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A DOUBLY NONLOCAL LAPLACE OPERATOR AND ITS CONNECTION TO THE CLASSICAL LAPLACIAN

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ABSTRACT. Motivated by the state-based peridynamic framework, we introduce a new nonlocal Laplacian that exhibits double nonlocality through the use of iterated integral operators. The operator introduces additional degrees of flexibility that can allow for better representation of physical phenomena at different scales and in materials with different properties. We study mathematical properties of this state-based Laplacian, including connections with other nonlocal and local counterparts. Finally, we obtain explicit rates of convergence for this doubly nonlocal operator to the classical Laplacian as the radii for the horizons of interaction kernels shrink to zero.

1. Introduction. Physical phenomena that are beset by discontinuities in the solution, or in the domain, have been challenging to study in the context of classical partial differential equations (PDEs). Moreover, nonlocal or discrete material behavior provides an additional catalyst to investigate integral type models for which discontinuous solutions are allowable. In particular, successful predictions of dynamic fracture in different materials (homogeneous or heterogeneous) have been obtained through the peridynamic formulation introduced by Stewart Silling in [**20**]; see fiber-reinforced composites [**10**], composite laminates [**11**], orthotropic material [**8**], layered glass [**2**], concrete [**12**]. Over the past decades nonlocal theories have also been successfully employed in modeling various other phenomena, including nonlocal diffusion [**1**], porous media flow [**13**] and tumor growth [**14**].

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The peridynamic theory offers a unified approach to capture the deformation of the material as well as the propagation of the cracks. In its original bond-based formulation the system captures the cumulative effects of the interactions between a point and all its neighbors. These interactions are weighted by distance-dependent kernels along bonds, which are vectors that connect every two nearby points. A novelty of the theory was the introduction of a horizon of interaction, a physical constant which characterizes a model, a material, or a phenomenon. Mathematically, this constant measures the size of the interaction set for the kernel given by its support. The nonlocality takes the form of an integral operator which replaces the spatial differential operators used in classical PDEs, giving rise to partial integro-differential equations (PIDEs). In the integral framework little to no regularity of solutions is needed, thus the set of allowable solutions includes all functions for which an integral can be defined, even very irregular functions. The protagonist of the bond-based formulation is the nonlocal Laplacian, which for clarity we label as the bond-based Laplacian:

$$\mathcal{L}_\mu[\boldsymbol{u}](\boldsymbol{x}) = \int_{\mathcal{B}_\delta} (\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}(\boldsymbol{x}))\mu(\boldsymbol{y}-\boldsymbol{x})\, dy. \tag{1.1}$$

In the above formula, $\mathcal{B}_\delta := \mathcal{B}_\delta(0)$ is the ball of radius δ centered at zero, while the kernel μ measures the strength of the bond $\boldsymbol{y}-\boldsymbol{x}$. Observe that this operator is well-defined even for very rough functions $\boldsymbol{u} : \mathbb{R}^n \to \mathbb{R}^k$, $n, k \geq 1$, as long as the integration for each component of $\boldsymbol{u}$ is valid, ($\mathcal{L}_\mu \in \mathbb{R}^k$). The constant $\delta > 0$ is the radius of the horizon, and in applications it can vary from very small values (peridynamics) to very large ones ($\delta = \infty$ in nonlocal diffusion [**1**]). Of interest to us is the case of a finite horizon as well as the transition to infinitesimal values; in other words, we study the limiting behavior of nonlocal operators as δ goes to zero.

The convergence of the bond-based Laplacian to the classical Laplacian as the horizon δ shrinks to zero has been studied in several papers. It was shown that the rate of convergence for the nonlocal Laplacian to the classical Laplacian, whenever applied to a sufficiently smooth function u, is proportional to δ^2 (the proportionality constant depends on bounds for the fourth derivative of u); see [**4, 6, 16**], where the arguments are based on the work in [**3**]; see also [**24**] where the analysis for numerical error is performed. Furthermore, in [**16, 17**] the authors have shown convergence of nonlocal L^2 solutions of the peridynamic system to classical solutions (with H_0^1 Sobolev regularity) of the Navier system.

In bond-based models particles interact through a central potential, thus “seeing” only neighbors in their horizon. A consequence of this formulation gives a restriction on the Poisson ratio of $\frac{1}{4}$ for the class of elastic materials modeled. Moreover, the bond-based systems lack the generality of stress tensors that are usually considered in continuum mechanics as they impose only a pairwise force interaction on particles. For a more detailed discussion of these aspects and the motivation for a more general theory, see **[23]**. To overcome these deficiencies Silling et al. in **[23]** introduced the theory of state-based peridynamics, in which the force between points are expressed through general operators called states. A discussion of these states, as relevant to this paper, is given in Section 2.3. The bond-based theory becomes a particular case of the state-based setting where points interact not only with their immediate neighbors (direct interactions), but also with neighbors of the neighbors (indirect interactions). Thus, the behavior of a point $\boldsymbol{x}$ will be determined by the forces acting on $\boldsymbol{x}$ through its neighbors $\boldsymbol{y}$, as well as by the forces acting on $\boldsymbol{y}$ through $\boldsymbol{y}$'s neighbors. This composition of interactions could also be extended to model a broader range of phenomena, such as seen in nonlinear elasticity, viscoelasticity, and viscoplasticity (for bond-based formulations of these models see **[5, 26, 7]**).

The focus of this work is on the study of a newly introduced state-based Laplacian operator:

$$\begin{aligned}\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}](\boldsymbol{x}) = &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\gamma(\boldsymbol{p}-\boldsymbol{x})+\gamma(\boldsymbol{q}-\boldsymbol{x}))\eta(\boldsymbol{q}-\boldsymbol{p})[\boldsymbol{u}(\boldsymbol{q})-\boldsymbol{u}(\boldsymbol{x})]\,d\boldsymbol{q}\,d\boldsymbol{p}\\ &-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\gamma(\boldsymbol{x}-\boldsymbol{p})+\gamma(\boldsymbol{q}-\boldsymbol{p}))\eta(\boldsymbol{q}-\boldsymbol{x})[\boldsymbol{u}(\boldsymbol{q})-\boldsymbol{u}(\boldsymbol{p})]\,d\boldsymbol{q}\,d\boldsymbol{p},\end{aligned}$$

which is inspired the state-based formulation of peridynamics (again, the integration is performed on each component of $\boldsymbol{u}$). As motivated by the physical considerations above, this operator captures effects from a wider and more diverse range set of interactions by looking at cumulative effects modeled through two integral operators with two (possibly different) kernels, γ and η. By incorporating two kernels the operator gains an adjustable degree of flexibility that is important in applications, thus increasing the physical relevance of the model. The engineering and computational communities have provided us with many studies for state-based models (see [**25, 19, 15, 9**] and also the overview paper [**21**]), but the theoretical investigations of these doubly nonlocal operators are still in their early stages.

1.1. Why a new nonlocal Laplacian? Significance of this paper. This new nonlocal Laplacian was inspired by three particular choices for kernels given by Silling in [**22**]; the examples concern elastic materials (in bond-based framework), linear fluids, and linear isotropic solids. At a mathematical level the state-based Laplacian is a double convolution-type operator, which generalizes the operator (1.1), while also providing a "decomposition" of the operator with respect to the kernels γ and η. The role of each kernel is discussed from a physical, as well as a mathematical point of view. Additionally, by writing the state-based Laplacian in convolution form we obtain an operator that is well-defined on spaces of very irregular functions, even on the space of distributions; see Section 3.1.1.

To summarize, the main contributions of this paper are:

• At a *physical level* we introduce a mathematical operator that captures nonlocal effects in materials or phenomena that are more general than the ones modeled with the single integral, bond-based operator. While this operator arises naturally from the (very) general state-based formulation, it allows us through its specific form involving two kernels to incorporate a variety of examples. Thus we introduce a framework in which the nonlocal Laplacian can model very different materials, or even different behavior. In this more general context we have the ability to study transitional behavior from one class of phenomena to another, as well as the transition from one type of material to another.

• At a *mathematical level* the double convolution operator gives us a novel way to model physical behavior in the space of discontinuous functions or distributions. The transition to "smooth" behavior can be studied through convergence results of the nonlocal operator to the classical Laplacian as the horizons of interaction shrink to zero. We obtain explicit rates of convergence and we discuss below the importance of regularity for functions on which the nonlocal operator is applied.

Finally, we make note of a couple of distinctions between this operator and other operators. First, the structure of the state-based Laplacian resembles the nonlocal biharmonic introduced in [**18**], due to the presence of the double nonlocality. However, we show that the doubly nonlocal Laplacian approaches a second and not a fourth-order differential operator. This aspect will be discussed in more detail in Remark 3.3. Also, $\mathcal{L}^s_{\gamma\eta}$ is a nonlocal version of a Laplace type operator and not of the Navier operator from elasticity, as it is missing the nonlocal counterpart of the $\nabla \operatorname{div} \boldsymbol{u}$ term [**16**].

1.2. Organization of the paper. The paper is structured in the following way. In Section 2 we give a brief overview of the bond-based theory of periydnamics followed by the extension to the state-based version of peridynamics, and show how our new operator arises naturally in this setting. We introduce the state-based Laplacian in Section 3 where we also include a discussion about the convolution form of the operator, as well as the mathematical and physical significance of the kernels. Section 4 contains a derivation for the scaling of the operator and the main results regarding (interior) convergence of this operator to Δ, its classical counterpart. The main theorems presented give precise rates of convergence in terms of the two horizons δ and ε of the kernels γ and η, respectively. The results are proven in the one-dimensional case for analytic functions, as well as in higher dimensions under less restrictive regularity assumptions. We conclude in Section 5 with a discussion of the results obtained in a physical and mathematical context, and directions for future work.

2. Background: bond-based and state-based peridynamic theory.

2.1. Bond-based peridynamic theory. In the original formulation of peridynamics, introduced by Silling [**20**], each point $\boldsymbol{x}$ interacts with all its neighbors within a domain, $\mathcal{H}_{\boldsymbol{x}}$, taken to be a ball of radius δ centered at $\boldsymbol{x}$. If $\boldsymbol{p} \in \mathcal{H}_{\boldsymbol{x}}$ then $\boldsymbol{\zeta} = \boldsymbol{p} - \boldsymbol{x}$ is called a bond for the point $\boldsymbol{x}$.

In this context consider the cumulative force that is acting on $\boldsymbol{x}$ through its neighbors inside the horizon, a force that is expressed through integral operators. By replacing differential operators with integral operators we allow low-regularity solutions to satisfy elasticity models. The bond-based peridynamics equation of elasticity as introduced by Silling [**20**] is given by

$$\rho(\boldsymbol{x})\ddot{\boldsymbol{u}}(\boldsymbol{x},t) = \int_{\mathcal{H}_{\boldsymbol{x}}} \boldsymbol{f}(\boldsymbol{u}(\boldsymbol{q},t) - \boldsymbol{u}(\boldsymbol{x},t), \boldsymbol{q} - \boldsymbol{x})\, dV_{\boldsymbol{x}} + \boldsymbol{b}(\boldsymbol{x},t),$$

where ρ is material density, $\boldsymbol{u}$ is the displacement vector field, and $\boldsymbol{f}$ gives the force vector that the particle $\boldsymbol{q}$ exerts on the particle $\boldsymbol{x}$. The form of $\boldsymbol{f}$ embodies the constitutive information of the material. However, as pointed out in **[23]** this system assumes that any two particles interact only through a central potential, a prohibitive type of interaction which eventually restricts the Poisson ratio of the material to $\frac{1}{4}$. The state-based theory of peridynamics overcomes this issue and generalizes the bond-based theory. The connection between the Laplace type operators that appear in each of these formulations is one of the goals of this work and is studied further in Section 3.4.

2.2. State-based peridynamic theory. The state-based theory of peridynamics was introduced in **[23]** and it allows indirect force interactions of a neighbor with its neighbor's neighbors. A given point $\boldsymbol{x}$ will be affected directly by its neighbors $\boldsymbol{p}$, as well as indirectly, by the neighbors $\boldsymbol{q}$ of $\boldsymbol{p}$ through the point $\boldsymbol{p}$ (see Figure 1). Mathematically, the interactions of the point $\boldsymbol{x}$ will be expressed through double integrals over the product space $\mathcal{B}_\delta(\boldsymbol{x}) \times \mathcal{B}_\varepsilon(\boldsymbol{p})$, for every point $\boldsymbol{p}$ in the horizon of $\boldsymbol{x}$. (Here $\mathcal{B}_\delta(\boldsymbol{x})$ denotes the ball of radius δ centered at $\boldsymbol{x}$.) Thus, the points acting on $\boldsymbol{x}$ can be $\varepsilon + \delta$ distance away from $\boldsymbol{x}$. This setting allows a very general approach to modeling that can incorporate a wide variety of physical behavior, which is achieved through the use of very general operators called peridynamic states **[23]**.

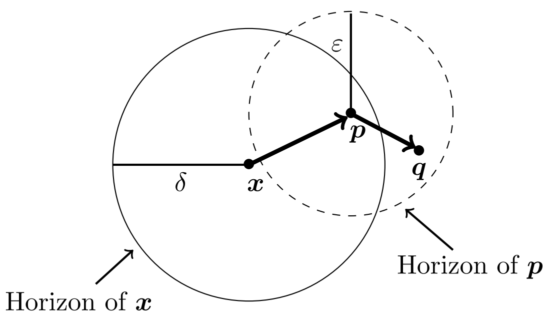

FIGURE 1. The indirect interaction of point $\boldsymbol{q}$ on point $\boldsymbol{x}$ through their common neighbor $\boldsymbol{p}$.

2.3. State-based peridynamic model and its linearization. The state-based Laplacian arises naturally from the state-based formulation which is described below, first in its most general form and then in its linearized form.

The displacement from the equilibrium position of a point $\boldsymbol{x}$ in the body $\mathcal{B}$ at time $t \geq 0$, denoted by $\boldsymbol{u}(\boldsymbol{x}, t)$, is described by the equation

$$\rho(\boldsymbol{x})\ddot{\boldsymbol{u}}(\boldsymbol{x}, t) = \int_{\mathcal{H}_{\S}} \{\boldsymbol{T}[\boldsymbol{x}, t]\langle \boldsymbol{q} - \boldsymbol{x}\rangle - \boldsymbol{T}[\boldsymbol{q}, t]\langle \boldsymbol{x} - \boldsymbol{q}\rangle\}\, dV_{\boldsymbol{q}} + \boldsymbol{b}(\boldsymbol{x}, t), \tag{2.1}$$

where ρ is material density and $\boldsymbol{b}$ is a prescribed body force density field. Above the operator $\boldsymbol{T}$ is called a vector state which when computed at the point $\boldsymbol{x}$ is applied to a bond $\boldsymbol{q} - \boldsymbol{x}$ whose resulting action is the force which $\boldsymbol{q}$ exerts on $\boldsymbol{x}$. Thus the right-hand side of (2.1) describes the cumulative effect of all action-reaction forces between $\boldsymbol{x}$ and its neighbors, and provides a very general framework for incorporating the material constitutive restrictions. A linearized version of this model is obtained by introducing a double state-kernel $\mathbb{K}$, at a point $\boldsymbol{x}$, scalar valued, which weighs the interactions between two bonds, $\boldsymbol{p} - \boldsymbol{x}$ and $\boldsymbol{q} - \boldsymbol{x}$, whose output is denoted by $\mathbb{K}[\boldsymbol{x}]\langle \boldsymbol{p} - \boldsymbol{x}, \boldsymbol{q} - \boldsymbol{x}\rangle$. For a detailed discussion of states and the linearization of the state-based formulation see **[22]**. The resulting equation after linearization is given by

$$\begin{aligned}\rho(\boldsymbol{x})\ddot{\boldsymbol{u}}(\boldsymbol{x}, t) = &\int_{\mathcal{B}}\int_{\mathcal{B}} \mathbb{K}[\boldsymbol{x}]\langle \boldsymbol{p}-\boldsymbol{x}, \boldsymbol{q}-\boldsymbol{x}\rangle(\boldsymbol{u}(\boldsymbol{q}, t)-\boldsymbol{u}(\boldsymbol{x}, t))\, dV_{\boldsymbol{q}}\, dV_{\boldsymbol{p}} \\ &-\int_{\mathcal{B}}\int_{\mathcal{B}} \mathbb{K}[\boldsymbol{p}]\langle \boldsymbol{x}-\boldsymbol{p}, \boldsymbol{q}-\boldsymbol{p}\rangle(\boldsymbol{u}(\boldsymbol{q}, t)-\boldsymbol{u}(\boldsymbol{p}, t))\, dV_{\boldsymbol{q}}\, dV_{\boldsymbol{p}} \\ &+\boldsymbol{b}(\boldsymbol{x}, t).\end{aligned} \tag{2.2}$$

For a simple material, the equation above represents a linearized state-based model for an elastic material if and only if $\mathbb{K}$ is symmetric, i.e., for any two bonds $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ which share the same application point, $\mathbb{K}[\boldsymbol{x}]\langle \boldsymbol{\xi}, \boldsymbol{\zeta}\rangle = \mathbb{K}[\boldsymbol{x}]\langle \boldsymbol{\zeta}, \boldsymbol{\xi}\rangle$. See the discussion in **[22,** Section 4.2, Proposition 4.1].

In **[22]** several choices of the state-kernel $\mathbb{K}$ are considered, each of them leading to a different physical model. For $\mathbb{K}[\boldsymbol{x}]\langle \boldsymbol{\xi}, \boldsymbol{\zeta}\rangle$ given in terms of a Dirac mass supported at $\boldsymbol{\zeta} = \boldsymbol{\xi}$, one recovers the peridynamic bond-based formulation. Below we introduce a generalization of this particular example which gives rise to a new Laplace-type operator.

The definition and properties of this new operator, together with the connections to nonlocal and local Laplacians are discussed below.

3. A doubly nonlocal Laplacian operator. Motivated by the discussion in the previous section we consider the state-kernel $\mathbb{K}$ given by

$$\mathbb{K}[\boldsymbol{x}]\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle := [\gamma(\boldsymbol{\xi})+\gamma(\boldsymbol{\zeta})]\,\eta(\boldsymbol{\zeta}-\boldsymbol{\xi}), \tag{3.1}$$

where γ and η are symmetric functions (i.e., $\gamma(-\boldsymbol{\zeta}) = \gamma(\boldsymbol{\zeta})$ and $\eta(-\boldsymbol{\xi}) = \eta(\boldsymbol{\xi})$). Taking $\boldsymbol{\xi} = \boldsymbol{p}-\boldsymbol{x}$ and $\boldsymbol{\zeta} = \boldsymbol{q}-\boldsymbol{x}$, (3.1) becomes

$$\mathbb{K}[\boldsymbol{x}]\langle\boldsymbol{p}-\boldsymbol{x},\boldsymbol{q}-\boldsymbol{x}\rangle = [\gamma(\boldsymbol{p}-\boldsymbol{x})+\gamma(\boldsymbol{q}-\boldsymbol{x})]\,\eta(\boldsymbol{q}-\boldsymbol{p}). \tag{3.2}$$

3.1. Introduction of the state-based Laplacian. We are now in position to formally introduce the new Laplace-type operator.

Definition 3.1. We define the nonlocal state operator $\mathcal{L}^s_{\gamma\eta}$ with kernels γ and η, to be the operator given by

$$\begin{aligned}\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}](\boldsymbol{x}) &= \sigma_{\gamma\eta}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\gamma(\boldsymbol{p}-\boldsymbol{x})+\gamma(\boldsymbol{q}-\boldsymbol{x}))\eta(\boldsymbol{q}-\boldsymbol{p})[\boldsymbol{u}(\boldsymbol{q})-\boldsymbol{u}(\boldsymbol{x})]\,d\boldsymbol{q}\,d\boldsymbol{p}\\ &\quad -\sigma_{\gamma\eta}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\gamma(\boldsymbol{x}-\boldsymbol{p})+\gamma(\boldsymbol{q}-\boldsymbol{p}))\eta(\boldsymbol{q}-\boldsymbol{x})[\boldsymbol{u}(\boldsymbol{q})-\boldsymbol{u}(\boldsymbol{p})]\,d\boldsymbol{q}\,d\boldsymbol{p},\end{aligned} \tag{3.3}$$

where $\sigma_{\gamma\eta}$ is a normalizing factor which is given by (4.1).

In section 4 the scaling $\sigma_{\gamma\eta}$ will be determined for kernels γ,η with finite radii of interaction, δ,ε such that

$$|\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}](\boldsymbol{x})-\Delta\boldsymbol{u}(\boldsymbol{x})|\to 0 \quad \text{as } \delta,\varepsilon\to 0,$$

for $\boldsymbol{u}$ sufficiently smooth and for every point $\boldsymbol{x}$ in the domain situated at a distance larger than $\delta+\varepsilon$ away from the boundary.

3.1.1. Convolution structure of the operator. Assuming that γ and η are L^1 integrable then the state-based Laplacian defined in (3.3) can be expressed in terms of double and single convolutions as follows:

$$\frac{\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}]}{2\sigma_{\gamma\eta}} = (\gamma*\eta*\boldsymbol{u})-(\eta*\boldsymbol{u})\|\gamma\|_{L^1}+(\gamma*\boldsymbol{u})\|\eta\|_{L^1}-\boldsymbol{u}\|\gamma\|_{L^1}\|\eta\|_{L^1}. \tag{3.4}$$

The convolution above is performed component wise, so each component of a vector is convolved with the scalar kernels. The expression (3.4) is similar to the convolution form of the bond-based Laplacian from (1.1) as expressed by

$$\mathcal{L}_\mu[\boldsymbol{u}] = \mu * \boldsymbol{u} - \boldsymbol{u}\|\mu\|_{L^1}.$$

For a physical interpretation of the operator $\mathcal{L}_\mu$ when μ is a probability measure, in the context of nonlocal diffusion, see [**1**].

Remark 3.2. The convolution formulation (3.4) shows that the operator $\mathcal{L}^s_{\gamma\eta}$ can be conveniently defined for functions $\boldsymbol{u}$ of different smoothness levels depending on choices of γ and η. In particular, γ and η in C^∞ will allow choosing $\boldsymbol{u}$ less smooth (even a distribution), and vice-versa. The support for each of the kernels γ and η could be taken to be unbounded, but for applications linked to peridynamics, the finite horizon is the relevant choice. Moreover, γ and η can be chosen to be Dirac masses, or derivatives of Dirac masses, as shown below.

Remark 3.3. Note that although the double integral form of the operator in its (3.3) or (3.4) form implies similarity to the biharmonic operator

$$\mathcal{B}_\mu[\boldsymbol{u}] = \mathcal{L}^2_\mu[\boldsymbol{u}] = \boldsymbol{u} * \mu * \mu - 2\boldsymbol{u} * \mu\|\boldsymbol{u}\|_{L^1} + \boldsymbol{u}\|\mu\|^2_{L^1},$$

introduced in [**18**], the convolution formulation of $\mathcal{L}^s_{\gamma\eta}$ clearly shows that no choice of kernels γ and η will yield the nonlocal biharmonic. Indeed, in order to eliminate the single convolution term one would have to choose a kernel that would also eliminate the double convolution term (single convolution is associated with bond-based Laplacian, while double convolution is associated with the state-based Laplacian). Finally, the doubly nonlocal state-based Laplacian will be shown to converge to a second-order differential operator, while the nonlocal biharmonic provides and approximation to the classical biharmonic Δ^2.

3.2. Kernels of the state-based Laplacian. Note from (3.1) that while $\mathbb{K}$ is symmetric with respect to the bonds $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, i.e., $\mathbb{K}[\boldsymbol{x}]\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle = \mathbb{K}[\boldsymbol{x}]\langle\boldsymbol{\zeta},\boldsymbol{\xi}\rangle$, the kernels γ, η play different roles in describing the dynamics. Indeed, the kernel elongations of the bonds $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are measured by the kernel γ, while η accounts for the interdependence between $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$. Thus the choice $\eta(\boldsymbol{\zeta}-\boldsymbol{\xi}) = \delta_0(\boldsymbol{\zeta}-\boldsymbol{\xi})$, where δ_0 is the Dirac mass centered at the origin, will yield the bond-based model, [**22**]. With the same choice for η, and γ given by two derivatives of the Dirac mass, we obtain the classical Laplacian, [**4**]. These connections are strengthened below as we show convergence of the operator to the classical Laplacian.

As previously done for bond-based peridynamics models, we will consider bounded regions of interactions for both stretching and bond interdependence effects, as given by γ, respectively η. Our specific assumptions for the kernels are given below.

Assumption 1. Assume that γ and η are nonnegative radial functions, so with an abuse of notation we write $\gamma(\boldsymbol{\xi}) = \gamma(|\boldsymbol{\xi}|)$ and $\eta(\boldsymbol{\zeta}) = \eta(|\boldsymbol{\zeta}|)$. Assume that γ is supported inside the ball of radius δ and that η is supported inside the ball of radius ε so that we have

$$\gamma(|\boldsymbol{\xi}|) = 0 \text{ for } |\boldsymbol{\xi}| > \delta, \quad \text{and} \quad \eta(|\boldsymbol{\zeta}|) = 0 \text{ for } |\boldsymbol{\zeta}| > \varepsilon.$$

Assumption 2. We consider specific rational forms for γ and η that allow us to explicitly compute the scaling for the operator $\mathcal{L}^s_{\gamma\eta}$ which gives the convergence to the classical Laplacian. For $\varepsilon, \delta > 0$ and $\alpha, \beta < n$, the choices

$$\gamma(\boldsymbol{\xi}) = \begin{cases} \dfrac{1}{|\boldsymbol{\xi}|^\alpha}, & |\boldsymbol{\xi}| \leq \delta, \\ 0, & |\boldsymbol{\xi}| > \delta, \end{cases} \qquad \eta(\boldsymbol{\zeta}) = \begin{cases} \dfrac{1}{|\boldsymbol{\zeta}|^\beta}, & |\boldsymbol{\zeta}| \leq \varepsilon, \\ 0, & |\boldsymbol{\zeta}| > \varepsilon, \end{cases} \tag{3.5}$$

produce the state-kernel $\mathbb{K}$

$$\mathbb{K}[\boldsymbol{x}]\langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle = \left(\frac{1}{|\boldsymbol{\xi}|^\alpha} + \frac{1}{|\boldsymbol{\zeta}|^\alpha} \right) \frac{1}{|\boldsymbol{\zeta} - \boldsymbol{\xi}|^\beta}, \tag{3.6}$$

for all $|\boldsymbol{\xi}| < \delta$ and $|\boldsymbol{\zeta} - \boldsymbol{\xi}| < \varepsilon$.

Next we introduce two functions $\pi, \theta : (0, \infty) \to [0, \infty)$ related to our kernels γ and respectively η, which will be needed for the proof of our convergence result to allow us to move the derivatives on the function $\boldsymbol{u}$ through integration by parts. They are selected such that they satisfy

$$\nabla_{\boldsymbol{y}} \pi(|\boldsymbol{y}|) = \boldsymbol{y} \gamma(\boldsymbol{y}), \quad \pi(\delta) = 0, \tag{3.7}$$

and

$$\nabla_{\boldsymbol{r}} \theta(|\boldsymbol{r}|) = \boldsymbol{r} \eta(\boldsymbol{r}), \quad \eta(\varepsilon) = 0. \tag{3.8}$$

With the same abuse of notation for radial functions, we have that π and θ are given explicitly by

$$\begin{aligned} \pi(\boldsymbol{y}) = \pi(|\boldsymbol{y}|) &:= \int_{\delta}^{|\boldsymbol{y}|} \lambda\gamma(\lambda)\, d\lambda, \\ \theta(\boldsymbol{r}) = \theta(|\boldsymbol{r}|) &:= \int_{\varepsilon}^{|\boldsymbol{r}|} \rho\eta(\rho)\, d\rho. \end{aligned} \tag{3.9}$$

Under Assumption 2 we obtain

$$\pi(\boldsymbol{\xi}) = \begin{cases} \dfrac{|\boldsymbol{\xi}|^{2-\alpha} - \delta^{2-\alpha}}{2-\alpha}, & \text{if } \alpha \neq 2, \\ \ln(|\boldsymbol{\xi}|/\delta), & \text{if } \alpha = 2, \end{cases} \tag{3.10}$$

$$\theta(\boldsymbol{\zeta}) = \begin{cases} \dfrac{|\boldsymbol{\zeta}|^{2-\beta} - \varepsilon^{2-\beta}}{2-\beta}, & \text{if } \beta \neq 2, \\ \ln(|\boldsymbol{\zeta}|/\varepsilon), & \text{if } \beta = 2. \end{cases} \tag{3.11}$$

We continue to use $\mathcal{B}_\delta$ and $\mathcal{B}_\varepsilon$ to denote the balls of respective radii centered at zero.

Lemma 3.4. *Under Assumption 1 with π and θ satisfying* (3.7) *and* (3.8), *we have*

$$\frac{1}{n}\int_{\mathcal{B}_\delta} \boldsymbol{y}^2\gamma(\boldsymbol{y})\, dy = -\int_{\mathcal{B}_\delta} \pi(\boldsymbol{y})\, dy, \tag{3.12}$$

$$\frac{1}{n}\int_{\mathcal{B}_\varepsilon} \boldsymbol{r}^2\eta(\boldsymbol{r})\, d\boldsymbol{r} = -\int_{\mathcal{B}_\varepsilon} \theta(\boldsymbol{r})\, d\boldsymbol{r}. \tag{3.13}$$

Proof. We prove the first equality, the second follows in a similar fashion. By taking the inner product of (3.7) with $\boldsymbol{y}$ we obtain

$$\boldsymbol{y} \cdot \nabla_{\boldsymbol{y}}\pi(\boldsymbol{y})\, dy = \boldsymbol{y}^2\gamma(\boldsymbol{y}).$$

Integration with respect to $\boldsymbol{y}$ on $\mathcal{B}_\delta$ yields

$$\int_{\mathcal{B}_\delta} \boldsymbol{y} \cdot \nabla_{\boldsymbol{y}}\pi(\boldsymbol{y})\, dy = \int_{\mathcal{B}_\delta} \boldsymbol{y}^2\gamma(\boldsymbol{y})\, dy.$$

By performing an integration by parts on the left side, where ν is the normal derivative in the $\boldsymbol{y}$ direction, we have

$$\int_{\mathcal{B}_\delta} \boldsymbol{y}\cdot\nabla_{\boldsymbol{y}}\pi(\boldsymbol{y})\,dy = \int_{\partial\mathcal{B}_\delta}\pi(\boldsymbol{y})\boldsymbol{y}\cdot\nu\,d\boldsymbol{y} - \int_{\mathcal{B}_\delta}\operatorname{div}(\boldsymbol{y})\cdot\pi(\boldsymbol{y})\,dy$$
$$= \int_{\partial\mathcal{B}_\delta}\pi(\boldsymbol{y})\boldsymbol{y}\cdot\nu\,d\boldsymbol{y} - n\int_{\mathcal{B}_\delta}\pi(\boldsymbol{y})\,dy.$$

For $\boldsymbol{y}\in\partial\mathcal{B}_\delta$ we have $\pi(\boldsymbol{y})=\pi(\delta)=0$; thus (3.12) holds. □

In order to employ the functions π and θ in the proof of our convergence result, we will need a different formulation for the state-based Laplacian, which is obtained in the next subsection.

3.3. Rewriting the Laplacian. The new expression for the state-based Laplacian will more easily allow us to identify the domains of integration for the variables and simplify the integrand. This more convenient form is given by Proposition 3.1.

Proposition 3.1. *Under Assumption 1, the state-based Laplacian can be written in the following form for all* $\boldsymbol{x}\in\mathbb{R}^n$*:*

$$\begin{aligned}(3.14)\quad &\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}](\boldsymbol{x})\\ &= 2\sigma(\varepsilon,\delta)\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(\boldsymbol{y})\eta(\boldsymbol{r})\\ &\quad\times[\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{r})-\boldsymbol{u}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r})+\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y})]\,d\boldsymbol{r}\,d\boldsymbol{y}.\end{aligned}$$

Proof. By rearranging (3.3) we obtain

$$\begin{aligned}(3.15)\quad \frac{\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}]}{2\sigma(\varepsilon,\delta)}(\boldsymbol{x}) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\gamma(\boldsymbol{p}-\boldsymbol{x})\eta(\boldsymbol{q}-\boldsymbol{p})[\boldsymbol{u}(\boldsymbol{q})-\boldsymbol{u}(\boldsymbol{x})]\,d\boldsymbol{q}\,d\boldsymbol{p}\\ &\quad-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\gamma(\boldsymbol{x}-\boldsymbol{p})\eta(\boldsymbol{q}-\boldsymbol{x})[\boldsymbol{u}(\boldsymbol{q})-\boldsymbol{u}(\boldsymbol{p})]\,d\boldsymbol{q}\,d\boldsymbol{p}\\ &\quad+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\gamma(\boldsymbol{q}-\boldsymbol{x})\eta(\boldsymbol{q}-\boldsymbol{p})[\boldsymbol{u}(\boldsymbol{q})-\boldsymbol{u}(\boldsymbol{x})]\,d\boldsymbol{q}\,d\boldsymbol{p}\\ &\quad-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\gamma(\boldsymbol{q}-\boldsymbol{p})\eta(\boldsymbol{q}-\boldsymbol{x})[\boldsymbol{u}(\boldsymbol{q})-\boldsymbol{u}(\boldsymbol{p})]\,d\boldsymbol{q}\,d\boldsymbol{p}.\end{aligned}$$

We perform a change of variables in each of the above integrals:

- $\boldsymbol{y} := \boldsymbol{p} - \boldsymbol{x}$ and $\boldsymbol{r} := \boldsymbol{q} - \boldsymbol{p}$, in the first integral,
- $\boldsymbol{y} := \boldsymbol{p} - \boldsymbol{x}$ and $\boldsymbol{r} := \boldsymbol{q} - \boldsymbol{x}$, in the second integral,
- $\boldsymbol{y} := \boldsymbol{q} - \boldsymbol{x}$ and $\boldsymbol{r} := \boldsymbol{p} - \boldsymbol{q}$, in the third integral,
- $\boldsymbol{y} := \boldsymbol{q} - \boldsymbol{p}$ and $\boldsymbol{r} := \boldsymbol{q} - \boldsymbol{x}$ in the fourth integral.

The resulting form is

$$\begin{aligned}
&\frac{\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}]}{2\sigma(\varepsilon,\delta)}(\boldsymbol{x}) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \gamma(\boldsymbol{y})\eta(\boldsymbol{r})[\boldsymbol{u}(\boldsymbol{y}+\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) + \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y})]\, d\boldsymbol{r}\, d\boldsymbol{y} \\
&+ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \gamma(\boldsymbol{y})\eta(\boldsymbol{r})[\boldsymbol{u}(\boldsymbol{y}+\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{r}+\boldsymbol{x}) + \boldsymbol{u}(\boldsymbol{r}+\boldsymbol{x}-\boldsymbol{y})]\, d\boldsymbol{r}\, d\boldsymbol{y}.
\end{aligned}$$

A final change of variables in

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \gamma(\boldsymbol{y})\eta(\boldsymbol{r})\boldsymbol{u}(\boldsymbol{r}+\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{r}\, d\boldsymbol{y},$$

and the fact that $\gamma(\boldsymbol{y}) = \gamma(-\boldsymbol{y})$, together with Assumption 1, gives (3.14). □

4. Convergence of the operators. The main result of this section shows that the state-based Laplacian applied to sufficiently smooth functions provides an approximation for the classical Laplacian applied to the same function, for δ, and ε close to zero. In fact, under C^4 assumptions for the function, we exhibit a rate of convergence for the error of this approximation that is quadratic with respect to the kernel horizons (Theorem 4.2). For C^2 functions we obtain simple convergence (of unspecified order), while for $C^{2,\alpha}$ with $0 < \alpha < 1$ the rate of convergence is proportional to the horizon raised to exponent α (see Theorem 4.5).

The scaling of the state-based Laplacian needed for this approximation will be shown to satisfy

$$\sigma(\varepsilon,\delta) = -\frac{1}{2\int_{\mathcal{B}_\varepsilon}\eta(\boldsymbol{r})\, d\boldsymbol{r}\int_{\mathcal{B}_\delta}\pi(\boldsymbol{y})\, dy}, \tag{4.1}$$

where η is the kernel in $\mathcal{L}^s_{\gamma\eta}$ and π is the function associated with γ given by (3.7). From Lemma (3.4) we find that the scaling is equivalent to

$$\sigma(\varepsilon,\delta) = \frac{n}{2\int_{\mathcal{B}_\varepsilon}\eta(\boldsymbol{r})\,d\boldsymbol{r}\int_{\mathcal{B}_\delta}\boldsymbol{y}^2\gamma(\boldsymbol{y})\,dy}. \tag{4.2}$$

In particular, under Assumption 2 for the specific kernels of (3.5), if $\alpha \neq 2$, we have

$$\sigma(\varepsilon,\delta) = \frac{(n-\beta)(n-\alpha+2)n\varepsilon^{\beta-n}\delta^{\alpha-n-2}}{2w_{n-1}^2}, \tag{4.3}$$

where w_{n-1} is the volume of the ball in $n-1$ dimensions. We begin by showing that in one dimension the difference between the nonlocal Laplacian and the classical Laplacian, when applied to analytic functions decays at the rate $\varepsilon^2+\delta^2$.

Theorem 4.1. *Let $\Omega = (a,b) \subset \mathbb{R}$ be a bounded interval and let u be analytic in Ω, with*

$$M := \sup_{x\in\Omega} |u^{(k)}(x)| < \infty,\ k \geq 4. \tag{4.4}$$

Let γ and η satisfy Assumption 1 above, the scaling $\sigma(\varepsilon,\delta)$ be given by (4.2) with $n=1$, and let

$$\Omega' := (a+\delta+\varepsilon,\ b-\delta-\varepsilon).$$

We have

$$\|\mathcal{L}^s_{\gamma\eta}[u] - \Delta u\|_{L^\infty(\Omega')} < C(\varepsilon^2+\delta^2), \tag{4.5}$$

as $\delta,\varepsilon \to 0$, where the constant C depends on M given by (4.4).

Proof. From (3.14) we have

$$\frac{\mathcal{L}^s_{\gamma\eta}[u]}{2\sigma(\varepsilon,\delta)}(x) = \int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)\big[u(x+y+r)-u(x)-[u(x+r)-u(x+y)]\big]\,dr\,dy.$$

Using the analytic expansion for u around x in the first term and around $x+y$ in the third term we obtain

$$\begin{aligned}
&\frac{\mathcal{L}^s_{\gamma\eta}[u]}{2\sigma(\varepsilon,\delta)}(x)\\
&=\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)\bigg[u'(x)(y+r)+u''(x)\frac{(y+r)^2}{2}+u'''(x)\frac{(y+r)^3}{3!}\\
&\qquad\qquad+u^{(4)}(x)\frac{(y+r)^4}{4!}+\sum_{n=5}^{\infty}u^{(n)}(x)\frac{(y+r)^n}{n!}\bigg]\,dr\,dy\\
&-\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)\bigg[u'(x+y)(r-y)+u''(x+y)\frac{(r-y)^2}{2}+u'''(x+y)\frac{(r-y)^3}{3!}\\
&\qquad\qquad+u^{(4)}(x+y)\frac{(r-y)^4}{4!}+\sum_{n=5}^{\infty}u^{(n)}(x+y)\frac{(r-y)^n}{n!}\bigg]\,dr\,dy.
\end{aligned}$$

Since $\gamma(y)$ and $\eta(r)$ are symmetric, each of the term that is an odd power in y or r in the first integral on the right-hand side of the preceding display is antisymmetric, with respect to y, or respectively r; hence, they vanish after integration. Similarly, in the second integral the terms containing odd power of r are therefore they also disappear (note that the same does not hold for y due to the presence of y in $u(x+y)$). We obtain

$$\begin{aligned}
&\frac{\mathcal{L}^s_{\gamma\eta}[u]}{2\sigma(\varepsilon,\delta)}\\
&=\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)\bigg[u''(x)\frac{y^2+r^2}{2}+u^{(4)}(x)\frac{y^4+6y^2r^2+r^4}{4!}\\
&\qquad\qquad+\sum_{n=3}^{\infty}u^{(2n)}(x)\sum_{i=0}^{n}\binom{2n}{2i}\frac{y^{2n-2i}r^{2i}}{(2n)!}\bigg]\,dr\,dy\\
&-\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)\bigg[-u'(x+y)y+u''(x+y)\frac{r^2+y^2}{2}.\\
&\qquad\qquad-u'''(x+y)\frac{3r^2y+y^3}{3!}+u^{(4)}(x+y)\frac{r^4+6r^2y^2+y^4}{4!}\\
&\qquad\qquad-\sum_{n=3}^{\infty}u^{(2n-1)}(x+y)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{y^{2n-1-2i}r^{2i}}{(2n-1)!}\\
&\qquad\qquad+\sum_{n=3}^{\infty}u^{(2n)}(x+y)\sum_{i=0}^{n}\binom{2n}{2i}\frac{y^{2n-2i}r^{2i}}{(2n)!}\bigg]\,dr\,dy.
\end{aligned}$$

Gathering the even derivative terms we have

$$\begin{aligned}
&\frac{\mathcal{L}^s_{\gamma\eta}[u]}{2\sigma(\varepsilon,\delta)}\\
&=\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)\Bigg[(u''(x)-u''(x+y))\frac{y^2+r^2}{2}\\
&\qquad\qquad + (u^{(4)}(x)-u^{(4)}(x+y))\frac{y^4+6y^2r^2+r^4}{4!}\\
&\qquad\qquad + \sum_{n=3}^{\infty}(u^{(2n)}(x)-u^{(2n)}(x+y))\sum_{i=0}^{n}\binom{2n}{2i}\frac{y^{2n-2i}r^{2i}}{(2n)!}\Bigg]\,dr\,dy\\
&+\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)\Bigg[u'(x+y)y+u'''(x+y)\frac{3r^2y+y^3}{3!}\\
&\qquad\qquad + \sum_{n=3}^{\infty}u^{(2n-1)}(x+y)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{y^{2n-1-2i}r^{2i}}{(2n-1)!}\Bigg]\,dr\,dy.
\end{aligned}$$

Employing analytic expansions in each of the even derivative terms near x gives

$$\begin{aligned}
\frac{\mathcal{L}^s_{\gamma\eta}[u]}{2\sigma(\varepsilon,\delta)} &= -\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)\Bigg[\left(u'''(x)y+\sum_{j=2}^{\infty}u^{(2+j)}(x)\frac{y^j}{j!}\right)\frac{y^2+r^2}{2}\\
&\qquad\qquad + \left(u^{(5)}(x)y+\sum_{j=2}^{\infty}u^{(4+j)}(x)\frac{y^j}{j!}\right)\frac{y^4+6y^2r^2+r^4}{4!}\\
&\qquad\qquad + \sum_{n=3}^{\infty}\left(\sum_{j=1}^{\infty}u^{(2n+j)}(x)\frac{y^j}{j!}\right)\sum_{i=0}^{n}\binom{2n}{2i}\frac{y^{2n-2i}r^{2i}}{(2n)!}\Bigg]\,dr\,dy\\
&+\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)\Bigg[u'(x+y)y+u'''(x+y)\frac{3r^2y+y^3}{3!}\\
&\qquad\qquad + \sum_{n=3}^{\infty}u^{(2n-1)}(x+y)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{y^{2n-1-2i}r^{2i}}{(2n-1)!}\Bigg]\,dr\,dy.
\end{aligned}$$

As before, each of the odd power terms (in y) in the first integral are antisymmetric and vanish. Simplifying produces

$$\frac{\mathcal{L}^s_{\gamma\eta}[u]}{2\sigma(\varepsilon,\delta)}$$

$$= -\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon} \gamma(y)\eta(r)\left[\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} u^{(2n+2j)}(x)\frac{y^{2j}}{(2j)!}\right)\sum_{i=0}^{n}\binom{2n}{2i}\frac{y^{2n-2i}r^{2i}}{(2n)!}\right] dr\,dy$$

$$+\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon} \gamma(y)\eta(r)\left[u'(x+y)y + u'''(x+y)\frac{3r^2y+y^3}{3!} + \sum_{n=3}^{\infty} u^{(2n-1)}(x+y)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{y^{2n-1-2i}r^{2i}}{(2n-1)!}\right] dr\,dy.$$

Next, we perform an analytic expansion around x for each of the odd derivatives in the second integral to obtain

$$\frac{\mathcal{L}^s_{\gamma\eta}[u]}{2\sigma(\varepsilon,\delta)}$$
$$= -\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon} \gamma(y)\eta(r)\left[\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} u^{(2n+2j)}(x)\frac{y^{2j}}{(2j)!}\right)\sum_{i=0}^{n}\binom{2n}{2i}\frac{y^{2n-2i}r^{2i}}{(2n)!}\right] dr\,dy$$
$$+\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon} \gamma(y)\eta(r)\left[\left(u'(x) + u''(x)y + \sum_{j=2}^{\infty} u^{(1+j)}(x)\frac{y^j}{j!}\right)y + \left(u'''(x) + u^{(4)}(x)y + \sum_{j=2}^{\infty} u^{(3+j)}(x)\frac{y^j}{j!}\right)\frac{(3r^2y+y^3)}{3!} + \sum_{n=3}^{\infty}\left(\sum_{j=0}^{\infty} u^{(2n-1+j)}(x)\frac{y^j}{j!}\right)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{y^{2n-1-2i}r^{2i}}{(2n-1)!}\right] dr\,dy.$$

Once again, the odd power terms (in y) in the second integral are antisymmetric and vanish. Simplifying and moving $2\sigma(\varepsilon,\delta)$ to the right side of the equation we obtain

$$\mathcal{L}^s_{\gamma\eta}[u](x) = -2\sigma(\varepsilon,\delta)\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon} \gamma(y)\eta(r)\left[\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} u^{(2n+2j)}(x)\frac{y^{2j}}{(2j)!}\right) \cdot \sum_{i=0}^{n}\binom{2n}{2i}\frac{y^{2n-2i}r^{2i}}{(2n)!}\right] dr\,dy$$
$$+ 2\sigma(\varepsilon,\delta)\left(\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon} \gamma(y)\eta(r)y^2\,dr\,dy\right)u''(x) + \text{ [continues]}$$

$$+ 2\sigma(\varepsilon,\delta)\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)\sum_{j=1}^{\infty}u^{(2+2j)}(x)\frac{y^{2j+1}}{(2j+1)!}y\,dr\,dy$$
$$+ 2\sigma(\varepsilon,\delta)\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)\Bigg[\sum_{n=2}^{\infty}\Bigg(\sum_{j=0}^{\infty}u^{(2n+2j)}(x)\frac{y^{2j+1}}{(2j+1)!}\Bigg)$$
$$\cdot\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{y^{2n-1-2i}r^{2i}}{(2n-1)!}\Bigg]\,dr\,dy.$$

The scaling given by (4.2) normalizes the coefficient of u'', so the error of the approximation is given by the remaining terms:

$$|\mathcal{L}^s_{\gamma\eta}[u](x) - u''(x)|$$
$$\leq 2M\sigma(\varepsilon,\delta)\times$$
$$\Bigg\{\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)\Bigg[\sum_{n=1}^{\infty}\Bigg(\sum_{j=1}^{\infty}\frac{|y|^{2j}}{(2j)!}\Bigg)\sum_{i=0}^{n}\binom{2n}{2i}\frac{|y|^{2n-2i}|r|^{2i}}{(2n)!}\Bigg]\,dr\,dy$$
$$+\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)\sum_{j=1}^{\infty}\frac{|y|^{2j+1}}{(2j+1)!}|y|\,dr\,dy$$
$$+\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)\Bigg[\sum_{n=2}^{\infty}\Bigg(\sum_{j=0}^{\infty}\frac{|y|^{2j+1}}{(2j+1)!}\Bigg)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{|y|^{2n-1-2i}|r|^{2i}}{(2n-1)!}\Bigg]\,dr\,dy\Bigg\},$$

where M is defined in (4.4). Since $|y| < \delta$ and $|r| < \varepsilon$, we get

$$|\mathcal{L}^s_{\gamma\eta}[u](x) - u''(x)|$$
$$\leq 2M\sigma(\varepsilon,\delta)\times$$
$$\Bigg\{\Bigg[\sum_{n=1}^{\infty}\Bigg(\sum_{j=1}^{\infty}\frac{\delta^{2j-2}}{(2j)!}\Bigg)\sum_{i=0}^{n}\binom{2n}{2i}\frac{\delta^{2n-2i}\varepsilon^{2i}}{(2n)!}\Bigg]\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)y^2\,dr\,dy$$
$$+\sum_{j=1}^{\infty}\frac{\delta^{2j}}{(2j+1)!}\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)y^2\,dr\,dy$$
$$+\Bigg[\sum_{n=2}^{\infty}\Bigg(\sum_{j=0}^{\infty}\frac{\delta^{2j}}{(2j+1)!}\Bigg)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{\delta^{2n-2-2i}\varepsilon^{2i}}{(2n-1)!}\Bigg]\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\gamma(y)\eta(r)y^2\,dr\,dy\Bigg\}.$$

Using $\sigma(\varepsilon,\delta)$ as given in (4.2) we have

$$
\begin{aligned}
&|\mathcal{L}^s_{\gamma\eta}[u](x) - u''(x)| \\
&\quad \le M\left[\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{\delta^{2j-2}}{(2j)!}\right)\sum_{i=0}^{n}\binom{2n}{2i}\frac{\delta^{2n-2i}\varepsilon^{2i}}{(2n)!}\right] + M\sum_{j=1}^{\infty}\frac{\delta^{2j}}{(2j+1)!} \\
&\qquad\qquad + M\left[\sum_{n=2}^{\infty}\left(\sum_{j=0}^{\infty}\frac{\delta^{2j}}{(2j+1)!}\right)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{\delta^{2n-2-2i}\varepsilon^{2i}}{(2n-1)!}\right].
\end{aligned}
$$

Separating the $n=1$ terms in the first set of summations, those with $j=1$ in the second summation and those with $n=2$ in the third set of summations, we obtain

$$
\begin{aligned}
&|\mathcal{L}^s_{\gamma\eta}[u](x) - u''(x)| \\
&\quad \le M\,\frac{\delta^2+\varepsilon^2}{2}\sum_{j=1}^{\infty}\frac{\delta^{2j-2}}{(2j)!} + M\left[\sum_{n=2}^{\infty}\left(\sum_{j=1}^{\infty}\frac{\delta^{2j-2}}{(2j)!}\right)\sum_{i=0}^{n}\binom{2n}{2i}\frac{\delta^{2n-2i}\varepsilon^{2i}}{(2n)!}\right] \\
&\qquad\qquad + M\frac{\delta^2}{6} + M\sum_{j=2}^{\infty}\frac{\delta^{2j-2}}{(2j)!} + M\,\frac{\delta^2+3\varepsilon^2}{6}\sum_{j=0}^{\infty}\frac{\delta^{2j}}{(2j+1)!} \\
&\qquad\qquad + M\left[\sum_{n=3}^{\infty}\left(\sum_{j=0}^{\infty}\frac{\delta^{2j}}{(2j+1)!}\right)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{\delta^{2n-2-2i}\varepsilon^{2i}}{(2n-1)!}\right].
\end{aligned}
$$

Since all of the above series are convergent for $\delta, \varepsilon < 1$ we note that each term is of order δ^2 or ε^2. Thus as δ and ε shrink to zero, $\mathcal{L}^s_{\gamma\eta}[u]$ converges to u'' at a rate of $\delta^2+\varepsilon^2$. □

Next we will present a much more general convergence result that holds in any dimension, under less regularity for $\boldsymbol{u}$, however we add an additional restriction on the support of the kernels. The ideas follow the method developed in **[6]** to show convergence of the bond-based Laplacian to the classical Laplacian.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 1$, be possibly unbounded, and for $0 < \varepsilon \le \delta$ let*

$$\Omega' := \Omega \setminus \{\boldsymbol{y} \in \Omega \mid \operatorname{dist}(\boldsymbol{y}, \partial\Omega) \le \varepsilon^2 + \delta^2\}.$$

For $\boldsymbol{u} \in C^4(\Omega)$ assume

$$M_4 := \sup_{\boldsymbol{x}\in\Omega} |\boldsymbol{u}^{(4)}(\boldsymbol{x})| < \infty. \tag{4.6}$$

Let γ and η satisfy Assumption 1, with the additional restriction that

$$c_1|\boldsymbol{y}|^{-\alpha} \leq \gamma(\boldsymbol{y}) \leq c_2|\boldsymbol{y}|^{-\alpha} \quad \text{for} \quad 0 \leq \alpha < n, \text{ and } 0 < c_1 \leq c_2.$$

Then $\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}]$ with scaling factor $\sigma(\varepsilon, \delta)$ given by (4.1) satisfies:

$$\|\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}] - \Delta\boldsymbol{u}\|_{L^\infty(\Omega')} < C\delta^2,$$

where C depends on M_4 given above.

Proof. From (3.14) we have that for every $\boldsymbol{x} \in \Omega'$

$$\frac{\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}]}{2\sigma(\varepsilon,\delta)}(\boldsymbol{x}) = 2\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon} \gamma(\boldsymbol{y})\eta(\boldsymbol{r})\big(\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x}) - [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y})]\big)\, d\boldsymbol{r}\, d\boldsymbol{y}. \tag{4.7}$$

Applying the fundamental theorem of calculus we obtain

$$\begin{aligned}\frac{\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}]}{2\sigma(\varepsilon,\delta)}(\boldsymbol{x}) &= 2\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon} \gamma(\boldsymbol{y})\eta(\boldsymbol{r})\int_0^1 [\nabla\boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r}))](\boldsymbol{y}+\boldsymbol{r})\, ds\, d\boldsymbol{r}\, d\boldsymbol{y} \\ &\quad - 2\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon} \gamma(\boldsymbol{y})\eta(\boldsymbol{r})\int_0^1 [\nabla\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))](\boldsymbol{r}-\boldsymbol{y})\, ds\, d\boldsymbol{r}\, d\boldsymbol{y},\end{aligned}$$

where $\nabla\boldsymbol{u}$ is the Jacobian matrix for $\boldsymbol{u}$. Expanding and collecting similar terms produces

$$\begin{aligned}\frac{\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}]}{2\sigma(\varepsilon,\delta)}(\boldsymbol{x}) &= \int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon} \gamma(\boldsymbol{y})\eta(\boldsymbol{r})\int_0^1 (\nabla\boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r})) + \nabla\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y})))\boldsymbol{y}\, ds\, d\boldsymbol{r}\, d\boldsymbol{y} \\ &\quad + \int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon} \gamma(\boldsymbol{y})\eta(\boldsymbol{r})\int_0^1 (\nabla\boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r})) - \nabla\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))\boldsymbol{r}\, ds\, d\boldsymbol{r}\, d\boldsymbol{y}.\end{aligned}$$

Using π and θ as defined in (3.7) and (3.8) we obtain

$$\frac{\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}]}{2\sigma(\varepsilon,\delta)} =: I_1 + I_2 \tag{4.8}$$

where

$$I_1 := \int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\int_0^1 \eta(\boldsymbol{r})\big(\nabla \boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r})) + \nabla \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))\big) \cdot \nabla_{\boldsymbol{y}}\pi(\boldsymbol{y})\, ds\, d\boldsymbol{r}\, d\boldsymbol{y},$$

$$I_2 := \int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}\int_0^1 \gamma(\boldsymbol{y})\big(\nabla \boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r})) - \nabla \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))\big) \cdot \nabla_{\boldsymbol{r}}\theta(\boldsymbol{r})\, ds\, d\boldsymbol{r}\, d\boldsymbol{y}.$$

Note that I_1 and I_2 are vector-valued quantities. Integration by parts in I_1 yields

$$\begin{aligned} I_1 = &-\int_{\mathcal{B}_\varepsilon}\int_0^1\int_{\mathcal{B}_\delta} \eta(\boldsymbol{r})\pi(\boldsymbol{y})\, \mathrm{div}_{\boldsymbol{y}}[\nabla \boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r}))]\, d\boldsymbol{y}\, ds\, d\boldsymbol{r} \\ &-\int_{\mathcal{B}_\varepsilon}\int_0^1\int_{\mathcal{B}_\delta} \eta(\boldsymbol{r})\pi(\boldsymbol{y})\, \mathrm{div}_{\boldsymbol{y}}[\nabla \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))]\, d\boldsymbol{y}\, ds\, d\boldsymbol{r} \\ &+\int_{\mathcal{B}_\varepsilon}\int_0^1\int_{\mathcal{B}_\delta} \eta(\boldsymbol{r})\pi(\boldsymbol{y})\nabla \boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r}))\frac{\boldsymbol{y}}{\delta}\, d\boldsymbol{y}\, ds\, d\boldsymbol{r} \\ &+\int_{\mathcal{B}_\varepsilon}\int_0^1\int_{\mathcal{B}_\delta} \eta(\boldsymbol{r})\pi(\boldsymbol{y})\nabla \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))\frac{\boldsymbol{y}}{\delta}\, d\boldsymbol{y}\, ds\, d\boldsymbol{r}. \end{aligned}$$

For $\boldsymbol{y} \in \partial\mathcal{B}_\delta$, we have $\pi(\boldsymbol{y}) = \pi(\delta) = 0$; thus the last two terms vanish, and I_1 becomes

$$\begin{aligned} I_1 = &-\int_{\mathcal{B}_\varepsilon}\int_0^1\int_{\mathcal{B}_\delta} \eta(\boldsymbol{r})s\Delta \boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r}))\pi(\boldsymbol{y})\, dy\, ds\, d\boldsymbol{r} \\ &-\int_{\mathcal{B}_\varepsilon}\int_0^1\int_{\mathcal{B}_\delta} \eta(\boldsymbol{r})(1-s)\Delta \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))\pi(\boldsymbol{y})\, dy\, ds\, d\boldsymbol{r}, \end{aligned}$$

which, after adding and subtracting $\Delta \boldsymbol{u}(\boldsymbol{x})$, we can write as

$$\begin{aligned} I_1 = &-\int_{\mathcal{B}_\varepsilon}\int_0^1\int_{\mathcal{B}_\delta} \eta(\boldsymbol{r})s[\Delta \boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r})) - \Delta \boldsymbol{u}(\boldsymbol{x})]\pi(\boldsymbol{y})\, dy\, ds\, d\boldsymbol{r} \\ &-\int_{\mathcal{B}_\varepsilon}\int_0^1\int_{\mathcal{B}_\delta} \eta(\boldsymbol{r})(1-s)[\Delta \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y})) - \Delta \boldsymbol{u}(\boldsymbol{x})]\pi(\boldsymbol{y})\, dy\, ds\, d\boldsymbol{r} \\ &- \Delta \boldsymbol{u}(\boldsymbol{x})\int_{\mathcal{B}_\varepsilon}\int_0^1\int_{\mathcal{B}_\delta} \eta(\boldsymbol{r})\pi(\boldsymbol{y})\, dy\, ds\, d\boldsymbol{r}. \end{aligned}$$

We use the same approach for I_2; we first integrate by parts, using the fact that $\theta(\boldsymbol{r}) = \theta(\varepsilon) = 0$ for $r \in \partial\mathcal{B}_\varepsilon$, and then add and subtract $\Delta\boldsymbol{u}(\boldsymbol{x})$ to obtain

$$I_2 = -\int_{\mathcal{B}_\delta}\int_0^1\int_{\mathcal{B}_\varepsilon} \gamma(\boldsymbol{y})s[\Delta\boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r})) - \Delta\boldsymbol{u}(\boldsymbol{x})]\theta(\boldsymbol{r})\,d\boldsymbol{r}\,ds\,d\boldsymbol{y}$$
$$+\int_{\mathcal{B}_\delta}\int_0^1\int_{\mathcal{B}_\varepsilon} \gamma(\boldsymbol{y})s[\Delta\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y})) - \Delta\boldsymbol{u}(\boldsymbol{x})]\theta(\boldsymbol{r})\,d\boldsymbol{r}\,ds\,d\boldsymbol{y}.$$

In order to make the coefficient of the Laplacian in the third integral of I_1 equal to 1, we take $\sigma(\varepsilon,\delta)$ as given by (4.1). With this choice of scaling we write

$$\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}](\boldsymbol{x}) - \Delta\boldsymbol{u}(\boldsymbol{x}) =: 2\sigma(\varepsilon,\delta)(J_1+J_2+J_3+J_4), \tag{4.9}$$

where

$$J_1 = -\int_{\mathcal{B}_\varepsilon}\int_0^1\int_{\mathcal{B}_\delta} \eta(\boldsymbol{r})\pi(\boldsymbol{y})s[\Delta\boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r})) - \Delta\boldsymbol{u}(\boldsymbol{x})]\,d\boldsymbol{y}\,ds\,d\boldsymbol{r},$$
$$J_2 = -\int_{\mathcal{B}_\varepsilon}\int_0^1\int_{\mathcal{B}_\delta} \eta(\boldsymbol{r})\pi(\boldsymbol{y})(1-s)[\Delta\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y})) - \Delta\boldsymbol{u}(\boldsymbol{x})]\,d\boldsymbol{y}\,ds\,d\boldsymbol{r},$$
$$J_3 = -\int_{\mathcal{B}_\delta}\int_0^1\int_{\mathcal{B}_\varepsilon} \gamma(\boldsymbol{y})\theta(\boldsymbol{r})s[\Delta\boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r})) - \Delta\boldsymbol{u}(\boldsymbol{x})]\,d\boldsymbol{r}\,ds\,d\boldsymbol{y},$$
$$J_4 = \int_{\mathcal{B}_\delta}\int_0^1\int_{\mathcal{B}_\varepsilon} \gamma(\boldsymbol{y})\theta(\boldsymbol{r})s[\Delta\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y})) - \Delta\boldsymbol{u}(\boldsymbol{x})]\,d\boldsymbol{r}\,ds\,d\boldsymbol{y}.$$

Again, J_1, J_2, J_3 and J_4 are vector-valued. We now look to bound each integral; we begin with J_1. Integrating by parts with respect to s and using antisymmetry of the integrands we obtain

$$J_1 = \int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta} \frac{1-s^2}{2}(\Delta\boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r})) - \Delta\boldsymbol{u}(\boldsymbol{x}))\eta(\boldsymbol{r})\pi(\boldsymbol{y})\Big|_{s=0}^{s=1}\,d\boldsymbol{y}\,d\boldsymbol{r}$$
$$-\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1 \frac{1-s^2}{2}\Delta\nabla\boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r}))(\boldsymbol{y}+\boldsymbol{r})\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,ds\,d\boldsymbol{y}\,d\boldsymbol{r},$$

which after evaluating at $s=0$ and $s=1$ in the first term gives,

$$J_1 = -\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1 \frac{(1-s^2)}{2}\Delta\nabla\boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r}))(\boldsymbol{y}+\boldsymbol{r})\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,ds\,d\boldsymbol{y}\,d\boldsymbol{r}.$$

Integrating by parts with respect to s again we obtain

$$J_1 = \int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\Big(\frac{1}{3}-\frac{s}{2}+\frac{s^3}{6}\Big)\Delta\nabla \boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r}))(\boldsymbol{y}+\boldsymbol{r})\eta(\boldsymbol{r})\pi(\boldsymbol{y})\Big|_{s=0}^{s=1} d\boldsymbol{y}\, d\boldsymbol{r}$$
$$-\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1\Big(\frac{1}{3}-\frac{s}{2}+\frac{s^3}{6}\Big)\Delta[\nabla^2 \boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r}))(\boldsymbol{y}+\boldsymbol{r})](\boldsymbol{y}+\boldsymbol{r})$$
$$\cdot\,\eta(\boldsymbol{r})\pi(\boldsymbol{y})\, ds\, d\boldsymbol{y}\, d\boldsymbol{r},$$

where ∇^2 is the Hessian tensor. Evaluating the first integral at $s=1$ yields a factor of zero, while evaluating at $s=0$ produces an anti-symmetric function which vanishes after integration. Hence, we have

$$J_1 = -\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1\Big(\frac{1}{3}-\frac{s}{2}+\frac{s^3}{6}\Big)\Delta[\nabla^2 \boldsymbol{u}(\boldsymbol{x}+s(\boldsymbol{y}+\boldsymbol{r}))(\boldsymbol{y}+\boldsymbol{r})](\boldsymbol{y}+\boldsymbol{r})$$
$$\cdot\,\eta(\boldsymbol{r})\pi(\boldsymbol{y})\, ds\, d\boldsymbol{y}\, d\boldsymbol{r},$$

Taking M_4 as defined in (4.6) we estimate the magnitude of J_1 as follows:

$$|J_1| \le M_4\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1\Big(\frac{1}{3}-\frac{s}{2}+\frac{s^3}{6}\Big)(|\boldsymbol{y}|^2+|\boldsymbol{r}|^2+2|\boldsymbol{y}\boldsymbol{r}|)\eta(\boldsymbol{r})|\pi(\boldsymbol{y})| ds\, d\boldsymbol{y}\, d\boldsymbol{r}$$
$$\le \frac{M_4}{4}\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}(|\boldsymbol{y}|^2+|\boldsymbol{r}|^2)\eta(\boldsymbol{r})|\pi(\boldsymbol{y})|\, d\boldsymbol{y}\, d\boldsymbol{r}.$$

Using the coarea formula we obtain

$$|J_1| \le \frac{M_4}{4}\bigg(n\omega_{n-1}\int_{\mathcal{B}_\varepsilon}\eta(\boldsymbol{r})\, d\boldsymbol{r}\int_0^\delta \lambda^{n+1}|\pi(\lambda)|\, d\lambda + n\omega_{n-1}\int_0^\varepsilon \rho^{n+1}\eta(\rho)\, d\rho\int_{\mathcal{B}_\delta}|\pi(\boldsymbol{y})|\, d\boldsymbol{y}\bigg) \tag{4.10}$$
$$= \frac{M_4}{4}(\delta^2+\varepsilon^2)\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\eta(\boldsymbol{r})|\pi(\boldsymbol{y})|\, d\boldsymbol{y}\, d\boldsymbol{r},$$

where in the last equality we used also the fact that

$$\lambda^n \le \lambda^{n-1}\delta \text{ and } \rho^n \le \rho^{n-1}\varepsilon. \tag{4.11}$$

Multiplying by $2\sigma(\varepsilon,\delta)$ given by (4.1) gives the bound

$$|2\sigma(\varepsilon,\delta)J_1| \le \frac{M_4}{4}(\delta^2+\varepsilon^2).$$

To bound on J_2, we first integrate by parts with respect to s to obtain

$$
\begin{aligned}
J_2 &= \\
&-\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta} \frac{2s-s^2-1}{2}(\Delta\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))-\Delta\boldsymbol{u}(\boldsymbol{x}))\eta(\boldsymbol{r})\pi(\boldsymbol{y})\Big|_{s=0}^{s=1} d\boldsymbol{y}\,d\boldsymbol{r}\\
&+\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1 \frac{2s-s^2-1}{2}(\Delta\nabla\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))(\boldsymbol{r}-\boldsymbol{y}))\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,dy\,d\boldsymbol{r},
\end{aligned}
$$

after which evaluation at $s=0$ and $s=1$ gives

$$
\begin{aligned}
J_2 = -\frac{1}{2}&\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}(\Delta\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y})-\Delta\boldsymbol{u}(\boldsymbol{x}))\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,dy\,d\boldsymbol{r}\\
&+\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1 \frac{2s-s^2-1}{2}\big(\Delta\nabla\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))-\Delta\nabla\boldsymbol{u}(\boldsymbol{x})\big)\\
&\qquad\qquad\cdot(\boldsymbol{r}-\boldsymbol{y})\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,ds\,d\boldsymbol{y}\,d\boldsymbol{r}.
\end{aligned}
$$

In the last line we have added the last term, which is zero by the antisymmetry of the integrand. Using the fundamental theorem of calculus for the first integral and and integrating by parts with respect to s in the second integral we have

$$
\begin{aligned}
J_2 = -\frac{1}{2}&\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1(\Delta\nabla\boldsymbol{u}(\boldsymbol{x}+s\boldsymbol{y})\boldsymbol{y})\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,ds\,d\boldsymbol{y}\,d\boldsymbol{r}\\
&+\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\Big(\frac{s^2-s}{2}+\frac{1-s^3}{6}\Big)\big(\Delta\nabla\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))-\Delta\nabla\boldsymbol{u}(\boldsymbol{x})\big)\Big|_{s=0}^{s=1}\\
&\qquad\qquad\cdot(\boldsymbol{r}-\boldsymbol{y})\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,d\boldsymbol{y}\,d\boldsymbol{r}\\
&-\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1\Big(\frac{s^2-s}{2}+\frac{1-s^3}{6}\Big)\Delta[\nabla^2\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))(\boldsymbol{r}-\boldsymbol{y})]\\
&\qquad\qquad\cdot(\boldsymbol{r}-\boldsymbol{y})\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,ds\,d\boldsymbol{y}\,d\boldsymbol{r}.
\end{aligned}
$$

Without changing the value of the integral we can insert again an antisymmetric integrand in the first integral. We also evaluate the second integral at $s=0$ and $s=1$, to produce

$$
\begin{aligned}
J_2 = -\frac{1}{2}&\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1(\Delta\nabla\boldsymbol{u}(\boldsymbol{x}+s\boldsymbol{y})-\Delta\nabla\boldsymbol{u}(\boldsymbol{x}))\boldsymbol{y}\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,ds\,d\boldsymbol{y}\,d\boldsymbol{r}\\
&-\frac{1}{6}\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}(\Delta\nabla\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y})-\Delta\nabla\boldsymbol{u}(\boldsymbol{x}))(\boldsymbol{r}-\boldsymbol{y})\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,dy\,d\boldsymbol{r}\\
&-\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1\Big(\frac{s^2-s}{2}+\frac{1-s^3}{6}\Big)\Delta[\nabla^2\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))(\boldsymbol{r}-\boldsymbol{y})](\boldsymbol{r}-\boldsymbol{y})\\
&\qquad\qquad\cdot\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,ds\,d\boldsymbol{y}\,d\boldsymbol{r}.
\end{aligned}
$$

Now, integrating by parts with respect to s in the first integral and applying the fundamental theorem of calculus in the second integral produces

$$\begin{aligned} J_2 = \frac{1}{2}\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}&(1-s)(\Delta\nabla \boldsymbol{u}(\boldsymbol{x}+s\boldsymbol{y}) - \Delta\nabla \boldsymbol{u}(\boldsymbol{x}))\boldsymbol{y}\eta(\boldsymbol{r})\pi(\boldsymbol{y})\Big|_{s=0}^{s=1}\,d\boldsymbol{y}\,d\boldsymbol{r} \\ &- \frac{1}{2}\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1 (1-s)\Delta[\nabla^2 \boldsymbol{u}(x+s\boldsymbol{y})\boldsymbol{y}]\boldsymbol{y}\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,dy\,d\boldsymbol{r} \\ &- \frac{1}{6}\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1 \Delta[\nabla^2 \boldsymbol{u}(\boldsymbol{x}+s\boldsymbol{y})\boldsymbol{y}](\boldsymbol{r}-\boldsymbol{y})\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,dy\,d\boldsymbol{r} \\ &- \int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1 \Big(\frac{s^2-s}{2}+\frac{1-s^3}{6}\Big)\Delta[\nabla^2 \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))(\boldsymbol{r}-\boldsymbol{y})](\boldsymbol{r}-\boldsymbol{y}) \\ &\qquad\qquad \cdot\,\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,ds\,d\boldsymbol{y}\,d\boldsymbol{r}. \end{aligned}$$

Evaluating the first integral at $s=0$ and $s=1$ gives

$$\begin{aligned} J_2 = -\frac{1}{2}\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1&(1-s)\Delta[\nabla^2 \boldsymbol{u}(x+s\boldsymbol{y})\boldsymbol{y}]\boldsymbol{y}\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,dy\,d\boldsymbol{r} \\ &- \frac{1}{6}\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1 \Delta[\nabla^2 \boldsymbol{u}(\boldsymbol{x}+s\boldsymbol{y})\boldsymbol{y}](\boldsymbol{r}-\boldsymbol{y})\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,dy\,d\boldsymbol{r} \\ &- \int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\int_0^1 (\frac{s^2-s}{2}+\frac{1-s^3}{6})\Delta[\nabla^2 \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}+s(\boldsymbol{r}-\boldsymbol{y}))(\boldsymbol{r}-\boldsymbol{y})](\boldsymbol{r}-\boldsymbol{y}) \\ &\qquad\qquad \cdot\,\eta(\boldsymbol{r})\pi(\boldsymbol{y})\,ds\,d\boldsymbol{y}\,d\boldsymbol{r}. \end{aligned}$$

By bounding the fourth-order derivatives as we did with J_1 and using $|\boldsymbol{y}|<\delta$ and $|\boldsymbol{r}|<\varepsilon$ we obtain

$$|J_2| \le \frac{7M_4}{12}(\delta^2+\varepsilon^2)\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}\eta(\boldsymbol{r})|\pi(\boldsymbol{y})|\,d\boldsymbol{y}\,d\boldsymbol{r},$$

and hence

$$|2\sigma(\varepsilon,\delta)J_2| \le \frac{7M_4}{12}(\delta^2+\varepsilon^2).$$

Using approaches similar to the ones employed to bound J_1 and J_2, we find the following bounds for J_3 and J_4:

$$\begin{aligned} |J_3| &\le \frac{M_4}{4}\int_{\mathcal{B}_\delta}\int_{\mathcal{B}_\varepsilon}(|\boldsymbol{y}|^2+|\boldsymbol{r}|^2)|\theta(\boldsymbol{r})|\gamma(\boldsymbol{y})\,d\boldsymbol{r}\,d\boldsymbol{y}, \\ |J_4| &\le \frac{5M_4}{6}\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta}(|\boldsymbol{y}|^2+|\boldsymbol{r}|^2)|\theta(\boldsymbol{r})|\gamma(\boldsymbol{y})\,dy\,d\boldsymbol{r}. \end{aligned}$$

Using Lemma 3.4 we have

$$|J_3| \le \frac{M_4}{4}\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta} |\boldsymbol{r}|^2\eta(\boldsymbol{r})|\pi(\boldsymbol{y})|\,d\boldsymbol{y}\,d\boldsymbol{r} + \frac{M_4}{4n}\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta} |\boldsymbol{r}|^4\eta(\boldsymbol{r})\gamma(\boldsymbol{y})\,dy\,d\boldsymbol{r}.$$

Then using (4.11) we find

$$|J_3| \le \frac{M_4}{4}\varepsilon^2\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta} \eta(\boldsymbol{r})|\pi(\boldsymbol{y})|\,d\boldsymbol{y}\,d\boldsymbol{r} + \frac{M_4}{4n}\varepsilon^4\int_{\mathcal{B}_\varepsilon}\int_{\mathcal{B}_\delta} \eta(\boldsymbol{r})\gamma(\boldsymbol{y})\,dy\,d\boldsymbol{r};$$

thus,

$$|2\sigma(\varepsilon,\delta)J_3| \le \frac{M_4}{4}\varepsilon^2 + \frac{M_4}{4n}\varepsilon^4\frac{\int_{\mathcal{B}_\delta}\gamma(\boldsymbol{y})\,dy}{|\int_{\mathcal{B}_\delta}\pi(\boldsymbol{y})\,dy|}. \tag{4.12}$$

Using the assumption that $c_1|y|^{-\alpha} \le \gamma(|y|) \le c_2|y|^{-\alpha}$ where $0 \le \alpha < n$ and $0 < c_1 \le c_2$, we get

$$|2\sigma(\varepsilon,\delta)J_3| \le \frac{M_4}{4}\varepsilon^2 + \frac{M_4c_2n(n+2-\alpha)}{4c_1n(n-\alpha)}\frac{\varepsilon^4}{\delta^2}.$$

Hence, under the assumption $\varepsilon \le \delta$, we have

$$|2\sigma(\varepsilon,\delta)J_3| \le \frac{M_4c_1n(n-\alpha)+M_4c_2n(n+2-\alpha)}{4c_1n(n-\alpha)}\varepsilon^2.$$

Similarly, we find the bound on J_4 to be

$$|2\sigma(\varepsilon,\delta)J_4| \le \frac{5M_4c_1n(n-\alpha)+5M_4c_2n(n+2-\alpha)}{6c_1n(n-\alpha)}\varepsilon^2.$$

Putting all of these together we find

$$\begin{aligned}|\mathcal{L}^s_{\gamma\eta}(\boldsymbol{u}) - \Delta\boldsymbol{u}(x)| &\le |2\sigma(\varepsilon,\delta)J_1| + |2\sigma(\varepsilon,\delta)J_2| + |2\sigma(\varepsilon,\delta)J_3| + |2\sigma(\varepsilon,\delta)J_4|\\ &\le C(\delta^2+\varepsilon^2) \le C\delta^2,\end{aligned}$$

where the value of the constant C changes from line to line and depends on M_4, n, α, c_1 and c_2. This estimate shows that our nonlocal state-based Laplacian with the scaling of (4.3) converges to the classical Laplacian at a rate of δ^2 independent of the dimension. □

Remark 4.3. The growth assumption on γ from Theorem 4.2 is easily guaranteed by Assumption 2, in which case we have an explicit value for $\sigma(\varepsilon,\delta)$ in terms of $\alpha, \beta, \delta, \varepsilon$ as given by (4.3).

Remark 4.4. In Theorem 4.2 we can relax the growth restrictions on γ by assuming instead that there exists a $C_1 > 0$ such that

$$\int_{\mathcal{B}_\delta} \gamma(\boldsymbol{y})\, dy \le \frac{C_1}{\delta^2} \left| \int_{\mathcal{B}_\delta} \pi(\boldsymbol{y})\, dy \right|. \tag{4.13}$$

Since $\varepsilon \le \delta$, (4.12) combined with (4.13) implies

$$|2\sigma(\varepsilon,\delta)J_3| \le \frac{M_4(n+C_1)}{8n}\varepsilon^2.$$

Similarly,

$$|2\sigma(\varepsilon,\delta)J_4| \le \frac{5M_4(n+C_1)}{6n}\varepsilon^2,$$

and the rate of convergence in the theorem holds.

Furthermore, we can replace the condition $\varepsilon \le \delta$ by the assumption that there exists a $C_2 > 0$ such that

$$\int_{\mathcal{B}_\delta} \gamma(\boldsymbol{y})\, dy \le \frac{C_2}{\varepsilon^2} \left| \int_{\mathcal{B}_\delta} \pi(\boldsymbol{y})\, dy \right|,$$

which becomes a condition that links the growth of γ with the growth of η. We then obtain from (4.12) that

$$|2\sigma(\varepsilon,\delta)J_3| \le \frac{M_4(n+C_2)}{8n}\varepsilon^2,$$

and

$$|2\sigma(\varepsilon,\delta)J_4| \le \frac{5M_4(n+C_2)}{6n}\varepsilon^2.$$

The resulting rate of convergence will be $\delta^2 + \varepsilon^2$.

We conclude this section with a more general convergence result requiring only twice differentiability, which yields also weaker convergence rates.

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 1$ be possibly unbounded, and for $0 < \varepsilon \le \delta$ let*

$$\Omega' := \Omega \setminus \{\boldsymbol{y} \in \Omega \,|\, \operatorname{dist}(\boldsymbol{y}, \partial\Omega) \le \varepsilon^2 + \delta^2\}.$$

Let γ, η satisfy Assumption 1, with the additional restriction that

$$c_1|\boldsymbol{y}|^{-\alpha} \le \gamma(\boldsymbol{y}) \le c_2|\boldsymbol{y}|^{-\alpha} \qquad \text{for } 0 \le \alpha < n \text{ and } 0 < c_1 \le c_2.$$

Then $\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}]$ *with scaling factor* $\sigma(\varepsilon,\delta)$ *given by* (4.1) *satisfies the following convergence estimates*

(i) *If* $\boldsymbol{u}\in C^{2,a}(\Omega)$ *with* $0<a<1$, *then*

$$\|\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}]-\Delta\boldsymbol{u}\|_{L^\infty(\Omega')}<C\delta^a.$$

(ii) *If* $\boldsymbol{u}\in C^2(\Omega)$, *then*

$$\|\mathcal{L}^s_{\gamma\eta}[\boldsymbol{u}]-\Delta\boldsymbol{u}\|_{L^\infty(\Omega')}\to 0 \qquad \text{as } \delta\to 0.$$

Proof. Both proofs follow along the same lines as that of Theorem 4.2. To show convergence and obtain explicit rates for part (i) we use again the expression (4.9) with terms J_1, J_2, J_3, J_4 defined on page 400 Note that for $\boldsymbol{u}\in C^{2,a}$ with $0<a<1$ we have

$$|\Delta\boldsymbol{u}(\boldsymbol{x})-\Delta\boldsymbol{u}(\boldsymbol{y})|<C|\boldsymbol{x}-\boldsymbol{y}|^a. \tag{4.14}$$

Applying (4.14) to estimate J_1 we obtain

$$|J_1|\le\int_{\mathcal{B}_\varepsilon}\int_0^1\int_{\mathcal{B}_\delta}\eta(\boldsymbol{r})\pi(\boldsymbol{y})s^{1+a}|\boldsymbol{y}+\boldsymbol{r}|^a\,d\boldsymbol{y}\,ds\,d\boldsymbol{r}.$$

Since

$$|\boldsymbol{y}+\boldsymbol{r}|^a\le|\boldsymbol{y}|^a+|r|^a,$$

after using exactly the same argument as in obtaining (4.10) we get

$$|2\sigma(\varepsilon,\delta)J_1|\le C(\delta^a+\varepsilon^a).$$

Similarly, estimating J_2, J_3, J_4 we obtain

$$|2\sigma(\varepsilon,\delta)J_2|\le C(\delta^a+\varepsilon^a), |2\sigma(\varepsilon,\delta)J_3|\le C\varepsilon^a, |2\sigma(\varepsilon,\delta)J_4|\le C\varepsilon^a,$$

so by adding all these inequalities the claim of (i) follows.

For (ii) the same process is followed, with the exception that for $\boldsymbol{u}\in C^2$ we have by the continuity of second-order derivatives that

$$|\Delta\boldsymbol{u}(\boldsymbol{x})-\Delta\boldsymbol{u}(\boldsymbol{y})|=o(|\boldsymbol{x}-\boldsymbol{y}|), \tag{4.15}$$

so

$$|2\sigma(\varepsilon,\delta)(J_1+J_2+J_3+J_4)|=o(\delta+\varepsilon)$$

which gives the conclusion of (ii). $\square$

5. Conclusions. To summarize, the main ideas of this paper revolve around the introduction of a new nonlocal Laplace-type operator, which is intimately connected to the state-based theory of peridynamics. The newly introduced state-based Laplacian offers an approximation of the classical Laplace operator for functions that are sufficiently smooth, however, it can be applied even to discontinuous functions or distributions; also the operator provides a lot more flexibility in modeling diffusion type phenomena. Note that the operator does not approximate the Navier operator from the system of elasticity **[16]** as we do not recover the term $\nabla \operatorname{div} \boldsymbol{u}$ in the limit as the horizon goes to zero. Our operator is applied to vector-valued functions, as it acts on each component. A nonlocal generalization to introduce a state-based Navier operator will be proposed in a future paper, by considering a vectorial or tensorial structure of the kernels. As this paper also points out, there are numerous nonlocal counterparts to a single local operator, so it will be nontrivial work to introduce a doubly nonlocal Navier operator with clear physical, as well as mathematical, significance that will connect it to applications and existing results in local theory.

Regarding convergence results presented in this manuscript, we would like to point out that the interior L^∞ bounds obtained for the error

$$(\mathcal{L}^s_{\gamma\eta} - \Delta)(\boldsymbol{u})$$

depend on the norm in a much smaller space (C^2 or C^4, depending on the convergence rate) of $\boldsymbol{u}$. Also, our bounds do not hold within distance $\delta + \varepsilon$ away from the boundary, where the state-based Laplacian capture information from outside the domain. Finally, our proof also shows that the quadratic rate of this convergence with respect to the horizons is optimal for functions $\boldsymbol{u} \in C^4$.

Finally, in the proof of Theorem 4.2 it is not clear if $\varepsilon \leq \delta$ is a necessary condition for convergence, so we are working to produce a counterexample to the convergence result for $\varepsilon > \delta$. Also, work in progress further explores the convolution structure of the operator and provides estimates for the solution of the Cauchy problem associated with the state-based Laplacian.

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