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SUFFICIENT CONDITIONS FOR A GROUP TO BE A DIRECT SUM OF CYCLIC GROUPS[1]

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1. **Introduction.** This paper, except possibly the last section, may be regarded as expository. Although substantial extensions of the known results are obtained herein, these new results are more or less immediate corollaries of the author's uniqueness theorem for totally projective groups [5]. However, the direct proofs given here are much simpler than the proof of the uniqueness theorem.

Sufficient conditions on a commutative group in order that it be a direct sum of cyclic groups have been sought for a long time. Although several such conditions have been found, better and more refined conditions are still desired. The oldest and probably the best known condition that implies that a commutative group is a direct sum of cyclic groups is that the group be finite. In searching for conditions that make an infinite commutative group a direct sum of cyclic groups, one may, of course, immediately restrict the problem to the primary and torsion-free cases. We concentrate in this paper on the primary case, that is, on finding sufficient conditions for a primary group to be a direct sum of cyclic groups. However, we remark that in the torsion-free case, as well as the primary case, there are some deep and important problems in abelian groups that amount basically to the question of whether or not a certain condition is sufficient for a group to be a direct sum of cyclic groups. Consider, for example, the famous problems of Baer and Whitehead; two recent papers of interest on these problems are [2] and [3].

An element g in an additively written p-primary group G is said to be divisible by p^n if there is a solution in G to the equation $p^n x = g$. The element g has *height* n if g is divisible by p^n but not by p^{n+1}. If g is divisible by p^n for each positive integer n, then we say that g has infinite height. Naturally, we say that G is without elements of infinite height if zero is the only element that has infinite height. Observe that in order for G to be a direct sum of cyclic groups it is necessary that G be without elements of infinite height. The following result was a major contribution to the theory of infinite commutative groups in its early development.

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THEOREM (PRÜFER [10]). *Let G be a primary group without elements of infinite height. If G is the union of an ascending chain*

$$0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$$

of finite subgroups, then G is a direct sum of cyclic groups.

The chain condition in the above theorem is obviously equivalent to G being countable, but we have cast the theorem this way to suggest generalizations.

An important generalization of Prüfer's theorem was made by Kulikov in [9].

THEOREM (KULIKOV). *Let G be a primary group without elements of infinite height. If G is the union of an ascending sequence*

$$0 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$$

of subgroups A_n of G whose elements have only a finite number of different heights in G, then G is a direct sum of cyclic groups.

We have now already reviewed the major classical results concerning our problem; however, still other results can be found in [6], [8] and [11]. In §3 we present a new condition for a group of cardinality $\aleph_1$ to be a direct sum of cyclic groups. This condition may be considered as an extension of Prüfer's. In §4 a condition more like that of Kulikov's, but in some ways more general, is shown to imply that a primary group is a direct sum of cyclic groups.

2. **Preliminaries.** Let G be a p-primary group without elements of infinite height; recall from the introduction the definition of height. A subgroup H of G is *pure* if each element of H has the same height in H as it does in G. Some elementary but important properties of purity are the following; for proofs, see [1]. The union of a chain of pure subgroups is again a pure subgroup. Any infinite subgroup is contained in a pure subgroup having the same cardinality as the given subgroup. Direct sums of cyclic groups are pure projectives, that is, H is a direct summand of G if H is pure in G and G/H is a direct sum of cyclic groups.

For a positive integer n, let $p^nG = \{p^nx : x \in G\}$. Introduce a topology on G by letting the subgroups $pG, p^2G, \cdots, p^nG, \cdots$ be a basis for the neighborhoods of zero. This topology is called the p-adic topology, and all topological references that follow will be to the p-adic topology. Note that G is Hausdorff since G has no elements of infinite height. Note also that there is a very simple algebraic test to determine whether or not a subgroup H of G is closed: H is closed in G if and only if G/H is without elements of infinite height. In our extension of Prüfer's theorem, we can replace the finite subgroups in the chain by countable ones as long as the countable subgroups are closed. However, it is necessary to put a condition on the nature of the chain itself.

We call a chain $\{A_i\}_{i\in I}$, indexed by I, of subgroups of G *complete* if the chain has the following property: for each subset $J \subseteq I$, the subgroup $A(J) = \bigcup_{j\in J} A_j$ belongs to the chain provided that it is contained in a member of the chain. Note that an ascending sequence

$$A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots, \qquad n < \omega,$$

is always a complete chain. However, a chain indexed by a longer initial segment of the ordinals may not be complete. In this connection, one should observe that the ascending chain in Prüfer's theorem is necessarily a sequence (indexed by the nonnegative integers) because the subgroups are all finite. Thus in Prüfer's theorem the chain is necessarily complete.

3. **Groups of power $\aleph_1$.** The main result of this section is the following theorem.

THEOREM 3.1. *Let G be a primary group without elements of infinite height. If G is the union of a complete chain*

$$(\mathcal{C}) \qquad 0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_i \subseteq \cdots, \qquad i \in I,$$

of countable, closed subgroups of G, then G is a direct sum of cyclic groups.

PROOF. If G is countable, then G is a direct sum of cyclic groups according to Prüfer. Thus we shall assume that G is uncountable. It is clear that the cardinality of G must be exactly $\aleph_1$, the first uncountable cardinal. Therefore, we can put $G = \{g_0, g_1, \cdots, g_\alpha, \cdots\}$, $\alpha < \Omega$, into one-to-one correspondence with the countable ordinals.

Let β be a countable ordinal and suppose that we have an ascending, complete chain

$$(\mathcal{C}') \qquad 0 = B_0 \subset B_1 \subset \cdots \subset B_\alpha \subset \cdots, \qquad \alpha < \beta,$$

of pure subgroups of G such that B_α is a member of $(\mathcal{C})$ for each $\alpha < \beta$ and such that $g_\alpha \in B_{\alpha+1}$ for each α satisfying $\alpha + 1 < \beta$. We wish to define B_β in such a way as to keep the induction hypotheses alive.

Case 1. β is a limit ordinal. Define $B_\beta = \bigcup_{\alpha<\beta} B_\alpha$. Then B_β is countable and is a member of $(\mathcal{C})$ since $(\mathcal{C})$ is complete and since B_α belongs to $(\mathcal{C})$ for each $\alpha < \beta$. Furthermore, B_β is a pure subgroup of G since the union of an ascending chain of pure subgroups of G is itself pure in G. The chain $\{B_\alpha\}_{\alpha\leqq\beta}$, of course, remains complete with this definition of B_β. Since β is a limit, there is no change at all, going from $\alpha < \beta$ to $\alpha \leqq \beta$, on the condition $g_\alpha \in B_{\alpha+1}$.

Case 2. $\beta - 1$ exists. Let $\gamma = \beta - 1$. Choose a countable pure subgroup C_1 of G that properly contains $\{B_\gamma, g_\gamma\}$. Now C_1 may not be a member of the chain $(\mathcal{C})$, but C_1 is contained in some member D_1 of $(\mathcal{C})$. In turn D_1 is contained in a countable pure subgroup C_2 of G, etc. Hence there exists a sequence

$$B_\gamma \subset C_1 \subseteq D_1 \subseteq C_2 \subseteq D_2 \subseteq \cdots,$$

where C_n is pure in G and D_n is a member of the chain $(\mathcal{C})$ for each positive integer n. Defining $B_\beta = \bigcup_{n<\omega} C_n = \bigcup_{n<\omega} D_n$, we observe that B_β is pure and belongs to $(\mathcal{C})$. Furthermore $g_\gamma \in B_\beta$, so all our conditions continue to hold.

From what we have done, it follows that G is the union of an ascending, complete chain

$$0 = B_0 \subset B_1 \subset \cdots \subset B_\alpha \subset \cdots, \qquad \alpha < \Omega,$$

of countable, closed, pure subgroups of G. Consider the quotient group $B_{\alpha+1}/B_\alpha$; it is countable. Since B_α is closed, G/B_α has no elements of infinite height, so certainly $B_{\alpha+1}/B_\alpha$ has no elements of infinite height. By Prüfer's theorem, $B_{\alpha+1}/B_\alpha$ is a direct sum of cyclic groups. But B_α is pure in G and therefore is pure in $B_{\alpha+1}$, so B_α is a direct summand of $B_{\alpha+1}$. Let $B_{\alpha+1} = B_\alpha + K_\alpha$. The final point is that $G = \sum_{\alpha<\Omega} K_\alpha$ since G is the union of the chain $\{B_\alpha\}_{\alpha<\Omega}$ and since this chain is complete. Thus G is a direct sum of cyclic groups.

A group is called a Fuchs five group (see Problem 5 in [1]) if it has the property that any infinite subgroup is contained in a direct summand of the group having the same cardinality as the subgroup. Let G be a primary group without elements of infinite height. If G is a Fuchs five group, must G be a direct sum of cyclic groups? A recent result of Hill [4] shows that the answer is negative, but one could ask does Fuchs five plus something else imply that G is a direct sum of cyclic groups. In this direction, we have the following positive result.

THEOREM 3.2. *Let Ω be the first uncountable ordinal and let $B = \sum_{\alpha<\Omega} B_\alpha$, where B_α is a countable direct sum of cyclic p-groups for each $\alpha < \Omega$. Denote by $\bar{B}_\alpha$ the torsion completion of B_α. If G is a pure subgroup of $\sum_{\alpha<\Omega} \bar{B}_\alpha$ containing B, then the following are equivalent:*

(1) *G is a direct sum of cyclic groups.*

(2) *G is the union of a chain of countable summands.*

(3) *G is a Fuchs five group.*

(4) *The closure of each infinite subgroup of G has the same cardinality as the subgroup.*

PROOF. If G is a direct sum of cyclic groups, then $G \cong B$ because B is a basic subgroup of G. However, B is obviously the union of a chain of countable summands, so (1) implies (2). Likewise, trivially (2) implies (3), and (3) implies (4). The only problem is showing that (4) implies (1); for this, we shall employ Theorem 3.1. For each $\beta < \Omega$, define $A_\beta = G \cap \sum_{\alpha<\beta} \bar{B}_\alpha$. Since $G \subseteq \sum_{\alpha<\Omega} \bar{B}_\alpha$, G is the union of the ascending chain

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\beta \subseteq \cdots, \qquad \beta < \Omega.$$

Obviously, the chain is complete; if β is a countable limit ordinal, $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$. Furthermore, for each $\beta < \Omega$,

$$\begin{aligned} G/A_\beta &= G \Big/ \Big(G \cap \sum_{\alpha<\beta} \bar{B}_\alpha \Big) \cong \Big\{ G, \sum_{\alpha<\beta} \bar{B}_\alpha \Big\} \Big/ \sum_{\alpha<\beta} \bar{B}_\alpha \\ &\subseteq \Big(\sum_{\alpha<\Omega} \bar{B}_\alpha \Big) \Big/ \sum_{\alpha<\beta} \bar{B}_\alpha \cong \sum_{\beta \leqq \alpha < \Omega} \bar{B}_\alpha , \end{aligned}$$

so G/A_β is without elements of infinite height; hence A_β is closed in G. In order to show that A_β is countable for each $\beta < \Omega$, define $C_\beta = \sum_{\alpha<\beta} B_\alpha$ and observe that A_β is contained in the closure of C_β; each element of A_β/C_β has infinite height in G/C_β. Hence A_β is countable for each $\beta < \Omega$, and G is a direct sum of cyclic groups by Theorem 3.1.

We remark that with condition (4) deleted the above theorem was proved by Williams [**12**]. For an example of a group G such that condition (4) does not imply condition (1), see the author's paper [**4**].

4. **Groups of arbitrary power.** Kulikov's theorem applies to groups of arbitrary power. Also we mention that the following result has been used, for example, by Kaplansky as a substitute for the Kulikov criterion. Define $G[p] = \{x \in G : px = 0\}$.

THEOREM 4.1 (KAPLANSKY [7]). *Let G be a primary group without elements of infinite height. If $G[p]$ is the union of an ascending sequence*

$$0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n \subseteq \cdots$$

of subgroups S_n of $G[p]$ whose elements have only a finite number of different heights in G, then G is a direct sum of cyclic groups.

Our purpose here is to establish the following

THEOREM 4.2. *Let G be a primary group without elements of infinite height. Then G is a direct sum of cyclic groups if (and only if) there* *exists a collection $\mathcal{C}$ of subgroups of $G[p]$ such that:*

(i) *Each member of $\mathcal{C}$ is closed in G.*

(ii) 0 *is a member of* $\mathcal{C}$.

(iii) *The group union in G of any number of subgroups belonging to $\mathcal{C}$ again belongs to $\mathcal{C}$.*

(iv) *If $S \in \mathcal{C}$ and if $T \subseteq G[p]$ is such that $\{S, T\}/S$ is countable, then there exists $S' \in \mathcal{C}$ such that $S' \supseteq \{S, T\}$ and such that S'/S is countable.*

Proof. First, observe that if $G = \sum_{i \in I} A_i$ is a direct sum of cyclic groups A_i, then the collection $\mathcal{C} = \{S_J\}_{J \subseteq I}$ where $S_J = \sum_{j \in J} A_j[p]$ and J ranges over all the subsets of I satisfies the given conditions.

Now suppose that $\mathcal{C}$ is a collection of subgroups of $G[p]$ satisfying conditions (i) — (iv). By an argument similar to the proof of Theorem 3.1, there exists an ascending chain, indexed by an appropriate initial segment of the ordinals,

$$0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_\alpha \subseteq \cdots$$

such that: $S_\alpha \in \mathcal{C}$ for each α, $S_{\alpha+1}/S_\alpha$ is countable, and the chain is a complete chain leading up to $G[p]$. We have not yet used the fact that the S_α's are closed in G, but we shall use it presently to obtain a "natural" splitting of S_α out of $S_{\alpha+1}$. Since the S_α's are vector spaces over Z/pZ, S_α is always a direct summand of $S_{\alpha+1}$ but such a decomposition may ignore the structure of G. A decomposition $S_{\alpha+1} = S_\alpha + K_\alpha$ is said to be natural if for each x in $S_{\alpha+1}$ written as $y + z$ where $y \in S_\alpha$ and $z \in K_\alpha$ has height in G equal to the minimum of the height of y and z. Now, since S_α is closed in G we know that G/S_α has no elements of infinite height. Note that $S_{\alpha+1}/S_\alpha$ is a countable subgroup of $(G/S_\alpha)[p]$. Thus we can write (see [6])

$$S_{\alpha+1}/S_\alpha = T_0 + T_1 + \cdots + T_n + \cdots, \qquad n < \omega,$$

where each nonzero element of T_n has height exactly n in G/S_α. If $T_n = \sum\{x_i + S_\alpha\}$, we have, for each i, an element $y_i \in G$ such that $p^n(y_i + S_\alpha) = x_i + S_\alpha$. Thus we may choose the representative x_i itself to have height n in G. With this choice, all the elements of $R_n = \sum\{x_i\}$ have height n in G and we have a natural decomposition.

$$S_{\alpha+1} = S_\alpha + (R_0 + R_1 + \cdots + R_n + \cdots) = S_\alpha + R_{(\alpha)}.$$

Since the chain $\{S_\alpha\}$ is complete and since it leads up to $G[p]$, we have from the natural decompositions $S_{\alpha+1} = S_\alpha + R_{(\alpha)}$ a natural decomposition of all of $G[p] = \sum R_{(\alpha)}$ into countable summands. From a result of Hill and Megibben [6, Lemma 1.11], it follows that G is (summable and therefore) a direct sum of cyclic groups.

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