begin{aligned} Sq^m w_n &= \sum \{a(t) w_{m-t} w_{n+t} \mid 0 \leqq t \leqq m\} + x \\ &= a(m) w_{m+n} + p + x. \end{aligned}$$

We have

$$\begin{aligned} \Delta x = {} & \Delta a(m) w_{m+n} + \Delta p + Sq^m \Delta w_n \\ = {} & a(m)\, \Delta w_{m+n} + \Delta p + Sq^m w_n \otimes 1 + 1 \otimes Sq^m w_n \\ & + Sq^m \sum \{w_i \otimes w_j \mid i + j = n \text{ and } 2 \leqq i \leqq n - 2\}. \end{aligned}$$

Inductively, the nontrivial terms (terms not of the form $y \otimes 1$ and $1 \otimes y$) cancel out in the last expression. Thus x is primitive.

Then if $m+n$ is even, we know $Sq^m w_n \in P[w_j \mid w_j \geqq 2]$ because the only even dimensional primitive x of the *BBSO* part of *BSJ* is 0. Hence, $m+n$ is odd. Then $\sigma Sq^m w_n \notin H^*(SO) \subset H^*(SJ)$ because the primitives x not in $P[w_j \mid w_j \geqq 2]$ are indecomposable, hence suspend nontrivially. Thus $\sigma w_n \notin H^*(SO)$. We note that n is therefore even and $\geqq 5$.

Now let q be the greatest 2^i such that $2^i \leqq n$ and suppose that $q < n$. Let $r = n - q$. In *BSO* there is a Wu relation $Sq^r w_q + p = w_n$. Since $q < n$, we know inductively that

$$\begin{aligned}\Delta(Sq^r w_q + p) = Sq^r \Delta w_q + \Delta p &= (Sq^r w_q + p) \otimes 1 \\ &+ \sum \{w_i \otimes w_j \mid i + j = n \text{ and } 2 \leqq i \leqq n-2\} \\ &+ 1 \otimes (Sq^r w_q + p)\end{aligned}$$

in *BSJ*. Then, in *BSJ*, we have $w_n = Sq^r w_q + p$ modulo the primitives of the *BBSO* part of *BSJ*. Since there are no even dimensional primitives of *BBSO*, and since σp must be 0, we have $w_n = Sq^r w_q + p$. But $Sq^r w_q + p \in P[w_j \mid j \geqq 2]$ inductively. Thus we cannot have $q < n$. Then $n = q$ is a power of 2 and, since $n \geqq 5$, we have $n \geqq 8$.

Let $u = q/2$. In *BSO* there is a Wu relation $Sq^{u,2} w_u = w_{q+2} + p$. We routinely (as above) calculate that $Sq^{u,2} w_u + p = w_{q+2}$ in *BSJ*, using again the primitive structure of *BSJ*. On the other hand, *BSO* has a Wu relation $Sq^2 w_q = w_2 w_q + w_{q+2}$. But

$$\begin{aligned}\Delta Sq^2 w_q + \Delta w_2 w_q = Sq^2 \textstyle\sum w_i \otimes w_j + \Delta w_2 \, \Delta w_q = Sq^2 w_q \otimes 1 \\ + \textstyle\sum \{w_i \otimes w_j \mid i + j = q + 2 \text{ and } 2 \leqq i \leqq q\} + 1 \otimes Sq^2 w_q.\end{aligned}$$

Thus $w_{q+2} = Sq^2 w_q + w_2 w_q = Sq^{u,2} w_u + p$ in *BSJ*. Now $w_{q+2} = Sq^{u,2} w_u + p = Sq^u(w_{u+2} + w_2 w_u) + p$ is in $P[w_j \mid j \geqq 2]$ inductively. Thus $Sq^2 w_q \in P[w_j \mid j \geqq 2]$. We cannot have $\sigma w_q = p_{q-1} + p'_{q-1}$ where p_{q-1} and p'_{q-1} are the primitives of the *SO* and *BSO* parts of *SJ*, because p'_{q-1} is indecomposable and $Sq^2 w_{q-1}$ involves w_{q+1}. This would contradict the fact that $Sq^2 w_q$ suspends into $H^*(SO)$. Then $\sigma w_q \in H^*(SO)$. But this contradicts our previous deduction that $\sigma w_n \notin H^*(SO)$. Thus our very original assumption that $Sq^m w_q \notin P[w_j \mid j \geqq 2]$ is false.

We wish to acknowledge that the basic idea for the proof of Theorem 2 appears in the work of Peterson and Toda [7].

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