ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 1, Number 1, Winter 1971

THE DECAY-SCATTERING SYSTEM [1]

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ABSTRACT. A decay-scattering system is defined in order to unify the mathematical treatment of scattering resonances and unstable particles. The decay law of an unstable particle, the inverse decay problem, and the effect of the presence of unstable particles on the scattering process are discussed from this point of view. A characterization of phenomenological descriptions of decay and resonance is given in terms of the analytic structure of the reduced resolvent. Next, the concept of symmetry and symmetry breaking is introduced, and simplified, but rigorous, formulations of K-meson and Σ-particle decays are proposed as physical applications. In conclusion we make some global remarks about the physical interpretation of the decay-scattering systems.

I. The concept of a decay-scattering system.

I. 1. What is the relation between a resonance and an unstable particle? In what sense are they alternative aspects of one and the same physical phenomenon?

A priori the two concepts are different: A resonance refers to the energy distribution of the outgoing particles in a scattering process and can be characterized by such parameters as the central energy and the width, whereas an unstable particle is described in a time-dependent picture by its mass and lifetime. Operationally the difference is as follows: In order to measure a resonance we make a scattering experiment; two particles collide, stick together for a while and separate, whereas in the other case we create an unstable particle at time $t = 0$ and look into its decay.

And yet the difference is quantitative rather than qualitative. It is true that, for example, the existence of a ρ-meson may be inferred from a resonance in the scattering cross-section of π-mesons and its decay cannot be directly measured by a track in a bubble chamber due to the shortness of its lifetime. But still the ρ-meson has a mass, definite quantum numbers, and other characteristics of a particle. Conversely, although the K-mesons can actually be produced in a beam and their decay observed, they can also be considered, in principle, as resonances

Received by the editors March 12, 1970.

AMS 1970 *subject classifications.* Primary 81A48, 81A30, 47A20.

[1]Work supported in part by a summer Research Grant from the University of Denver Research Institute.

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of a π-meson system. Hence resonance and decay are two aspects of one and the same physical object.

It is our purpose to clarify the relation between these two aspects with the help of a suitable mathematical model which describes resonance and decay in a unified manner.

I. 2. Consider a selfadjoint operator H_0 on a Hilbert space $\mathcal{H}$ whose spectrum consists of a continuum from 0 to ∞ and a finite set of discrete eigenvalues $m_i > 0$ embedded in it. Let P be the projection of $\mathcal{H}$ on the discrete part of H_0 and $\overline{P} = 1 - P$ the projection on the continuous part. Now consider an interaction V which does not commute with P, i.e. such that $\overline{P}VP \neq 0$. The system $\{H_0, H = H_0 + V\}$ then describes the phenomenon of decay in the following sense: The states $\varphi \in P\mathcal{H}$ are stable under the partial evolution $U_0(t) = e^{-iH_0t}$ since they are discrete eigenstates of H_0, but they decay under V since the total evolution $U(t) = e^{-iHt}$ no longer conserves the eigenspace $P\mathcal{H}$. We call $P\mathcal{H}$, therefore, the space of unstable particles. Our viewpoint is that the unstable particles would be stable with well-defined masses m_i under the unperturbed evolution $U_0(t)$, and that they decay solely through the perturbation V. Such a decomposition of H into H_0 and V is particularly natural whenever the forces under which the unstable particles are produced differ qualitatively from the forces under which they decay. This is, for instance, the case for the K-mesons. The theory is, however, not restricted in its application to structures of this type.

So far the system $\{H_0, H\}$ is thus suitable for the description of the decay of unstable particles. In order to make clear its alternative aspect as a resonance scattering system we have to postulate that in a certain way a scattering matrix exists. In the ordinary description of scattering it is assumed that H_0 has no bound states and that the scattering matrix S is unitary in $\mathcal{H}$. Here we make the modified postulate that $\{\overline{P}H_0\overline{P}, H\}$ is a scattering system and that S is unitary only on the space $\overline{P}\mathcal{H}$ of the decay products. Physically this means that the decay products span the whole space asymptotically.

I. 3. We summarize this discussion in the following:

Definition. Let H_0 be a selfadjoint operator in a separable Hilbert space $\mathcal{H}$ with continuous spectrum $\Lambda_0 = [0, \infty]$ and a discrete spectrum $\{m_i > 0\}$ embedded in Λ_0. Let $\overline{P}$, P (with $P+\overline{P} = I$) be the projections on the corresponding parts of $\mathcal{H}$, and let V be a selfadjoint operator such that $\overline{P}VP \neq 0$. Let finally $\{\overline{P}H_0\overline{P}, H\}$ be a scattering system [1]. Then the pair $\{H_0, H\}$ is called a *decay-scattering system*, P the *space of unstable particles (resonances)* and $\overline{P}\mathcal{H}$ *the space of decay products (scattering states).*

According to this definition, the continuous spectrum Λ of H is automatically identical with the continuous spectrum Λ_0 of H_0. Bound states (and singular parts within the continuum) are, however, not excluded a priori. Their existence is considered in Chapter V below.

I. 4. Our main goal is to establish the precise relation between resonance and decay. We formulate this question by stating three mathematically and physically distinct problems:

(1) *The decay problem.* How does an unstable particle, created at time $t = 0$, decay?

(2) *The inverse decay problem.* Under what conditions is it possible to reconstruct the scattering process from the exact knowledge of the decay law of the unstable particles?

(3) *The resonance problem.* How does the presence of unstable particles affect the scattering process? Conversely, under what conditions does a resonance in the scattering cross-section correspond to an unstable particle?

I. 5. After a short exposition of the elementary properties of decay-scattering systems and the definition of the relevant physical quantities (Chapter II) we shall investigate the above problems from a global viewpoint (Chapter III). Next we introduce an algebraically soluble model of a decay-scattering system (Chapter IV) and treat in some detail the simplest case (Chapter V).

In Chapter VI we investigate more thoroughly the analytical structure of the reduced resolvent which leads to an exact characterization of approximate (phenomenological) descriptions of decay and resonance within our theory (Chapter VII).

Next we introduce the concept of symmetry and symmetry breaking (Chapter VIII) and propose, as physical applications, a simplified but rigorous theory of K-meson and Σ-particle decay (Chapter IX).

The concluding part (Chapter X) contains some general remarks about the physical significance of the decay-scattering systems.

II. The relevant physical quantities.

II. 1. For the decay problem the basic quantity is the *reduced motion*

$$U'(t) = PU(t)P \tag{1}$$

which governs the time-evolution of the subspace $P\mathcal{H}$ of the unstable particles. From this we can derive the so-called *decay law of the unstable particles* (φ_i orthonormal basis in $P\mathcal{H}$)

$$p(t) = \sum_i |(\varphi_i, U(t)\varphi)|^2 , \qquad \varphi \in P\mathcal{H}, \tag{2}$$

which expresses the probability that an unstable particle φ, created at time $t = 0$, is in the subspace $P\mathcal{H}$ of unstable particles at time t.

(2) can also be written

$$p(t) = \mathrm{Tr}_P(U'^{\dagger}(t)U'(t)P_\varphi) \tag{2'}$$

where P_φ is the projection on the subspace spanned by φ.

II. 2. The total time-evolution $U(t) = e^{-iHt}$ and the resolvent $R(z) = (z - H)^{-1}$ are related to each other by the (inverse) Laplace transform

$$U(t) = \frac{1}{2\pi i}\oint R(z)\, e^{-izt}\, dz, \tag{3}$$

where the integration path is around the spectrum Λ of H [11]. If we project this into $P\mathcal{H}$, we can express the reduced motion $U'(t)$ by the *reduced resolvent* $R'(z) = PR(z)P$:

$$U'(t) = \frac{1}{2\pi i}\oint R'(z)\, e^{-izt}\, dz. \tag{4}$$

II. 3. The scattering system $\{\bar{P}H_0\bar{P}, H\}$ has the following elementary properites [1]:

(1) The *wave operators* $\Omega_\pm$ exist and are partial isometries between $\bar{P}\mathcal{H}$ and $\bar{Q}\mathcal{H}$ (with $\bar{Q}$ the projection on the continuous part of $\mathcal{H}$ with respect to H) linking the spectra Λ_0 and Λ.

$$\Omega_\pm\Omega_\pm^\dagger = \bar{Q}, \qquad \Omega_\pm^\dagger\Omega_\pm = \bar{P}, \qquad \bar{Q}H\bar{Q} = \Omega_\pm\bar{P}H_0\bar{P}\Omega_\pm^\dagger.$$

(2) The *scattering operator* S exists on $\bar{P}\mathcal{H}$, commutes with $\bar{P}H_0\bar{P}$ and is unitary in $\bar{P}\mathcal{H}$:

$$S = \Omega_+^\dagger\Omega_-, \qquad [\bar{P}H_0\bar{P}, S] = 0, \qquad SS^\dagger = S^\dagger S = \bar{P}.$$

(3) The *scattering amplitude* $T = S - I$ also commutes with $\bar{P}H_0\bar{P}$ and satisfies the relation

$$T^\dagger T = TT^\dagger = -T - T^\dagger.$$

II. 4. In order to define the experimental quantities of scattering, we introduce the spectral representation of $\bar{P}\mathcal{H}$ with respect to $\bar{P}H_0\bar{P}$. If the spectrum Λ_0 of $\bar{P}H_0\bar{P}$ is absolutely continuous, we have the direct integral representation $\mathcal{H} \to \{\mathcal{H}(\lambda)\}$:

$$\psi \in \bar{P}\mathcal{H} \to \{\langle\lambda|\psi\rangle\}; \qquad \bar{P}H_0\bar{P}\psi \to \{\lambda\langle\lambda|\psi\rangle\};$$
$$\|\psi\|^2 = \int_{\Lambda_0} \|\langle\lambda|\psi\rangle\|^2 d\lambda,$$

where $\|\langle\lambda|\psi\rangle\|^2$ is the norm in $\mathcal{H}(\lambda)$ [2]. Since T commutes with $\bar{P}H_0\bar{P}$ it is reduced by $\{\mathcal{H}(\lambda)\}$ and we have

$$T\psi \to \{T(\lambda)\langle\lambda|\psi\rangle\}$$

where the $T(\lambda)$ are operators in $\mathcal{H}(\lambda)$. If the interaction V satisfies suitable conditions, $T(\lambda)$ is Hilbert-Schmidt and can be expressed in terms of V and $R(z)$ as follows [3]

$$T(\lambda) = -2\pi i \lim_{\epsilon \downarrow 0} \langle \lambda | V + VR(\lambda + i\epsilon) V | \lambda \rangle. \tag{5}$$

II. 5. The scattering operator $S = T + I$, being also reduced by $\{\mathcal{H}(\lambda)\}$ and hence unitary in each $\mathcal{H}(\lambda)$, can be written

$$S(\lambda) = e^{2i\delta(\lambda)} = T(\lambda) + 1_\lambda \tag{6}$$

where the selfadjoint operators $\delta(\lambda)$ in $\mathcal{H}(\lambda)$ are the *phase-shift operators.* $\delta(\lambda)$ is determined modulo $2\pi \cdot 1_\lambda$ by continuity.

Furthermore we introduce the *total scattering cross-section* $\sigma(\lambda)$ (neglecting kinematical factors) by

$$\sigma(\lambda) = \frac{1}{4} \mathrm{Tr}_\lambda (T^\dagger(\lambda) T(\lambda)), \tag{7}$$

the trace being taken over $\mathcal{H}(\lambda)$. Using the unitarity condition, we can also write

$$\sigma(\lambda) = -\frac{1}{4} \mathrm{Tr}_\lambda (T(\lambda) + T^\dagger(\lambda)) = -\frac{1}{2} \mathrm{Re}\, \mathrm{Tr}_\lambda\, T(\lambda). \tag{7'}$$

This is the so-called *optical theorem.*

From (6) and (7) we see that the scattering amplitude and cross-section can be expressed in terms of the phase-shift by

$$T(\lambda) = 2i\, e^{i\delta(\lambda)} \sin \delta(\lambda); \quad \sigma(\lambda) = \mathrm{Tr}_\lambda \sin^2 \delta(\lambda). \tag{8}$$

III. **The main problems.**

A. *The decay problem.*

III. 1. Theorem. *The reduced motion $U'(t)$ satisfies the integro-differential equation*

$$i \frac{d}{dt} U'(t) = PHPU'(t) - i \int_0^t d\tau\, PH\bar{P}\, e^{-i\bar{P}H\bar{P}\tau} \bar{P}HPU'(t-\tau). \tag{9}$$

Proof. Projecting the identity $zR(z) - 1 = HR(z)$ into the subspaces $P\mathcal{H}$, $\bar{P}\mathcal{H}$, we obtain the coupled system

$$zR'(z) - P = PHR'(z) + PH\bar{P}R(z)P,$$

$$z\bar{P}R(z)P = \bar{P}HR'(z) + \bar{P}H\bar{P}R(z)P,$$

and eliminating $\bar{P}R(z)P$:

$$zR'(z) - P = PHR'(z) + PH\bar{P}(z - \bar{P}H\bar{P})^{-1}\bar{P}HR'(z).$$

The result (9) then follows from the inverse Laplace transform (4) by using the convolution theorem.

III. 2. Equation (9) is called the *Master Equation for the reduced motion* and applies not only to decay-scattering systems, but in general for evolutions obtained from a unitary group $\{U(t)\}$ by projection into a subspace [4]. The simplest nontrivial example is the following:

$$U(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{P} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad U'(t) = \cos t,$$

$U(t)$ starts out, at $t = 0$, with vanishing gradient and, owing to the finiteness of the complementary space $\bar{P}\mathcal{H}$, its motion is strictly periodic. The evolution of a subsystem $P\mathcal{H}$ becomes truly aperiodic only in the case where the generator H of $U(t)$ has a continuous spectrum. It can be verified easily that $U'(t)$ defined in this example satisfies the Master Equation (9).

III. 3. Although the continuity of the spectrum of H forces the decay of the finite subspace $P\mathcal{H}$ to be approximately exponential (as is shown below in Chapter VII), some essential characteristics of the trivial example in III. 2 remain generally valid. For instance:

THEOREM. *The reduced motion of $U'(t)$ of the space $P\mathcal{H}$ is not a group.*

PROOF. Suppose it were. Then $PU(t_1)U(t_2)P = PU(t_1)P \cdot PU(t_2)P$ and hence $PU(t_1)\bar{P}U(t_2)P = 0$. Now let $t_2 = -t_1 = t$, and φ an arbitrary element in $P\mathcal{H}$. Noting that $U(-t) = U^{\dagger}(t)$, we have $\|\bar{P}U(t)P\varphi\|^2 = (\varphi, PU(-t)\bar{P}U(t)P\varphi) = 0$ for all $\varphi \in P\mathcal{H}$ and therefore $\bar{P}U(t)P = 0 = PU(t)\bar{P}$, for all t. Hence $U(t)$ is reduced by P, $\bar{P}$ which contradicts the definition of a decay-scattering system.

THEOREM. *The decay rate $(d/dt)p(t)$ vanishes at $t = 0$.*

PROOF. From (2′) and the unitarity of $U(t)$ follows

$$\frac{d}{dt} \operatorname{Tr}_P(PU^{\dagger}(t)PU(t)P_{\varphi})\Big|_{t=0}$$

$$= \operatorname{Tr}_P(iPHU^{\dagger}(t)PU(t)P_{\varphi} - iPU^{\dagger}(t)PHU(t)P_{\varphi})\Big|_{t=0}$$

$$= i\operatorname{Tr}_P(PHP_{\varphi} - PHP_{\varphi}) = 0.$$

III. 4. A general study of reduced motions could start from the Master Equation (9), for instance, by setting up the iterative series

$$i\frac{d}{dt}U'^{(\nu)}(t) = PHPU'^{(\nu-1)}(t) - i\int_0^t d\tau\, PH\bar{P}\, e^{-i\bar{P}H\bar{P}\tau}\, \bar{P}HPU'^{(\nu-1)}(t-\tau)$$

with $U'^{(0)}(t)$ satisfying $i(d/dt)\,U'^{(0)}(t) = PHPU'^{(0)}(t)$. We do not pursue this approach here, but turn instead to a study of the analytic structure of the reduced resolvent $R'(z)$. $U'(t)$ can then be discussed by the help of equation (4).

B. *The inverse decay problem.*

III. 5. A suitable starting point for an investigation of this problem is given by the theory of extensions of Hilbert spaces.

DEFINITION. A one-parameter family of bounded operators $U'(t)$ [t real] in a Hilbert space $\mathcal{H}'$ is of *positive type*, if for every family $\{\varphi_t\}$ of elements in $\mathcal{H}'$ such that $\varphi_t = 0$ for almost all t, we have

$$\sum_t \sum_{t'} (\varphi_t, U'(t-t')\varphi_{t'}) \geqq 0.$$

THEOREM (BY B. SZ.-NAGY [5]). *If $\{U'(t)\}$ is a one-parameter, weakly continuous, positive type family of operators in $\mathcal{H}'$, then there exists a unique minimal extension $\mathcal{H} \supset \mathcal{H}'$ and a continuous group of unitary operators $\{U(t)\}$ in $\mathcal{H}$ such that $\mathcal{H}' = P\mathcal{H}$ and $U'(t) = PU(t)P$ for all t. The minimality of the extension is defined in the sense that the set $\{U(t)\varphi\}[\forall t, \forall \varphi \in \mathcal{H}']$ spans the space $\mathcal{H}$.*

III. 6. We now assume that we are given a one-parameter, weakly continuous, positive type family of operators $\{U'(t)\}$ describing the evolution of unstable particles in a finite dimensional Hilbert space $\mathcal{H}'$, and we further assume the validity of the following:

Postulate. The selfadjoint generator H of the unitary continuous group $\{U(t)\}$ obtained from $\{U'(t)\}$ by extension has a positive continuous spectrum $\Lambda = [0, \infty]$.

Then we define a decay-scattering system $\{H_0, H\}$ in the extended space by the decomposition

(10) $H = H_0 + V;\qquad H_0 = \overline{P}H\overline{P} + PHP;\qquad V = \overline{P}HP + PH\overline{P}.$

It is obvious that in this decay-scattering system the given space $\mathcal{H}' = P\mathcal{H}$ of the unstable particles appears indeed as the discrete part of $\mathcal{H}$ with respect to H_0 and the masses of the unstable particles are the discrete positive eigenvalues $\{m_i\}$ of the operator PHP in $P\mathcal{H}$.

We note, however, that the decomposition (10), and hence the masses m_i, are not uniquely determined by the sole requirement that the given quantities $\mathcal{H}'$, $U'(t)$ be obtained from a decay-scattering system by projection into the discrete part of H_0. In fact, the term PHP could, for example, be included in V rather than H_0, without changing either $P\mathcal{H}$ and $U'(t)$ or the scattering amplitude. But the discrete space $\mathcal{H}' = P\mathcal{H}$ then becomes a zero-energy eigenspace of H_0 and the unstable particles have mass zero.

Among all possible decompositions of H, (10) can be characterized by the additional requirement that the perturbation V be minimal in the sense that V contains only those forces which are responsible for the decay of the unstable particles, i.e., the forces which relate $P\mathcal{H}$ to $\bar{P}\mathcal{H}$ (cf. IV.1). The only effect of the term PHP in (10) is to shift around the masses of the unstable particles.

The above postulate about the positivity of the spectrum of H restricts implicitly the class of decay laws $\{U'(t)\}$ compatible with a decay-scattering system. This leads to the (unsolved)

Problem. Characterize among all continuous positive type operator families $\{U'(t)\}$ in $\mathcal{H}'$ those which yield in the extended space $\mathcal{H}$ a unitary group $\{U(t)\}$ whose generator has a continuous positive spectrum.

C. *The resonance problem.*

III. 7. In entire generality this problem is difficult. We shall take it up in special cases later (V. 3, VII. 5, VIII. 3); here we restrict ourselves to some heuristic remarks.

Let the continuous spectrum Λ_0 of H_0 be simple and let n and n' be the dimensions of the discrete parts of $\mathcal{H}$ with respect to H_0 and H. A generalization of *Levinson's theorem* then yields the relation

$$\delta(\infty) - \delta(0) = \pi(n - n')$$

for the phase-shift $\delta(\lambda)$ which in this case is a numerical, real-valued and continuous function on Λ_0. There exist therefore at least $|n - n'|$ points $\lambda_{p_i} \in \Lambda_0$ for which $\delta(\lambda_{p_i})$ is an odd multiple of $\pi/2$, and according to (8) $\sigma(\lambda_{p_i}) = 1$. In other words, the scattering cross-section $\sigma(\lambda)$ has at least $|n - n'|$ maxima with value 1. If we call the maxima of $\sigma(\lambda)$ with value 1 true resonances, in distinction to accidental relative maxima ("bumps") in the background scattering, we then have the

THEOREM. *In a decay-scattering system with n unstable particles* (dim $P = n$), *simple energy spectrum and no bound states in H, there are at least n true resonances in* $\sigma(\lambda)$.

III. 8. The converse problem is more complicated. There can, in principle, exist more than n true resonances in $\sigma(\lambda)$, if $\delta(\lambda)$ is not monotonically increasing. It can, however, be shown [7] that causality requires a lower bond for the derivative of $\delta(\lambda)$; the occurrence of more than n true resonances can then be excluded for a class of sufficiently weak interactions V.

III. 9. In cases involving higher multiplicities of Λ_0 and symmetry breaking, no analogue to Levinson's theorem is known and the precise link between unstable particles and resonances is not easy to establish. (See VII. 3 for an approximate treatment of this question.) In the presence of symmetries (Chapter VIII) the system reduces, however, into 1-dimensional systems, in which the above characterization of the resonances can again be applied.

IV. An algebraically soluble model.

IV. 1. Let $\{H_0, H\}$ be a decay-scattering system in which the following additional conditions are valid:

$$\text{(11)} \qquad \dim P = n < \infty; \qquad \bar{P}V\bar{P} = 0.$$

We refer to such a decay-scattering system as the n-dimensional Friedrichs' model [8]. We note that the rank of $V = PVP + PV\bar{P} + \bar{P}VP$ is finite ($2n$) in this model. Hence, according to a theorem of Kato [9], $\{\bar{P}H_0\bar{P}, H\}$ forms indeed a scattering system and the last condition of definition I.3 is a consequence of (11).

The Friedrichs' model is algebraically soluble because of the following two facts:

(1) The reduced resolvent $R'(z)$ can be explicitly expressed in terms of the free resolvent $R_0(z)$ and the interaction V.

(2) The scattering amplitude depends only on $R'(z)$ and not on the other parts of the total resolvent $R(z)$.

IV. 2. Theorem. *The reduced resolvent $R'(z)$ is expressed by V and $R_0(z)$ as follows*:

$$\text{(12)} \quad R'(z) = h^{-1}(z); \qquad h(z) = \sum_{i=1}^{n}(z - m_i)P_i - PVP - PV\bar{P}\,R_0(z)\bar{P}VP.$$

Here $h(z)$ is an operator function in $P\mathcal{H}$, and $\{P_i\}$ is a set of n orthogonal, 1-dimensional projectors corresponding to the discrete eigenvalues m_i of PH_0P (which may coincide).

Proof. Projection of the second resolvent equation $R(z) = R_0(z) + R_0(z)VR(z)$ into the subspaces $P\mathcal{H}$, $\bar{P}\mathcal{H}$ yields the coupled system

$$PR(z)P = PR_0(z)P + PR_0(z)PV\bar{P}\,R(z)P + PR_0(z)PVP\,R(z)P,$$
$$\bar{P}\,R(z)P = \bar{P}\,R_0(z)\bar{P}VPR(z)P.$$

Eliminating $\bar{P}R(z)P$ and multiplying from the left by $P(z - H_0)P = \Sigma_i(z - m_i)P_i$, we obtain

$$\text{(13)} \qquad \Big[\sum_i (z - m_i)P_i - PVP - PV\bar{P}\,R_0(z)\bar{P}VP\Big]R'(z) = P,$$

from which (12) follows by taking the inverse within the subspace P.

IV. 3. According to (5) and (11) the scattering amplitude in the spectral representation with respect to $\bar{P}H_0\bar{P}$ becomes

$$\text{(14)} \qquad T(\lambda) = -2\pi i\langle\lambda|VR'(\lambda + i0)V|\lambda\rangle$$

and hence depends only on the reduced resolvent $R'(z)$.

Let $\{\varphi_i\}$ be the orthonormal basis in $P\mathcal{H}$ corresponding to the projectors P_i. In this basis the function $h(z)$ defined in (12) takes on the matrix form

$$h_{ij}(z) = (z - m_i)\delta_{ij} - V_{ij} - \int_{\Lambda_0} \frac{\mathrm{Tr}_\lambda X_{ij}(\lambda)}{z - \lambda}\, d\lambda, \tag{15}$$

where the interaction function $X_{ij}(\lambda)$ is defined by

$$X_{ij}(\lambda) = \overline{\langle\lambda|V\varphi_i\rangle}\langle\lambda|V\varphi_j\rangle. \tag{16}$$

We also note the relations

$$h(\lambda - i0) = h^\dagger(\lambda + i0), \qquad h(\lambda + i0) - h(\lambda - i0) = 2\pi i\, \mathrm{Tr}_\lambda\, X(\lambda). \tag{17}$$

With the help of these functions the scattering amplitude (14) reads

$$T(\lambda) = -2\pi i\, \mathrm{Tr}_P[X(\lambda)h^{-1}(\lambda + i0)] \tag{18}$$

and the cross-section, using (7), (17), and (18):

$$\sigma(\lambda) = \frac{i\pi}{2} \mathrm{Tr}_\lambda\, \mathrm{Tr}_P[X(\lambda)(h^{-1}(\lambda + i0) - h^{-1t}(\lambda + i0))]\,. \tag{19}$$

Finally the reduced motion (4) becomes

$$U'(t) = \frac{1}{2\pi i} \oint h^{-1}(z)e^{-izt}dz. \tag{20}$$

In equations (18) and (20), scattering and decay are entirely expressed in terms of the interaction functions $X(\lambda)$ and $h(\lambda)$ which in turn are explicit functions of the interaction V alone, according to (15) and (16).

V. **Analysis in the simplest case.**

V. 1. In this chapter we analyze the exact solutions of the Friedrichs' model in the simplest case where

(1) H_0 has a single bound state φ with eigenvalue m_0 (i.e. $\dim P = 1$),

(2) The absolutely continuous spectrum of H_0 is simple,

(21) (3) $V = PV\bar{P} + \bar{P}VP$ (no self interaction PVP!).

The purpose of this review of Friedrichs' original article [8] is to illustrate some of the abstract results obtained in Chapter III and to provide an intuitive basis for the next chapter where we shall treat in more generality the analytic structure of the reduced resolvent in a decay-scattering system.

From IV. 3 we obtain the following expressions for the reduced resolvent, the scattering cross-section and the reduced motion:

$$(22)\ R'(z) = h^{-1}(z) = \frac{1}{z - m_0 - \int_{\Lambda_0} \frac{X(\lambda)}{z-\lambda} d\lambda}; \quad X(\lambda) = |\langle\lambda|V\varphi)|^2,$$

$$(23) \qquad \sigma(\lambda) = \frac{(\pi X(\lambda))^2}{|h(\lambda + i0)|^2} = \frac{(\pi X(\lambda))^2}{\left(\lambda - m_0 - \mathrm{P}\!\!\int \frac{X(\lambda')}{\lambda-\lambda'} d\lambda'\right)^2 + (\pi X(\lambda))^2},$$

$$(20) \qquad U'(t) = \frac{1}{2\pi i} \oint R'(z)\, e^{-izt}\, dz.$$

V. 2. First we investigate the existence of a *bound state in H*.

THEOREM. *If* $X(\lambda) > 0$ *for all* $\lambda > 0$, *H has a bound state if and only if* $h(0) \geqq 0$; *its eigenvalue* $\tilde{\lambda}$ *is then necessarily negative (it cannot be embedded in* Λ_0).

PROOF. The spectral representation of $\mathcal{H}$ with respect to H_0 consists of a number (the component $\psi_0 = (\varphi,\psi)$ of ψ in the subspace $P\mathcal{H}$) and a Lebesgue square-integrable function $\langle\lambda|\psi)$:

$$\psi \to \{\psi_0, \langle\lambda|\psi)\}, \qquad H_0\psi \to \{m_0\psi_0, \lambda\langle\lambda|\psi)\}.$$

The interaction (21) is represented by

$$V\psi \to \left\{ \int_{\Lambda_0} \overline{\langle\lambda|V\varphi)}\langle\lambda|\psi)d\lambda\, ,\ \psi_0\langle\lambda|V\varphi) \right\}.$$

Therefore the eigenfunction equation $H\psi = \tilde{\lambda}\psi$ leads to the equations

$$(\tilde{\lambda} - m_0)\psi_0 = \int_{\Lambda_0} \overline{\langle\lambda|V\varphi)}\langle\lambda|\varphi)d\lambda; \qquad (\tilde{\lambda} - \lambda)\langle\lambda|\psi) = \psi_0\langle\lambda|V\varphi)$$

and, substituting the second into the first, to

$$(24) \qquad \tilde{\lambda} - m_0 = \int \frac{X(\lambda)}{\tilde{\lambda} - \lambda} d\lambda.$$

Since $X(\lambda) > 0$ for $\lambda > 0$, the right-hand side diverges unless $\tilde{\lambda} \leqq 0$. (24) then implies the inequality

$$- m_0 \geqq \int_{\Lambda_0} \frac{X(\lambda)}{\tilde{\lambda} - \lambda}$$

or $h(0) \geqq 0$.

V. 3. Next we test the conclusions from *Levinson's theorem* stated in III. 7. From (23) we see that $\sigma(\lambda)$ has a true resonance if and only if the function $h(\lambda) = \lambda - m_0 - ⨍ (X(\lambda')d\lambda'/(\lambda - \lambda'))$ vanishes. Since $h(\lambda)$ is continuous and $h(\lambda) \to \infty$ for $\lambda \to \infty$, there must exist a zero if $h(0) < 0$, in other words, according to the above theorem, if H has no bound state. This leads to the

COROLLARY. *$\sigma(\lambda)$ has a true resonance if H has no bound state (in accordance with the theorem in* III. 7).

V. 4. We now consider the *analytic structure of the reduced resolvent.*

THEOREM. *If H has no bound state, the reduced resolvent $R'(z)$ is regular analytic in the entire complex plane cut by the spectrum Λ_0 of H_0. If H has a bound state, then $R'(z)$ has a pole at its eigenvalue $\tilde{\lambda}$ on the negative real axis.*

Note that, according to [15] and [17], $X(\lambda)$ can be interpreted as the difference between two Hilbert transforms and is therefore defined pointwise (for the present purpose).

PROOF. From (22) it follows that $R'(z)$ is regular except for poles ($X(\lambda)$ is integrable!). But poles cannot occur off the real axis since for $\operatorname{Im} z \neq 0$

$$\operatorname{Im} h(z) = \operatorname{Im} z\left(1 + \int \frac{X(\lambda)}{|z-\lambda|^2} d\lambda\right) \neq 0.$$

Suppose there were a pole at $\tilde{\lambda} \leqq 0$. Then $h(\tilde{\lambda}) = 0$ implies

$$m_0 = \tilde{\lambda} - \int \frac{X(\lambda')}{\tilde{\lambda} - \lambda'} d\lambda' \leqq \int \frac{X(\lambda')}{\lambda'} d\lambda'$$

or $h(0) \geqq 0$, and hence H has a bound state (V. 2). Conversely, if $\tilde{\lambda}$ is a bound state of H, it satisfies (24), hence $h(\tilde{\lambda}) = 0$ and $R'(z)$ has a pole.

V. 5. For the *analytic continuation of $R'(z)$* we have the following

THEOREM. *$R'(z)$ can be continued from above into its second sheet if and only if the interaction function $X(\lambda)$ is analytic and can be continued into the lower half-plane. The continuation is*

$$R'^{\text{II}}(z) = \frac{1}{z - m_0 - \int \frac{X(\lambda)}{z-\lambda} d\lambda + 2\pi i X(z)}. \tag{25}$$

PROOF. If $X(z)$ is analytic, $R'^{\text{II}}(z)$ is also. Hence it is sufficient to show that $R'^{\text{II}}(\lambda - i0) = R'^{\text{I}}(\lambda + i0)$. But this follows immediately from the relation

$$\int \frac{X(\lambda')}{\lambda \pm i0 - \lambda'} d\lambda' = ⨍ \frac{X(\lambda')}{\lambda - \lambda'} d\lambda' \mp i\pi X(\lambda).$$

It can be seen from (25) that for sufficiently small interactions, i.e., $\|V\varphi\|^2 = \int_{\Lambda_0} X(\lambda)d\lambda << 1$, the reduced resolvent $R'(z)$ has exactly one simple pole z_p in the second sheet near the real axis. We shall use this as a postulate in the more general context of Chapter VI, although we should note that the uniqueness of the pole cannot be proved for completely arbitrary interactions V. (This question leads to the bifurcation problem which is in general difficult. Pole trajectories for increasing "coupling" have been discussed for this model in [**10**].)

V. 6. Let us finally briefly consider the *decay problem.* We have now the possibility to deform the integration path in (20) into the second sheet from above through Λ_0. In this process we pick up the contribution from the pole z_p in the second sheet plus some contributions from the branch points. As we shall see in more generality in Chapter VII, the pole contribution leads to an exponential decay while the other terms are significant only for small and very large times. The latter have been partly estimated in [**10**]. It turns out that the asymptotic behavior of $U'(t)$ for $t \to \infty$ is polynomial rather than exponential (see VII).

VI. Analytic structure of the reduced resolvent.

VI. 1. From the discussion of the explicit model in Chapter V it is apparent that an entirely general theory of the reduced resolvent in terms of the interaction V alone would be difficult to establish. There does not seem to exist a simple criterion for V which, for example, ensures that there is exactly one pole in the analytic continuation of $R'(z)$.

The theory presented here will be incomplete insofar as we shall impose some conditions on $R'(z)$ with the purpose of avoiding the specific difficulties connected with the bifurcation problem and other similar complications. These conditions thus amount to implicit restrictions on the class of potentials.

The first condition is concerned with the analytic continuation of $R'(z)$ into the second sheet, and the second with the existence and properties of poles in this continuation. Before stating them formally, we make some heuristic remarks.

VI. 2. It is well known [**11**] that Λ is a natural boundary of analyticity for the total resolvent $R(z)$. We now indicate how, on the other hand, the reduced resolvent $PR(z)P$ may be continuable, in particular if dim P is finite.

Continuability through Λ would imply that there exists an open interval of Λ in which the discontinuity

$$D(\lambda) = R(\lambda + i0) - R(\lambda - i0)$$

is analytic in the weak topology. We therefore calculate

$$(f, D(\lambda)g) = \lim_{\epsilon \downarrow 0} \int_\Lambda \left(\frac{1}{\lambda + i\epsilon - \lambda'} - \frac{1}{\lambda - i\epsilon - \lambda'} \right) d(f, E(\lambda')g)$$

$$= \lim_{\epsilon \downarrow 0} \int_\Lambda \frac{-2i\epsilon}{(\lambda - \lambda')^2 + \epsilon^2} \frac{d(f, E(\lambda')g)}{d\lambda'} d\lambda' = -2\pi i \frac{d}{d\lambda} (f, E(\lambda)g)$$

where $(d/d\lambda)(f, E(\lambda)g)$ is the Radon-Nikodym derivative of the (absolutely continuous) spectral family $E(\lambda)$ of H. Suppose now that we restrict ourselves to the reduced resolvent $R'(z)$ in $P\mathcal{H}$ only. If dim P is finite, there exists a finite basis φ_i in $P\mathcal{H}$, and if we postulate that the functions $(d/d\lambda)(\varphi_i, E(\lambda)\varphi_j)$ are bounded and admit an analytic continuation to complex λ's in an interval of Λ, then all finite linear combinations f, g of φ_i will have the same property and $R'(z)$ is continuable. The same argument clearly fails if $P\mathcal{H}$ is infinite dimensional.

VI. 3. As to the existence of poles in the second sheet and the rank of $R'(z)$, the following heuristic comment may provide some insight:

In the limit of vanishing perturbation we have

$$R'(z) \to PR_0(z)P = \sum_i \frac{P_i}{z - m_i} \tag{26}$$

where P_i are the orthogonal projections on the eigenspaces of m_i. Hence, in this limit, the reduced resolvent has exactly n poles at m_i (which may coincide) with orthogonal residua P_i, and the rank of $R'(z)$ is n if $z \neq m_i$. Under the perturbation these poles disappear if H has no eigenstates. But in the first sheet, $PR(z)P$ is regular everywhere except on Λ. Hence the poles are likely to wander into the second sheet and, in general, to separate, unless the couplings of the eigenspaces P_i with the continuum $\bar{P}$ are identical. For sufficiently weak perturbation we therefore expect exactly n (= dim P) distinct poles z_{p_i} in the second sheet. Furthermore, we expect the residua to remain linearly independent, and the rank of $R'(z)$ to remain n for $z \neq z_{p_i}$.

VI. 4. Based on these remarks we now formulate the assumptions underlying our discussion of the reduced resolvent as follows:

Assumptions. (1) If dim $P = n$ is finite, the reduced resolvent $R'(z)$ can be continued from above through the spectrum Λ of H and is regular analytic in the second sheet except for n distinct simple poles z_{p_i} situated in the lower half-plane near Λ.

(2) The rank of the analytic continuation of $R'(z)$ is n in the regularity domain.

VI. 5. An immediate consequence of these assumptions is the following

THEOREM. *The residues g_{p_i} of $R'(z)$ at the poles z_{p_i} are operators of rank one.*

PROOF. Since rank $R'(z) = n$ for z regular there exists the inverse $R'^{-1}(z)$ for z regular. At the poles z_{p_i} we have det $R'^{-1}(z_{p_i}) = 0$, and rank $R'^{-1}(z) = n - 1$, since the poles are simple. From $R'^{-1}(z)R'(z) = P$ we obtain by integration around the poles $R'^{-1}(z_{p_i})g_{p_i} = 0$. Hence rank $g_{p_i} \leqq$ dim kernel $R'^{-1}(z_{p_i}) = n -$ rank $R'^{-1}(z_{p_i}) = n - (n - 1) = 1$. Since $g_{p_i} \neq 0$, it follows that rank $g_{p_i} = 1$.

VI. 6. In order to obtain a constructive method to deal with the residues we introduce a "generalized generator" $W(z)$ in $P\mathcal{H}$ by

$$R'(z) = (z - W(z))^{-1}. \tag{27}$$

In analogy to (26) we then have the

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$$R'(z) = \sum_i \frac{Q_i(z)}{z - w_i(z)} \tag{28}$$

where $w_i(z)$ are the n eigenvalues of $W(z)$, $Q_i(z)$ are idempotents depending on z and satisfying the orthogonality and completeness relations

$$Q_i(z)Q_j(z) = \delta_{ij}Q_i(z) \ ; \qquad \sum_i Q_i(z) = P. \tag{29}$$

PROOF. The $Q_i(z)$ are constructed by forming the direct product of the left and right eigenvectors $v_i(z)$ and $u_i(z)$ of the matrix $W(z)$. In the limit of vanishing interaction, $v_i(z)$ goes over to the transpose conjugate of $u_i(z)$ (they become independent of z); for sufficiently small interaction, $u_i(z)$ is distinct from $u_j(z)$ for $i \neq j$.

The first part of equation (29) follows from

$$Q_i(z) = u_i(z) \otimes v_i(z)$$

and

$$W(z)u_i(z) = w_i(z)u_i(z), \qquad v_j(z)W(z) = w_j(z)v_j(z),$$

i.e.,

$$0 = [w_i(z) - w_j(z)]\,(v_j(z)u_i(z)),$$

so that $(v_j(z)u_i(z)) = 0$ for $w_i(z) \neq w_j(z)$. The idempotence property is valid since we normalize to $(v_i(z)u_i(z)) = 1$.

The completeness property is slightly more involved. Assuming the u_i to form a (not necessarily orthogonal) basis in $P\mathcal{H}$, the orthogonality and normalization conditions imply that

$$v_i = \sum_j m_{ij} u_j^\dagger$$

where $u_j^\dagger$ is the conjugate transpose of u_j and m_{ij} is the inverse of the matrix $(u_i^\dagger u_j)$. Hence

$$\sum_i u_i(z) \otimes v_i(z) = \sum_{ij} m_{ij} u_i(z) \otimes u_j(z)^\dagger$$

is Hermitian even though $Q_i(z)$ may not be. The third part of equation (29) then follows since this sum is idempotent, leaves every element of $P\mathcal{H}$ invariant, and vanishes on $\bar{P}\mathcal{H}$. We finally remark that $\mathrm{Tr}_P Q_i(z) = 1$, since a complete orthogonal set can be constructed of linear combinations of the u_j; if the first element of the set is chosen to be $u_i/|u_i|$, its contribution to the trace is unity and the remaining part vanishes by construction.

VI. 7. The construction of the pole residues g_p proceeds now as follows: The n poles z_{p_i} of $R'(z)$ are the (distinct) roots of the equations $z_{p_i} - w_i(z_{p_i}) = 0$. From (28) we obtain therefore

$$g_{p_i} = Q_i(z_{p_i})/(1 - w_i'(z_{p_i})) \tag{30}$$

i.e., the residues are porportional to the idempotents $Q_i(z_{p_i})$. Hence they are in general

(1) not of trace 1,
(2) not selfadjoint,
(3) not orthogonal.

Equation (29) does not imply $Q_i(z_{p_i})Q_j(z_{p_j}) = 0$ for $z_{p_i} \neq z_{p_j}$, and hence we have, in general, according to (30), $g_{p_i} g_{p_j} \neq 0$.

As a consequence, the residues g_p are in general not states. The one-dimensional projectors on the ranges of g_p are however pure states whose physical significance will now be established.

VI. 8. Let P_p be the one-dimensional projectors on the ranges of g_p. Then we have

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$$\mathrm{Tr}_P(W(z_p)P_p) = z_p; \qquad \mathrm{Tr}_P(W(z_p)g_p) = \frac{z_p}{1 - w_p'(z_p)}\,. \tag{31}$$

PROOF. Integrating the relation $R'^{-1}(z)R'(z) = (z - W(z))R'(z) = P$ around the pole z_p, we obtain $(z_p - W(z_p))g_p = 0$. Since g_p is rank 1, it is also true that $(z_p - W(z_p))P_p = 0$. The result follows from these two relations by taking the trace and by using (30).

If we split z_p into its real and imaginary parts and $W(z_p)$ into its Hermitian and skew Hermitian parts:

$$z_p = \lambda_p - i\Gamma_p/2; \qquad W(z_p) = W^{(+)}(z_p) + iW^{(-)}(z_p)$$

we obtain the more detailed result

$$(31') \qquad \mathrm{Tr}_P(W^{(+)}(z_p)P_p) = \lambda_p; \qquad \mathrm{Tr}_P(W^{(-)}(z_p)P_p) = -\Gamma_p/2.$$

We now introduce a physical terminology for these quantities; its justification will become clear in the next section.

DEFINITION. We call the projectors P_p on the ranges of the residues g_p the *resonant states,* the operators $W^{(+)}(z_p)$, $W^{(-)}(z_p)$ the *mass* and the *decay operators,* λ_p the *resonance mass* and Γ_p the *decay constant.* Then equation (31′) can be expressed in the following form:

THEOREM. *The expectation values of the mass and decay operators in the resonance states are the resonance mass and the decay constant for that state.*

VII. **Phenomenological approximations.**

VII. 1. **The simplest decay and resonant-scattering processes** can be described approximately in terms of a contractive semigroup for the reduced motion, and the so-called Breit-Wigner function for the scattering cross-section [**12**]. The approximations involved are of the following general types:

(1) pole approximation,
(2) weak interaction limit,
(3) simplifying assumptions about the structure of the residues.

We shall proceed in steps in order to exhibit the necessity of the various assumptions in deriving these phenomenological descriptions.

VII. 2. The *pole approximation* consists in retaining for the reduced resolvent $R'(z)$ only the first terms of its Laurent expansion:

$$(32) \qquad R'(z) \cong \sum_p \frac{g_p}{z - z_p}.$$

In this approximation $R'(z)$ is thus a meromorphic operator function with n isolated simple poles z_p; the branch cut Λ has disappeared. Inserting (32) into the expressions (4) for the reduced motion, we obtain

$$(33) \quad U'(t) \cong \sum_p g_p \exp(-iz_pt) = \sum_p g_p \exp\left(- i\left(\lambda_p - \frac{i\Gamma_p}{2}\right) t\right).$$

From this expression the physical significance of the real and imaginary parts of the poles z_p appears clearly as the resonance mass and the decay constant (cf. the definition in VI. 8). Furthermore, we see that for $U'(t)$ to be a contractive semigroup it is necessary and sufficient that the *residues are orthogonal idempotents*, i.e., that

$$g_p g_{p'} = \delta_{pp'} g_p. \tag{34}$$

VII. 3. In the study of the scattering cross-section we restrict ourselves to the Friedrichs' model in which $\sigma(\lambda)$ depends only on the reduced resolvent. In pole approximation (32) equation (19) becomes

$$\sigma(\lambda) \cong \frac{i\pi}{2} \sum_p \mathrm{Tr}_\lambda \left\{ \frac{\mathrm{Tr}_P(X(\lambda)g_p)}{\lambda - z_p} - \frac{\mathrm{Tr}_P(X(\lambda)g_p^\dagger)}{\lambda - \bar{z}_p} \right\}$$

$$\cong \frac{i\pi}{2} \sum_p \frac{1}{(\lambda - \lambda_p)^2 + (\Gamma_p/2)^2} \mathrm{Tr}_\lambda \{ (\lambda - \lambda_p) \mathrm{Tr}_P [X(\lambda)(g_p - g_p^\dagger)] - (i\Gamma_p/2) \mathrm{Tr}_P [X(\lambda)(g_p + g_p^\dagger)] \}. \tag{35}$$

(This approximation is fairly good in the neighborhood of λ_p. In the nondegenerate case, where all the m_i are different, the poles closest to λ make up the principal contributions.)

In order to get the Breit-Wigner form (in its restricted definition as a precise Lorentz form) we first suppose that the *residues are self-adjoint*: $g_p = g_p^\dagger$. Then the first term in (35) vanishes. Second we have to evaluate the term $\mathrm{Tr}_P(X(\lambda)g_p)$. Here we assume the *limit of weak interaction* to hold, in which higher order terms in the coupling $\|V\|$ are neglected. Using (15) and (27), we can then write for $W(z)$ at the pole z_p in the second sheet:

$$W(z_p) = \sum_i m_i P_i + PVP + \int \frac{\mathrm{Tr}_\lambda X(\lambda')}{z_p - \lambda'} d\lambda' - 2\pi i \, \mathrm{Tr}_\lambda \, X(z_p)$$

$$\cong \sum_i m_i P_i + PVP + \mathrm{P}\!\!\int \frac{\mathrm{Tr}_\lambda X(\lambda')}{\lambda_p - \lambda'} d\lambda' - i\pi \, \mathrm{Tr}_\lambda \, X(\lambda_p).$$

Therefore

$$\therefore W^{(-)}(z_p) \cong -\pi \mathrm{Tr}_\lambda \, X(\lambda_p) \quad ,$$

and hence, according to (31) ($\lambda \sim \lambda_p$)

$$\mathrm{Tr}_P \, \mathrm{Tr}_\lambda (X(\lambda)g_p) \cong \mathrm{Tr}_P \, \mathrm{Tr}_\lambda (X(\lambda_p)g_p)$$

$$= -\frac{1}{\pi} \mathrm{Tr}_P(W^{(-)}(z_p)g_p) = \frac{\Gamma_p}{2\pi(1 - w_p'(z_p))} \; .$$

Insertion of this into (35) yields the scattering cross-section in Breit-Wigner form:

$$(36) \qquad \sigma(\lambda) \cong \sum_p \frac{(\Gamma_p/2)^2}{(\lambda - \lambda_p)^2 + (\Gamma_p/2)^2} \left(\frac{1}{1 - w_p'(z_p)} \right).$$

(The last factor ("inelasticity") would be unity if, for example, (34) were satisfied.)

VII. 4. We summarize these findings in the following

THEOREM. *The reduced motion* $U'(t)$ *and the cross-section* $\sigma(\lambda)$ *have in pole approximation the forms* (33) *and* (35). *If the pole residues* g_p *are orthogonal idempotents in the sense of equation* (34), *then the reduced motion forms a contractive semigroup. If the pole residues* g_p *are selfadjoint and if the limit of weak interaction holds, then the scattering cross-section has the Breit-Wigner form* (36).

In the form (36) the cross-section appears as the superposition of n resonances with the resonant masses λ_p as the centers, and the decay constants $\Gamma_p/2$ as the widths, of their distributions.

VII. 5. The approximations considered here exhibit in a striking fashion the close link which exists in a decay-scattering system between the phenomena of decay and resonance. As we can see from equations (33) and (36), the reduced motion $U'(t)$ and the cross-section $\sigma(\lambda)$ are both characterized by the location of the poles z_p in the reduced resolvent $R'(z)$. Hence $\lambda_p = \operatorname{Re} z_p$ takes on the dual aspects of the decay and the resonance energies and $-\Gamma_p/2 = \operatorname{Im} z_p$ those of the decay constants and the widths of the resonances.

In pole approximation the decay is essentially exponential (equation (33)), whereas, according to the global remarks made in Chaper III, the initial decay rate vanishes. This raises the question of the validity of the pole approximation in the decay problem. We do not attempt here a general study of this question (see V. 6 for the 1-dimensional case).

VIII. Symmetry and symmetry breaking.

VIII. 1. Consider a scattering system $\{H_0, H\}$ with *spherically symmetric potential* V. This means that the angular momentum operator L commutes with both H_0 and H, and therefore with the scattering operator S. The projectors P_ℓ on the eigenspaces of L with definite angular momentum eigenvalues ℓ reduce the scattering system:

$$\mathcal{H} = \sum_{\ell=0}^{\infty} \oplus \mathcal{H}_\ell, \quad S = \sum_{\ell=0}^{\infty} S_\ell, \quad \{S(\lambda)\} = \left\{ \sum_{\ell=0}^{\infty} S_\ell(\lambda) \right\},$$

etc. Such an operator L is called a dynamical symmetry of the system.

A further dynamical symmetry is, for instance, the third component

L_3 of the angular momentum with eigenvalues $m = -\ell, \cdots, +\ell$, and we know that in a *system without spin* a state is completely determined if we specify its energy and its angular momentum eigenvalues ℓ, m. In this case we call the system $\{L, L_3\}$ a complete symmetry. We now formalize this in the following

DEFINITION. A selfadjoint operator A is called a *dynamical symmetry* of the decay-scattering system $\{H_0, H\}$ if it commutes with H_0 and H. A commuting set $\{A_\alpha\}$ of dynamical symmetries is called *complete* if the system $\{H_0, A_\alpha\}$ is maximal commuting, i.e. if $\{H_0, A_\alpha\}'' = \{H_0, A_\alpha\}'$, where $\{H_0, A_\alpha\}''$ is the von Neumann algebra generated by $\{H_0, A_\alpha\}$ and $\{H_0, A_\alpha\}'$ its commutant [13].

VIII. 2. We first consider the case of a complete system of dynamical symmetries. For simplicity we assume that the spectrum of H_0 consists of an absolutely continuous part Λ_0 with uniform multiplicity (possibly ∞), and a single point eigenvalue $m_0 > 0$ with corresponding projectors $\bar{P}$ and P, while the spectra of A_α are discrete. Let us denote by c_i the discrete points in the product spectrum $\prod_\alpha \Lambda_\alpha$ of $\{A_\alpha\}$ and by I_i the projectors on the corresponding eigenspaces. Since $\{A_\alpha\}$ is complete, the common eigenspaces I_iP of A_α and PH_0P are one-dimensional (with $\{\varphi_i\}$ the corresponding orthonormal basis in $P\mathcal{H}$), and the continuous spectra of $P_i\bar{P}H_0\bar{P}P_i$ are simple.

The spectral representation of $\mathcal{H}$ with respect to the set $\{H_0, A_\alpha\}$ has now the following form:

$$\mathcal{H} \to \sum_i P_i \mathcal{L}^2(\Lambda_0),$$

$$\psi \to \{\langle\lambda, c_i|\psi), \psi_i\}; \qquad H_0\psi \to \{\lambda\langle\lambda, c_i|\psi), m_0\psi_i\}$$

where $\psi_i = (\varphi_i, \psi)$ and where $\langle\lambda, c_i|\psi)$ are numerical valued, square integrable functions on Λ_0 such that

$$\sum_{c_i} \int |\langle\lambda, c_i|\psi)^2 d\lambda < \infty.$$

VIII. 3. Since the family $\{I_i\}$ reduces both H_0 and H, we have $I_iVI_j = 0$ for $i \neq j$, and the decay-scattering system $\{H_0, H\}$ splits into a direct sum of one-dimensional decay-scattering systems $\{I_iH_0I_i, I_iHI_i\}$ in $I_i\mathcal{H}$. In particular the pole residua g_{p_i} become orthogonal and Levinson's theorem (III. 7) holds in each subspace I_i.

THEOREM. *In the case of complete dynamical symmetry the decay-scattering system decomposes into a direct sum of one-dimensional systems. In each system the scattering cross-section has at least one true resonance, if P_iHP_i has no bound state. The pole residua are orthogonal, and hence in the absence of inelasticity, the semigroup* *property and the Breit-Wigner form for the resonances are valid in pole approximation.*

VIII. 4. If the potential V in our example of VII. 1 were not spherically symmetric, then the angular momentum operator L would not commute with H and S (although it still would with H_0). We call such an operator a broken symmetry.

DEFINITION. A selfadjoint operator A is called a *broken* (*dynamical*) *symmetry* of the decay-scattering system, if it commutes with H_0 but not with H. A is a *weakly broken symmetry*, if $\|I_iVI_j\| << \|I_iVI_i\|$ for $i \neq j$. (Here I_i are again the spectral projections of A.)

A decay-scattering system with a broken symmetry does not reduce into a direct sum of one-dimensional systems with orthogonal pole residua, and none of the nice conclusions of the theorem in VII. 3 hold. In particular, the nonorthogonality of the pole residua results in a system without the semigroup property, even in the pole approximation.

In the next chapter we shall evaluate the effect of a broken symmetry in a concrete example.

IX. Physical applications.

IX. 1. The decay of the neutral K-mesons in a slightly idealized description yields an almost perfect illustration of the formalism developed so far. It leads in fact to the simplest model of a multi-dimensional decay-scattering system with weak symmetry breaking.

In a second example (Σ^0-decay) we shall consider the situation of two bound states embedded in the continuum of H_0 of which only one is decaying whereas the other remains as a bound state of H. Such a situation is not incompatible with our general definition of the decay-scattering system, but it presents in a simple way the phenomenon of a so-called (finite) mass renormalization.

A. *The decay of the neutral K-mesons* [14].

IX. 2. The neutral K-mesons are produced by strong interactions and they decay only through weak interactions, the ratio between strong and weak interactions being approximately 10^{13}. It is therefore natural to split H into a free Hamiltonian H_0 which contains all the strong interactions (including final state interactions of the decay products), under which the K^0-mesons are stable, and a weak perturbation V which induces their decay.

The K^0-meson space is two-dimensional [15] and all K^0's have an identical well-defined free mass m_K ($\sim$500 Mev). Under the weak interactions they decay into either π-mesons or into channels with leptons. For simplicity, we do not discuss here the leptonic decay (involving electrons, neutrinos, etc.).

The CP-operator (charge conjugation times partiy) is a broken symmetry of the system, with two eigenvalues ± 1. The linear combinations of neutral K-mesons which are eigenstates of CP with eigenvalues $+1$ and -1 are called, respectively, K_1 and K_2. The 2π-states are CP $+$ 1 and the 3π-states CP-1 eigenstates.

Under the weak interaction V the K-mesons decay into the π's. If CP were a precise symmetry, K_1 would decay only into 2π, and K_2 into 3π, and since the coupling between K_1 and 2π is much larger than between K_2 and 3π, the K_1-decay would be much more rapid than the K_2-decay (ratio of about 100). For an arbitrary initial K-meson beam the K_1-component would therefore immediately decay into 2π's while the K_2-component would live much longer. After some time the beam should thus produce exclusively 3π's.

This was, however, contradicted by experiment [**16**] : There is a small ratio (about 0.2%) of 2π's which are produced at a time where, under the hypothesis of CP-symmetry, we would expect only 3π's. Hence the weak interaction V violates the CP-symmetry which becomes, therefore, a weakly broken symmetry.

We now try to reformulate this problem within the framework of our theory.

(1) The K^0-π system is a decay-scattering system. The space P of the unstable neutral K-mesons is two-dimensional. It is a discrete eigenspace of H_0 corresponding to the doubly degenerate eigenvalue m_K (= mass of K-mesons). The continuous spectrum Λ_0 of H_0 has multiplicity 2 and the corresponding space $\overline{P}\mathcal{H}$ of decay products contains the 2π- and 3π-states.

(2) There is a weakly broken symmetry CP with two eigenvalues $c_1 = 1$, $c_2 = -1$ with corresponding projectors I_1, I_2 $(I_1 + I_2 = 1)$. Since CP commutes with H_0, P and $\overline{P}$ are reduced by I_1 and I_2. The common eigenvectors of I_i and P are called K_1 and K_2; the space $I_1\overline{P}\mathcal{H}$ contains the π-meson pairs and $I_2\overline{P}\mathcal{H}$ the π-meson triplets.

(3) The CP-eigenstates K_1, K_2 are coupled to the continuum $\overline{P}$ with different strengths: The pole z_L of $R'(z)$ corresponding to K_2 is about 100 times nearer to the real axis than the pole z_S corresponding to K_1; hence the states K_L, K_S which span the ranges of the pole residua g_L, g_S decay with rates differing by about a factor 100. K_L is called the *longlived* and K_S the *shortlived* K-meson.

(4) The weak symmetry breaking manifests itself through the fact that $I_iVI_j \neq 0$ for $i \neq j$. In particular, we have $I_1\overline{P}VPI_2 \neq 0$, which means that K_2 can decay into a 2π-state although K_2 has CP $= -1$ and 2π has CP $= +1$.

(5) The CP-symmetry breaking could also be achieved by a term

I_1PVPI_2 linking directly K_1 and K_2. (A theory in which the entire breaking occurs in the term PVP is called a *superweak theory.*) We assume here to the contrary that PVP conserves the CP-symmetry and that the entire violation lies in the terms $PV\bar{P}$ and $\bar{P}VP$. We can then include PVP in H_0 (since it does not induce any decay or symmetry breaking). This assumption simplifies the results slightly.

IX. 4. In this theory the reduced motion of the K-meson system is now expressed, in the pole approximation, by

$$U'(t) \cong g_S\, e^{-iz_S t} + g_L\, e^{-iz_L t}.$$

The nonorthogonality of the pole residua g_S, g_L is of the order of the ratio between symmetry conserving and symmetry breaking terms of V, i.e., $g_S g_L = O(\alpha)$, where α is the order of CP violation (amplitude) [14].

The interaction functions are, in the basis $K_1, K_2 \in P$:

$$\mathrm{Tr}_\lambda\, X_{ij}(\lambda) = \overline{\langle \lambda, c_1|VK_i\rangle}\langle \lambda, c_1|VK_j\rangle + \overline{\langle \lambda, c_2|VK_i\rangle}\langle \lambda, c_2|VK_j\rangle,$$

$$h_{ij}(z) = (z - m_K)\delta_{ij} - \int_0^\infty \frac{\mathrm{Tr}_\lambda\, X_{ij}(\lambda)}{z - \lambda}\, d\lambda$$

from which the scattering amplitudes and cross-section are obtained by equations (18) and (19).

B. *The decay of* Σ^0.

IX. 5. The neutral Σ- and Λ-particles are formed by strong interactions with masses $m_\Sigma > m_\Lambda > 0$. The Σ^0, however, decays under electromagnetic interactions ($\sim$100 times weaker than strong interactions), into a Λ^0 and a photon γ, while Λ^0 is much more stable and decays only under weak interactions. The symmetry distinguishing Σ^0 and Λ^0 (except for their relatively small mass difference) is the so-called isotopic spin; it is broken by electromagnetic interactions. If we put the strong interactions into H_0, the electromagnetic ones into V and neglect the weak interactions, and if we note that the decay products (Λ, γ) have necessarily a higher energy than the free mass m_Λ of Λ, we obtain the following model:

(1) The $(\Sigma, \Lambda, \gamma)$-system forms a decay-scattering system in the following sense: The spectrum of H_0 consists of three parts: two point eigenvalues m_Λ, m_Σ with corresponding eigenvectors Λ, Σ and one-dimensional projectors P_Λ, P_Σ, and a continuum Λ_0 with $\bar{P}_{(\Lambda,\gamma)} = \bar{P}$ extending from m_Λ to infinity. Since $m_\Sigma > m_\Lambda$, m_Σ is embedded in Λ_0. The multiplicity of Λ_0 is 1. We have

$$H_0 = P_\Lambda H_0 P_\Lambda + P_\Sigma H_0 P_\Sigma + \bar{P} H_0 \bar{P}.$$

The corresponding spectral representation is

$$\psi \to \{\psi_\Lambda, \psi_\Sigma, \langle\lambda|\psi\rangle\},$$
(37)
$$H_0\psi \to \{m_\Lambda\psi_\Lambda, m_\Sigma\psi_\Sigma, \lambda\langle\lambda|\psi\rangle\}$$

where

$$\psi_\Lambda = (\Lambda, \psi); \quad \psi_\Sigma = (\Sigma, \psi).$$

(2) The electromagnetic interaction V couples Σ to the (Λ, γ)-continuum and to Λ but not Λ to the continuum (Λ is to be stable). We have

$$V = \bar{P}VP_\Sigma + P_\Sigma V\bar{P} + P_\Lambda VP_\Sigma + P_\Sigma VP_\Lambda,$$

or, in spectral representation:

$$V\psi \to \left\{ V_{\Lambda\Sigma}\,\psi_\Sigma, V_{\Sigma\Lambda}\,\psi_\Lambda + \int_{m_\Lambda}^{\infty} \overline{\langle\lambda|V\Sigma)}\langle\lambda|\psi\rangle d\lambda, \langle\lambda|V\Sigma)\psi_\Sigma \right\} \tag{38}$$

where

$$V_{\Lambda\Sigma} = (\Lambda, V\Sigma), \qquad V_{\Sigma\Lambda} = (\Sigma, V\Lambda).$$

IX. 6. Under this interaction the discrete eigenvalue m_Σ of H_0 disappears from the spectrum of H, while the eigenvalue m_Λ at the lower end of the (Λ, γ)-continuum shifts to the left and remains in the spectrum of H as a discrete point $m_{\Lambda'} < m_\Lambda$. In order to see this last point, we solve the eigenvalue equation $H\psi = m_\Lambda\,\psi$. From (37), (38), we obtain the componentwise equations:

$$\begin{aligned}(m_{\Lambda'} - m_\Lambda)\psi_\Lambda &= V_{\Lambda\Sigma}\,\psi_\Sigma,\\ (m_{\Lambda'} - m_\Sigma)\psi_\Sigma &= V_{\Sigma\Lambda}\,\psi_\Lambda + \int_{m_\Lambda}^{\infty} \overline{\langle\lambda|V\Sigma)}\langle\lambda|\psi\rangle d\lambda,\\ (m_{\Lambda'} - \lambda)\langle\lambda|\psi\rangle &= \langle\lambda|V\Sigma)\psi_\Sigma,\end{aligned} \tag{39}$$

whence the following equation for $m_{\Lambda'}$:

$$m_\Lambda = m_\Sigma + \int_{m_\Lambda}^{\infty} \frac{|\langle\lambda|V\Sigma)|^2}{m_{\Lambda'} - \lambda}\, d\lambda + \frac{|V_{\Lambda\Sigma}|^2}{m_{\Lambda'} - m_\Lambda}. \tag{40}$$

From this equation follows that $m_{\Lambda'} < m_\Lambda$. Furthermore, it can be proved from (39) that the corresponding eigenvector $\psi = \Lambda'$ of H differs from Λ only weakly, i.e., that $\|P_\Sigma\Lambda'\|$ and $\|\bar{P}\Lambda'\|$ are small of the order of $\|V\|$ compared to $\|P_\Lambda\Lambda'\|$. But note that Λ' does not lie entirely in the subspace $P\mathcal{H}$ and the corresponding projector $P_{\Lambda'}$ does not commute with P.

IX. 7. Next we discuss the poles of the reduced resolvent $R'(z)$. It follows from (15) and (16) for the interaction functions:

$$X(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & X_{\Sigma\Sigma}(\lambda) \end{pmatrix}, \tag{41}$$

$$(42) \qquad h(z) = (R'(z))^{-1} \begin{pmatrix} z - m_{\Lambda'}, & -V_{\Lambda\Sigma} \\ -V_{\Sigma\Lambda}, & z - m_{\Sigma} - \int \frac{|\langle\lambda|V\Sigma\rangle|^2}{z-\lambda} d\lambda \end{pmatrix}$$

The poles of $R'(z)$ occur at det $(h(z_p)) = 0$. It is easy to verify that one pole occurs at $m_{\Lambda'}$ in the first sheet by inserting (40) into (42). Let z_Σ be the other pole lying in the lower half-plane of the second sheet. From

$$z_\Sigma - m_\Sigma - \int \frac{X_{\Sigma\Sigma}(\lambda)}{z_\Sigma - \lambda} d\lambda + 2\pi i\, X(z_\Sigma) - \frac{|V_{\Lambda\Sigma}|^2}{z_\Sigma - m_\Lambda} = 0,$$

it follows that, in the weak interaction limit

$$\operatorname{Im} z_\Sigma = -\Gamma_\Sigma/2 \cong -\pi\, X(m_\Sigma)$$

for the decay constant (or the width of the resonance), and for the resonant mass we have

$$\operatorname{Re} z_\Sigma = m'_\Sigma \cong m_\Sigma + P\!\!\int \frac{X_{\Sigma\Sigma}(\lambda)}{m_\Sigma - \lambda} d\lambda + \frac{|V_{\Lambda\Sigma}|^2}{m_\Sigma - m_\Lambda}.$$

IX. 8. The residuum of $R'(z)$ at $m_{\Lambda'}$ is $PP_{\Lambda'}P$, where $P_{\Lambda'}$ is the projection on Λ'. Let g_Σ be the residuum at z_Σ. Then we obtain for the *reduced motion in pole approximation*:

$$U'(t) \cong PP_{\Lambda'}P\, e^{-im_{\Lambda'}t} + g_\Sigma\, e^{-iz_\Sigma t},$$

and for the evolution and asymptotic behavior of the initial states Λ and Σ:

$$U'(t)\Lambda \cong \Lambda'_\Lambda P\Lambda'\, e^{-im_{\Lambda'}t} + g_\Sigma \Lambda\, e^{-iz_\Sigma t} \xrightarrow[t >> (\operatorname{Im} z_\Sigma)^{-1}]{} \Lambda'_\Lambda P\Lambda'\, e^{-im_{\Lambda'}t},$$

$$U'(t)\Sigma \cong \Lambda'_\Sigma P\Lambda'\, e^{-im_{\Lambda'}t} + g_\Sigma \Sigma\, e^{-iz_\Sigma t} \xrightarrow[t >> (\operatorname{Im} z_\Sigma)^{-1}]{} \Lambda'_\Sigma P\Lambda'\, e^{-im_{\Lambda'}t}.$$

Since $\|P\Lambda'\| \sim 1$, $\|P_\Lambda \Lambda'\| \sim 1$, while $\|P_\Sigma \Lambda'\| = O(\|V\|) << 1$, we see that, under $U'(t)$, Λ tends to a state whose norm remains almost 1, while Σ decays almost to 0. This is so because a small part of Λ makes a transition into Σ which decays, while a small part of Σ goes into $P\Lambda'$ which is stable. The longlived component in $P\mathcal{H}$ has always the character of $P\Lambda'$, irrespective of the initial state. But the longlived Λ's arising from an initial Σ are not accompanied by a photon γ.

Finally, we treat the *scattering problem*. Starting from the precise general formula (19) for the scattering cross-section and noting that $h^{-1}(\lambda + i0) = [h^{-1}(\lambda - i0)]^*$, we have

$$\begin{aligned}\sigma(\lambda) &= \frac{i\pi}{2}\mathrm{Tr}_P[X(\lambda)(h^{-1}(\lambda + i0) - h^{-1}(\lambda - i0))] \\ &= -\pi \,\mathrm{Im}\, \mathrm{Tr}_P[X(\lambda)h^{-1}(\lambda + i0)] \\ &= \pi\, \mathrm{Im} \left\{\frac{1}{\det h(\lambda + i0)}\mathrm{Tr}_P[X(\lambda)\mathrm{cof}\, h(\lambda + i0)]\right\}\end{aligned}$$

and, inserting (41) and (42), the following exact expression:

$$\sigma(\lambda) = \frac{(\pi X_{\Sigma\Sigma}(\lambda))^2}{\left[\lambda - m_\Sigma - \dfrac{|V_{\Sigma\Lambda}|^2}{\lambda - m_\Lambda} - \displaystyle\fint_{m_\Lambda}^{\infty} \frac{|\langle\lambda'|V\Sigma\rangle|^2}{\lambda - \lambda'}\, d\lambda'\right]^2 + (\pi X_{\Sigma\Sigma}(\lambda))^2}$$

X. Physical significance of decay-scattering systems.

X. 1. In Chapter I we introduced decay-scattering systems in a purely mathematical way as a simple modification of the formal scattering systems defined by Jauch [1] and Kato [9]. A common feature of these abstract systems is that to a given unitary evolution $U(t)$, with generator H whose spectrum is positive and continuous, there exist many decompositions $H = H_0 + V$ such that the pair $\{H_0, H\}$ has the properties of a (decay-) scattering system. In particular it is always possible to define a decay-scattering system by

$$H = H_0 + V; \qquad H_0 = \bar{P}H\bar{P} + PHP$$

where P, $\bar{P}$ is an arbitrary partition of the unity with dim $P = n$ finite.

In order to relate these mathematical constructions to a given experimental situation involving decay and resonance one has to answer the following:

Questions. (1) Does the decay law obtained in a decay-scattering system yield a correct description of the time-evolution of those states we are willing to consider as the states of the unstable particles?

(2) What is the relation between the scattering amplitudes obtained in a decay-scattering system to the resonances measured in the laboratory?

We are still far from having a purely operational self-consistent procedure which would permit one to associate in a canonical manner decay-scattering systems to an arbitrary collection of experimental data. A physical theory is rarely inductive in this strict sense. What usually happens is that we possess a certain amount of "exterior" evidence of what a theory of given phenomena should look like. For example the fact that there should be two neutral K-mesons was originally obtained from a consideration of their decay properties with respect to the discrete operations of space reflection and charge conjugation [15]. In other cases, like the hydrogen atom or α-decay, we may use some a priori knowledge of the form of the potential responsible for scattering and decay.

In the absence of a universal bootstrap theory we confine ourselves here to a review of some instances in which decay-scattering systems may actually lead to a good phenomenological description.

X. 2. Question (1) obtained a partial answer in III.6 where the space $\mathcal{H}'$ of the unstable particles and their evolution $U'(t)$ was supposed to belong to our a priori knowledge. We outlined there a theoretical procedure (inverse decay problem), which, at least in principle, permits one to embed this system into a decay-scattering system $\{H_0, H\}$ defined in an enlarged space $\mathcal{H}$ such that $\mathcal{H}' = P\mathcal{H}$ and $U'(t) = Pe^{-iHt}P$.

In our deductive theory of the K-mesons (Chapter IX), the basic information consisted in a knowledge of the K-meson masses, and the fact that the interactions inducing their decay into the π-mesons are weak and break the CP-symmetry. The decay-scattering system results from including the strong interactions into the Hamiltonian H_0; the decay law then becomes a matter for experimental verification.

X. 3. As to Question (2) we should note that the scattering amplitude $T(\lambda)$ of a decay-scattering system never corresponds to the amplitude measured in an experiment, but represents rather a fictive experiment between "distorted waves." This comes about as follows:

A scattering experiment correlates the free outgoing to the free ingoing states. The experimental S-matrix therefore corresponds to a decomposition of the total Hamiltonian H into the operator H_{kin}, representing the free kinetic energy of the asymptotic states, and the total perturbation V_{total}, whereas in the decay-scattering system $\{H_0, H\}$ the operator H_0 includes all of the interaction between the free particles which leaves the "unstable particles" in bound states. If we suppose that H has no bound states, Levinson's theorem implies that, for the total phase-shift

$$\delta_{\text{total}}(\infty) - \delta_{\text{total}}(0) = 0$$

whereas, for the phase-shift of the decay-scattering system,

$$\delta(\infty) - \delta(0) = n\pi$$

with n the dimension of the discrete part of H_0. From the chain-rule for the wave operators we obtain

$$\Omega_{\text{total}} = \Omega(H \leftarrow H_{\text{kin}}) = \Omega(H \leftarrow \bar{P}H_0\bar{P}) \cdot \Omega(\bar{P}H_0\bar{P} \leftarrow H_{\text{kin}})$$

where $\Omega_d = \Omega(\bar{P}H_0\bar{P} \leftarrow H_{\text{kin}})$ leads from the free to the distorted waves. For the relation between the corresponding scattering operators this implies

$$S_{\text{total}}(\lambda) = S(\lambda)\, S_d(\lambda)$$

where S_{total} and S_d are in the spectral representation with respect to H_{kin} [17]. (L. Stodolsky [17] has used a similar construction to take into account the part $\bar{P}V\bar{P}$ of the potential (cf. X. 1).)

In the case of the K-mesons, for instance, the distortion is due to the strong interactions and S is related to the experimental S_{total} by the strong interaction phase-shifts in the π-meson space.

X. 4. Still a different problem arises if we assume that the potential $V_{\text{total}} = H - H_{\text{kin}}$ is known a priori. A typical example is provided by the customary description of the α-particle resonances due to the tunnel effect. Here the total scattering amplitude has a resonance, but the phase-shift satisfies $\delta(\infty) = \delta(0)$ since H_0 and H have no bound states.

The description in terms of a decay-scattering system would consist in introducing a state $\varphi \in \mathcal{H}$, called a compound nucleus, and then defining H_0 by (43), P being the projector on φ. A procedure which might lead to a unique definition of φ could consist in constructing the state which maximalizes the expectation value of Wigner's delay-time operator [7].

We note that the new potential $V = H - H_0$ obtained in the decay-scattering description would be nonlocal, since its rank is finite.

Acknowledgments. We wish to acknowledge interesting discussions with T. Kato, S. T. Kuroda, P. D. Lax, R. S. Phillips, and others attending the 1969 Seminar on the Mathematical Theory of Scattering at Flagstaff, Arizona, and to thank Professor C. Wilcox for his hospitality there. The point of view we have taken in the algebraic treatment of the resolvent equations is due in large part to C. Piron. We are also grateful to F. Coester, H. Ekstein, and M. Gourdin for discussions and very helpful suggestions.

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