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DIVISIBLE QUOTIENT GROUPS OF REDUCED ABELIAN GROUPS

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The groups under discussion here are all Abelian, so *group* means *Abelian group*. A group G is *divisible* if $nG = \{ng \mid g \in G\} = G$ for all nonzero integers n. Typical examples are the additive group Q of rational numbers, and its homomorphic images. In fact, every divisible group is a direct sum of such groups. If D is a divisible subgroup of G, then D is a summand of G. Every group G has a unique largest divisible subgroup D, and if $G = D \oplus H$, then H has no nonzero divisible subgroups. Such an H is called *reduced*. Reduced groups G can have divisible quotient groups G/A. In fact, free groups are reduced and certainly have divisible quotient groups. However, if G is reduced and G/A is divisible, then A cannot be too small compared to G. For example, suppose G is a reduced p-group and B is a *basic* subgroup of G. That is, B is a subgroup of G such that G/B is divisible, B is a direct sum of cyclic groups, and $B \cap nG = nB$ for all integers n. Then Fuchs shows [1, Theorem 30.1] that $|B|^{\aleph_0} \geqq |G|$. Fuchs proves this using his [2] quasibases of such G. He then uses $|B|^{\aleph_0} \geqq |G|$ to show that for reduced p-groups G, $|G/G^1|^{\aleph_0} \geqq |G|$, where $G^1 = \bigcap_{n=1}^{\infty} nG$ is the subgroup of elements of infinite height in G. The facts are important in the theory of p-groups. (They are crucial in establishing necessary and sufficient conditions for a well-ordered sequence of p-groups with no elements of infinite height to be the Ulm sequence of a reduced p-group. See [1, Chapter VI], for example.) Now these inequalities hold in general. That is, if G is any reduced group and G/A is divisible, then $|A|^{\aleph_0} \geqq |G|$, and $|G/G^1|^{\aleph_0} \geqq |G|$. The group G does not have to be a p-group and A does not have to be a basic subgroup of G. The second inequality is actually a consequence of the first, as we shall see. The inequality $|A|^{\aleph_0} \geqq |G|$ has a short homological proof as follows. The exact sequence $0 \to Z \to Q \to Q/Z \to 0$ yields the exact sequence $\mathrm{Hom}(Q, G) = 0 \to \mathrm{Hom}(Z, G) \approx G \to \mathrm{Ext}(Q/Z, G)$ so that $\mathrm{Ext}(Q/Z, G)$ contains a copy of G. The sequence $0 \to A \to G \to G/A \to 0$ yields the epimorphism $\mathrm{Ext}(Q/Z, A) \to \mathrm{Ext}(Q/Z, G) \to 0$; $\mathrm{Ext}(Q/Z, G/A) = 0$ since every extension of a divisible group splits. Thinking of $\mathrm{Ext}(Q/Z, A)$ as the group of factor systems (which are some of the

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maps $Q/Z \times Q/Z \to A$) modulo the group of principal factor systems gets $|A|^{\aleph_0} \geqq |\mathrm{Ext}(Q/Z, A)|$, and the epimorphism above then implies that $|A|^{\aleph_0} \geqq |G|$. This generalizes and simplifies the proof in [3] that if G is a reduced *torsion group* and G/A is divisible, then $|A|^{\aleph_0} \geqq |G|$. But the inequality $|A|^{\aleph_0} \geqq |G|$ should be elementary. It should be accessible without using the concept of basic subgroups, or of quasibases, and it certainly should not require homological methods. We will give an elementary proof of this inequality, and derive, again in an elementary manner, the inequality $|G/G^1|^{\aleph_0} \geqq |G|$ from it.

THEOREM. *If G is a reduced group and G/A is divisible, then* $|A|^{\aleph_0} \geqq |G|$.

PROOF. Let $g \in G$. Write

$$\begin{aligned} g &= 2!g_2 + a_1 \\ g_2 &= 3!g_3 + a_2 \\ &\vdots \\ g_n &= (n+1)!g_{n+1} + a_n \\ &\vdots \end{aligned}$$

with $a_i \in A$. Thus we get a map

$$G \to A^{\omega} : g \to \{a_i\}_{i=1}^{\infty} .$$

It suffices to show that this map is one-to-one. If g and h have the same image $\{a_i\}_{i=1}^{\infty}$, then

$$\begin{aligned} h &= 2!h_2 + a_1 \\ h_2 &= 3!h_3 + a_2 \\ &\vdots \end{aligned}$$

Therefore

$$\begin{aligned} g - h &= 2!(g_2 - h_2) \\ g_2 - h_2 &= 3!(g_3 - h_3) \\ &\vdots \end{aligned}$$

The set $\{g - h, g_2 - h_2, g_3 - h_3, \cdots\}$ generates a divisible subgroup of G, obviously. Since G is reduced, $g - h = 0$.

COROLLARY. *If G is a reduced group, then* $|G/G^1|^{\aleph_0} \geqq |G|$.

PROOF Let A be a subgroup of G maximal with respect to $A \cap G^1 = 0$. Then G/A is divisible [2, Theorem 12]. (The proof of this in [3, Theorem 12] is elementary and short.) Since $|G/G^1| \geqq |A|$, and $|A|^{\aleph_0} \geqq |G|$, the corollary follows.

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