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A HYBRID COLLOCATION METHOD FOR FRACTIONAL INITIAL VALUE PROBLEMS

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ABSTRACT. This paper is concerned with the application of a hybrid collocation method to a class of initial value problems for differential equations of fractional order. First, the fractional differential equation is converted to a nonlinear Volterra integral equation with a weakly singular kernel. Then, the Volterra integral equation is converted to a fixed point problem. A hybrid collocation algorithm is developed to solve the fixed point problem, and the optimal order of convergence of the proposed algorithm is obtained. Two numerical experiments are conducted to demonstrate the efficiency of the hybrid collocation algorithm.

1. Introduction. Fractional calculus is a generalization of classical integer-order calculus [**25**]. Fractional differential operators are non-local. As such, they can more accurately describe non-local phenomena such as sub-diffusions and seepage flows in porous media. Through fractional differential equations, fractional calculus has a wide range of applications in the areas of physics, engineering and life science, among other disciplines [**6, 26, 28**]. Recently, there have been numerous studies on the theory and numerical methods for fractional differential equations. We refer to [**7, 9, 11, 12, 13, 15, 22**] and the references therein for details.

In this paper, we consider the following class of fractional differential

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equations ([**17**]):

$$D_*^\alpha y(t) = f(t, y(t)), \quad t \in [0,1], \tag{1.1}$$

$$y(0) = y_0, \tag{1.2}$$

where $0 < \alpha < 1$ and the fractional differential operator D_*^α [**19**] is defined as follows. Let $T[y]$ represent the Taylor polynomial of degree $\lfloor \alpha \rfloor$ and $\lceil \alpha \rceil$ denote the largest integer smaller than or equal to α and the smallest integer greater than or equal to α, respectively, and let J^β denote the Riemann-Liouville integral operator:

$$J^\beta y(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds.$$

Also denote D^α the Riemann-Liouville differential operator of order α:

$$D^\alpha := D^{\lceil \alpha \rceil} J^{-\lceil \alpha \rceil + \alpha}.$$

The fractional differential derivative $D_*^\alpha y(t)$ in the Caputo sense is defined as

$$D_*^\alpha y(t) := D^\alpha (y - T[y])(t), \quad \alpha \notin \mathbb{N}.$$

A great deal of effort has been made to obtain numerical solutions to Problem (1.1)–(1.2) (see the work presented in [**7, 13, 15**], for example). In this work, we consider the approach of converting the problem to a Volterra integral equation ([**9, 17**]):

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\, ds. \tag{1.3}$$

In [**16**], the authors presented numerical experiments for (1.3) using the fractional Adams scheme of [**13**], the Grunward-Letnkov operator ([**15**]) based finite difference scheme, and the quadrature based fractional backward difference (see [**15**]). In [**17**], a nonpolynomial collocation method was proposed to acquire the numerical solution where the optimal convergence order was obtained for the linear case. The error estimate was later extended to the nonlinear case [**18**].

In this paper, we adopt the hybrid collocation (HC) method, which uses nonpolynomial collocation functions only on the first interval. In comparison with the approach of [**17, 18**], the dimension of the linear and nonlinear system of collocation equations in the HC method is significantly reduced while the optimal order of convergence is still maintained. The HC method was first proposed in [**4**] to solve linear Volterra integral equations with weakly singular kernels. This method uses the singularity preserving collocation method on the first subinterval and a graded piecewise polynomial collocation method on the rest of the time domain. As pointed out in [**4**], this method makes use of the strength of both the singularity preserving collocation and the graded collocation methods. The same idea was later extended to solve Fredholm integral equations with weakly singular kernels ([**5**]) and a special nonlinear Volterra integral equation with $f(t,y)=t^{1/3}y^4$ ([**27**]).

To use the HC method, we first convert (1.3) into a fixed point problem for z, as follows.

$$z(t)=f\left(t,y_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}z(s)\,ds\right). \tag{1.4}$$

Then, the solution y of (1.3) is obtained through

$$y(t)=y_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}z(s)\,ds.$$

The fixed point idea was presented and developed in [**21, 23**]. As explained in [**2, 20**], for linear problems, this process is the same as the iterated collocation method, which is more accurate than solving for y directly. To apply the HC method to (1.4), we first use a result of [**9**] to obtain the singularity expansion for z under the analytic assumption for f. Then, we prove that the hybrid method for (1.4) achieves the optimal order of convergence.

The paper is organized as follows. In Section 2, we establish the global existence, uniqueness and smoothness of the solution for Problem (1.1)–(1.2). In Section 3, we study the singularity structure of fractional initial value problem (1.1)–(1.2). Section 4 is devoted to constructing the hybrid collocation algorithm. In Section 5, we conduct an error analysis to show that the HC method achieves the optimal order of convergence. Finally, in Section 6, we conduct two numerical experiments to demonstrate our theoretical results.

2. Existence, uniqueness and smoothness of the solution. The local existence and uniqueness of the solution of (1.3), and hence, of Problem (1.1)–(1.2), which we state in the next theorem, was studied in [**10**]. The smoothness properties of the solution were discussed in

[8]. In this section, we first give the sufficient conditions on the "global" existence and uniqueness of (1.1)–(1.2), which is defined on a fixed interval $[0,1]$. Then, we discuss the regularity of the solution away from the origin.

Theorem 2.1 ([10]). *Assume that* $D=[0,\chi]\times[y_0-\eta,y_0+\eta]$, *where* χ *and* η *are positive constants, and* $f:D\to\mathbb{R}$ *is continuous. Define*

$$\chi^*=\min\left\{\chi,\left(\frac{\eta\Gamma(\alpha+1)}{\|f\|_\infty}\right)^{1/\alpha}\right\}.$$

Then, a solution to Problem (1.1)–(1.2) *exists. If, in addition,* f *is Lipschitz continuous, then Problem* (1.1)–(1.2) *has a unique solution.*

Under a stricter condition on f, we can derive the existence of a unique solution of Problem (1.1)–(1.2) on the entire interval $[0,1]$, which we state in the next theorem. The proof of the theorem, which we omit, is a simple practice of the contraction mapping theorem.

Theorem 2.2. *Assume* f *is Lipschitz continuous and* $L/(\Gamma(\alpha+1))<1$, *where* L *is the Lipschitz constant of* f. *Then, Problem* (1.1)–(1.2) *has a unique solution in* $C[0,1]$.

Next, we consider the regularity of the exact of solution y of Problem (1.1)–(1.2). In general, y may not be smooth even if the right-hand f is smooth. In the next theorem, we will give the regularity properties of the solution away from the origin. The next lemma from **[1, 3]** will play an important role in our analysis.

Lemma 2.3. *Consider the nonlinear Volterra integral equation*

$$Y(t')=F(t')+\int_0^{t'}K(t',s',Y(s'))\,ds',\quad t'\in[0,T]. \tag{2.1}$$

Assume that

($V1$) *the kernel* $K=K(t',s',Y)$ *is* m *times* ($m\geq 1$) *continuously differentiable with respect to* t', s', Y *for* $t'\in[0,T]$, $s'\in[0,t')$, $Y\in\mathbb{R}$, *and there exists a real number* $\nu\in(0,1)$ *such that, for* $0\leq s'<t'\leq T$, $Y\in\mathbb{R}$, *and for nonnegative integers* i, j, k *with* $i+j+k\leq m$, *the*

following inequalities hold:

$$\left|\left(\frac{\partial}{\partial t'}\right)^i\left(\frac{\partial}{\partial t'}+\frac{\partial}{\partial s'}\right)^j\left(\frac{\partial}{\partial Y}\right)^k K(t',s',Y)\right| \le b_1(|Y|)|t'-s'|^{-\nu-i} \tag{2.2}$$

and

$$\begin{aligned}\left|\left(\frac{\partial}{\partial t'}\right)^i\left(\frac{\partial}{\partial t'}+\frac{\partial}{\partial s'}\right)^j\left(\frac{\partial}{\partial Y}\right)^k K(t',s',Y_1)\right.\\ \left.-\left(\frac{\partial}{\partial t'}\right)^i\left(\frac{\partial}{\partial t'}+\frac{\partial}{\partial s'}\right)^j\left(\frac{\partial}{\partial Y}\right)^k K(t',s',Y_2)\right|\\ \le b_2(\max\{|Y_1|,|Y_2|\})|Y_1-Y_2||t'-s'|^{-\nu-i},\end{aligned} \tag{2.3}$$

where the functions $b_1 : [0,\infty) \to [0,\infty)$ *and* $b_2 : [0,\infty) \to [0,\infty)$ *are assumed to be monotonically increasing;*

$(V2)$ $F \in C^{m,\nu}(0,T]$, *i.e.,* $F(t')$ *is* m *times continuously differentiable for* $0 < t' \le T$ *and, for* $k = 0,1,\dots,m$

$$|F^{(k)}(t')| \le Ct'^{1-\nu-k}. \tag{2.4}$$

Then, $Y \in C^{m,\nu}(0,T]$.

Theorem 2.4. *Assume that* $f \in C^m$, $m \in \mathbb{N}$, *can be written in the form* $f(t,y) = \widetilde{f}(t^{1/q},y)$, *where* $q \ge 2$ *is an integer. Then, for all* $0 < \epsilon < 1$, *the solution* y *of Problem* (1.1)–(1.2) *belongs to* $C^{m,\alpha}(\epsilon,1]$.

Proof. Rewrite (1.3) in the form

$$y(t) = g(t) + \frac{1}{\Gamma(\alpha)}\int_\epsilon^t (t-s)^{\alpha-1} f(s,y(s))\,ds, \quad t > \epsilon, \tag{2.5}$$

where

$$g(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^\epsilon (t-s)^{\alpha-1} f(s,y(s))\,ds. \tag{2.6}$$

Setting $s' = s-\epsilon$, $t' = t-\epsilon$ and changing f to $\widetilde{f}$ in (2.5) yields

$$\begin{aligned}&y(t'+\epsilon) =\\ &g(t'+\epsilon) + \frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t'-s')^{\alpha-1}\widetilde{f}((s'+\epsilon)^{1/q}, y(s'+\epsilon))\,ds',\ t>0,\end{aligned} \tag{2.7}$$

where

$$g(t'+\epsilon) = y_0 + \frac{1}{\Gamma(\alpha)} \int_{-\epsilon}^{0} (t'-s')^{\alpha-1} \widetilde{f}((s'+\epsilon)^{1/q}, y(s'+\epsilon))\, ds'. \tag{2.8}$$

Note that (2.7) is in the form (2.1) with

$$Y(t') = y(t'+\epsilon),$$

$$K(t', s', Y(s')) = \frac{1}{\Gamma(\alpha)} (t'-s')^{\alpha-1} \widetilde{f}((s'+\epsilon)^{1/q}, Y(s')), \tag{2.9}$$

$$f(t') = y_0 + \frac{1}{\Gamma(\alpha)} \int_{-\epsilon}^{0} (t'-s')^{\alpha-1} \widetilde{f}((s'+\epsilon)^{1/q}, Y(s'))\, ds', \tag{2.10}$$

and $(V1)$ and $(V2)$ are satisfied with $\nu = \alpha$. Therefore, $y \in C^{m,\alpha}(\epsilon, 1]$. This completes the proof. □

3. Singularity expansions. In this section, we study the singularity expansion of the solution z of the fixed point problem (1.4). This expansion result will be the foundation for our hybrid collocation method.

First, we quote a result in **[9]** on the structure of the exact solution of Problem (1.1)–(1.2).

Lemma 3.1. *Let $\alpha = p/q$, where $p \geq 1$ and $q \geq 2$ are two relatively prime integers. Assume that f can be written in the form $f(t,y) = \widetilde{f}(t^{1/q}, y)$, and $\widetilde{f}$ is analytic in a neighborhood of $(0, y_0)$. Then, there exist $R > 0$ and a uniquely determined analytic function $\widetilde{y} : (-R, R) \to \mathbb{R}$ such that $y(t) = \widetilde{y}(t^{1/q})$ for $t \in [0, R)$.*

This lemma implies that we can write the solution y of Problem (1.1)–(1.2) as

$$y(t) = \sum_{i=0}^{\infty} a_i t^{i/q}, \quad t \in [0, R), \tag{3.1}$$

where a_i are constants and R is the radius of convergence of the power series.

Remark 3.2. From the proof of [**9,** Lemma 2], we can deduce that the series in (3.1) is also absolutely and uniformly convergent on subintervals of $[0, R)$.

In order to derive the singular expansion for the solution of Problem (1.1)–(1.2), we first analyze the structure of the solution of Problem (1.1)–(1.2).

Proposition 3.3. *Assume that the assumptions of Lemma* 2 *and Theorem* 2.4 *hold. Then, for* $0 < \widetilde{R} < R$, *the solution* y *of Problem* (1.1)–(1.2) *can be written as*

$$y(t) = y_1(t) + y_2(t), \quad 0 \leq t \leq \widetilde{R},$$

where $y_1 \in C^m[0, \widetilde{R}]$, *and* y_2 *is given by*

$$y_2(t) = \sum_{i=0}^{mq-1} a_i t^{i/q}. \tag{3.2}$$

Proof. Let

$$y_1(t) = \sum_{i=mq}^{\infty} a_i t^{i/q}. \tag{3.3}$$

It suffices to show that $y_1 \in C^m[0, \widetilde{R}]$.

Note that the series on the right hand side of (3.3) is absolutely and uniformly convergent for $t \in [0, \widetilde{R}]$; thus, $y_1 \in C[0, \widetilde{R}]$. Now, consider the series

$$\sum_{i=mq}^{\infty} \frac{i}{q} a_i t^{i/q-1}. \tag{3.4}$$

Let $\tau = t^{1/q}$. Then, (3.3) and (3.4) become, respectively,

$$\sum_{i=mq}^{\infty} a_i \tau^i, \tag{3.5}$$

and

$$\sum_{i=mq}^{\infty} \frac{i}{q} a_i \tau^{i-q}. \tag{3.6}$$

From the Cauchy-Hadamard theory, we know that series (3.5) and series (3.6) have the same radius of convergence, which implies that series (3.3) and series (3.4) have the same radius of convergence. Thus, series (3.4) is also absolutely convergent for $t \in [0, \widetilde{R}]$. In addition, for $t \in [0, \widetilde{R}]$, we have

$$\left| \frac{i}{q} a_i t^{i/q-1} \right| \leq \left| \frac{i}{q} a_i \widetilde{R}^{i/q-1} \right|.$$

Due to the fact that series (3.4) is absolutely convergent at $t = \widetilde{R}$, the majorant criterion indicates that series (3.4) is uniformly convergent for $t \in [0, \widetilde{R}]$. Therefore, we can exchange the order of summation and differentiation to obtain

$$y_1' = \left(\sum_{i=mq}^{\infty} a_i t^{i/q} \right)' = \sum_{i=mq}^{\infty} \frac{i}{q} a_i t^{i/q-1}.$$

The above series is absolutely and uniformly convergent for $t \in [0, \widetilde{R}]$. Moreover, each term in the series is continuous. Thus, $y_1 \in C^1[0, \widetilde{R}]$. Repeating the above argument, we conclude that $y_1 \in C^m[0, \widetilde{R}]$. □

The main result of this section is a singularity decomposition of the solution y for Problem (1.1)–(1.2) in $[0, 1]$.

Theorem 3.4. *Assume that the assumptions of Lemma* 3.1 *and Theorem* 2.4 *hold. Then, the solution y of Problem* (1.1)–(1.2) *can be written as*

$$y = y_1 + y_2, \tag{3.7}$$

where $y_1 \in C^m[0, 1]$ and y_2 is given by (3.2).

Proof. By Proposition 3.3, for $t \in [0, \widetilde{R}]$, $y_1(t) = \sum_{i=mq}^{\infty} a_i t^{i/q}$ and $y_1 \in C^m[0, \widetilde{R}]$. On the other hand, by Theorem 2.4, there exists an $\overline{\epsilon} \in (0, \widetilde{R})$ such that $y \in C^m(\overline{\epsilon}, 1]$. Since $\overline{\epsilon} < \widetilde{R}$, $y_1 \in C^m[0, 1]$. □

Next, we introduce a finite-dimensional subspace of $C[0, 1]$, where the nonsmooth part y_2 in the decomposition (3.7) for y belongs. Let N_0 be the set of nonnegative integers. For $q \geq 2$, let $\beta = 1/q$. Then, $0 < \beta \leq 1/2$.

For a given positive integer m, we define the finite-dimensional subspace V_m^β of $C[0,1]$ by

$$V_m^\beta := \operatorname{span}\{t^{i+j\beta}, i,j \in N_0,\ i+j\beta < m,\ 0<\beta\le 1/2\}. \tag{3.8}$$

From the proof of Theorem 3.4, we know that the nonsmooth part y_2 of the decomposition (3.7) of y belongs to V_m^β. For example, if $p=2$, then $q=3$, which implies $\alpha = 2/3$. Then, $y_1 \in C^2[0,1]$ and $y_2 \in V_2^{1/3} = \operatorname{span}\{1, t^{1/3}, t^{2/3}, t, t^{4/3}, t^{5/3}\}$.

In the last part of this section we consider the singularity expansion of the fixed point problem (1.4). Following the idea of Kumar and Sloan ([**23**]), we convert the Volterra integral equation (1.3) into a fixed point problem as follows. Define a function z by

$$z(t) := f(t, y(t)), \quad t\in[0,1]. \tag{3.9}$$

Substituting (3.9) into (1.3), we have that

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} z(s)\,ds, \quad t\in[0,1]. \tag{3.10}$$

It follows from (3.9) that z satisfies the nonlinear integral equation

$$z(t) = f\left(t, y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} z(s)\,ds\right), \quad t\in[0,1]. \tag{3.11}$$

The next proposition provides the singularity expansion of the exact solution of (3.11).

Proposition 3.5. *Assume that the conditions of Lemma* 3.1 *and Theorem* 2.4 *hold. Then, the solution z of equation* (3.11) *can be written as*

$$z = z_1 + z_2, \tag{3.12}$$

where $z_1 \in C^m[0,1]$ and $z_2 \in V_m^\beta$.

Proof. Let $y(t) = \widetilde{y}(t^{1/q})$. For $t \in [0,R)$, we find

$$z(t) = f(t,y(t)) = \widetilde{f}(t^{1/q}, y(t)) = \widetilde{f}(t^{1/q}, \widetilde{y}(t^{1/q})) =: \widetilde{z}(t^{1/q}).$$

Here, $\widetilde{z}(\cdot) = \widetilde{f}(\cdot, \overline{y}(\cdot))$. Since $\widetilde{f}$ is analytic in a neighborhood of $(0,y_0)$ and $\widetilde{y}$ is analytic in $[0,R)$, $\widetilde{z}$ is also analytic near 0. By Lemma 3.1, z

can be written as

$$z(t) = \widetilde{z}(t^{1/q}) = \sum_{i=0}^{\infty} b_i t^{i/q}, \quad t \in [0, R_1),$$

where the b_i are constants and $R_1 > 0$ is a real number. Following a similar argument as in the proof of Theorem 3.4, we can write z as $z = z_1 + z_2$ with $z_1 \in C^m[0,1]$ and $z_2 \in V_m^\beta$. $\square$

As in [**4**], the singularity expansion (3.12) will enable us to utilize the hybrid collocation (HC) to obtain the approximate solutions of the fractional differential equation problem (1.1)–(1.2).

4. Numerical approach. In this section, we will describe the HC method for (3.11), and consequently, the initial value problem (1.1)–(1.2).

Let l denote the cardinality of the set V_m^β. We use P_m to denote the space of polynomials of degree $\leq m-1$. It is clear that $P_m \subset V_m^\beta$.

For convenience, we introduce an index set

$$W_{\beta,m} = \{i + j\beta : i, j \in N_0, i + j\beta < m\} = \{v_k : k = 1, 2, \ldots, l\}.$$

With this notation, we can express V_m^β as

$$V_m^\beta = \text{span}\{t^{v_k} : k = 1, 2, \ldots, l\}.$$

Set $r := m/\beta$, and let i_0 be an integer such that

$$\left\lfloor \left(\frac{N}{i_0}\right)^r \right\rfloor = N,$$

and set $N' := N - i_0 + 1$. Then, the partition on $[0,1]$ is given by

$$t_0 = 0, t_i = \left(\frac{i_0 + i - 1}{N}\right)^r, \quad i = 1, 2, \ldots, N'.$$

For $i = 0, 1, \ldots, N' - 1$, denote by $\sigma_i = [t_i, t_{i+1}]$ the subintervals of $[0,1]$ with lengths $h_i = t_{i+1} - t_i$ and $h = \max_{i=0,1,2,\ldots,N'-1} h_i$.

A space of functions piecewise in $V \subseteq C[0,1]$ is defined by

$$(V)_h = \{v : v|_{[t_{i-1},t_i]} \in V|_{[t_{i-1},t_i]},\ i = 1, 2, \ldots, N'\}.$$

In particular, we denote

$$S_{m,h} := (P_m)_h, \qquad V_{m,h} := (V_m^\beta)_h.$$

Next, we define the finite-dimensional space S_h^m, to which our approximate solutions belong, by setting

$$S_h^m := \{y : y|_{\sigma_0} \in V_m^\beta|_{\sigma_0},\ y|_{\sigma_i} \in P_m|_{\sigma_i}, i = 1, 2, \ldots, N' - 1\}.$$

Now, we are ready to introduce the HC method for (3.11). For the first subinterval σ_0, we consider l collocation points

$$t_{0j} = t_0 + h_0 c_j, \quad j = 1, 2, \ldots, l,$$

where $0 \le c_1 < c_2 < \cdots < c_l \le 1$ are collocation parameters. For the rest of the subintervals σ_i, $i = 1, 2, \ldots, N' - 1$, we define the m collocation points by

$$t_{ij} := t_i + c_j h_i, \quad j = 1, 2, \ldots, m.$$

The HC method for the fixed point problem (3.11) is to seek $\overline{z} \in S_h^m$ such that

$$\overline{z}(t_{0j}) = f\left(t_{0j}, y_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_{0j}} (t_{0j} - s)^{\alpha-1} \overline{z}(s)\, ds\right), \quad j = 1, 2, \ldots, l, \tag{4.1}$$

and

$$\overline{z}(t_{ik}) = f(t_{ik}, y_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_{ik}} (t_{ik} - s)^{\alpha-1} \overline{z}(s)\, ds), \quad i = 1, 2, \ldots, N' - 1, \quad k = 1, 2, \ldots, m. \tag{4.2}$$

Substituting z with the approximation $\overline{z}$ on the right-hand side of (3.10), we obtain the HC approximation $\overline{y}$ to y:

$$\overline{y}(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \overline{z}(s)\, ds, \quad t \in [0, 1]. \tag{4.3}$$

To express the HC solution $\overline{z}(t)$ as a system of nonlinear equations for $\overline{z}(t_{ik})$, we introduce the Lagrange basis functions on the first subinterval σ_0 by

$$L_{0i}|_{\sigma_0} \in V_m^\beta|_{\sigma_0}, \quad i = 1, 2, \ldots, l. \tag{4.4}$$

such that

$$L_{0i}(t_{0j}) = \delta_{ij}, \quad j = 1, 2, \dots, l. \tag{4.5}$$

The existence and uniqueness of the Lagrange basis functions L_{0i}, $i = 1, 2, \dots, l$, were proven in **[24, 27]**. In fact, we may write

$$L_{0i}(t) = \begin{cases} \sum_{p=1}^{l} a_{ip} t^{v_p} & t \in \sigma_0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

where the coefficients a_{ip} are obtained by solving the linear system (4.5). For the rest of the subintervals σ_k, $k = 1, 2, \dots, N' - 1$, we define the Lagrange polynomial basis functions L_{ki}, $i = 1, 2, \dots, m$, by

$$L_{ki}(t) = \begin{cases} L_i(\frac{t - t_{k-1}}{h_k}) & t \in \sigma_k, \\ 0 & \text{otherwise,} \end{cases} \tag{4.7}$$

where $L_i \in P_m$, $i = 1, 2, \dots, m$, are the standard Lagrange polynomial basis functions on the partition $0 \le c_1 < \cdots < c_m \le 1$.

With these basis functions, we can express the solution $\overline{z}$ of the HC method as

$$\overline{z}_0(t) = \sum_{k=1}^{l} \overline{z}(t_{0k}) L_{0k}(t), \quad t \in \sigma_0, \tag{4.8}$$

$$\overline{z}_i(t) = \sum_{k=1}^{m} \overline{z}(t_{ik}) L_{ik}(t), \quad t \in \sigma_i, \tag{4.9}$$
$$i = 1, 2, \dots, N' - 1.$$

From (4.1) and (4.2), we have that for $j = 1, 2, \dots, l$,

$$\overline{z}(t_{0j}) = f\left(t_{0j}, y_0 + \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{l} \overline{z}(t_{0k}) \int_0^{t_{0j}} (t_{0j} - s)^{\alpha - 1} L_{0k}(s)\, ds\right), \tag{4.10}$$

and, for $i=1,2,\dots,N'-1$, $k=1,2,\dots,m$,

$$\overline{z}(t_{ik}) = f\Bigg(t_{ik}, y_0 + \frac{1}{\Gamma(\alpha)}\Bigg(\sum_{k=1}^{l}\overline{z}(t_{0k})\int_0^{t_1}(t_{ik}-s)^{\alpha-1}L_{0k}(s)\,ds + \sum_{r=1}^{i-1}\sum_{j=1}^{m}\overline{z}(t_{rj})\int_{t_r}^{t_{r+1}}(t_{ik}-s)^{\alpha-1}L_{rj}(s)\,ds + \sum_{j=1}^{m}\overline{z}(t_{ij})\int_{t_i}^{t_{ik}}(t_{ik}-s)^{\alpha-1}L_{ij}(s)\,ds\Bigg)\Bigg). \tag{4.11}$$

Finally, in this section, we express the approximate solution $\overline{z}$ as the solution of an operator equation. Towards this end, we first define two interpolation operators $P_{h,0}$ and $P_{h,1}$ as

$$(P_{h,0}f)(t) = \sum_{j=1}^{l} f(t_{0j})L_{0j}(t), \quad t\in[t_0,t_1], \tag{4.12}$$

and

$$(P_{h,1}f)(t) = \sum_{k=1}^{N'-1}\sum_{j=1}^{m} f(t_{kj})L_{kj}(t), \quad t\in[t_1,1]. \tag{4.13}$$

Following **[4]**, we define the hybrid interpolation operator Q_h by

$$(Q_hf)|_{[0,t_1]} = (P_{h,0}f)|_{[0,t_1]}, \qquad (Q_hf)|_{[t_1,1]} = (P_{h,1}f)|_{[t_1,1]}. \tag{4.14}$$

Next, we define operator $T^\alpha : C[0,1]\to C[0,1]$ by

$$(T^\alpha y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds, +y_0,$$

and operator F by

$$F(y)(t) = f(t, y(t)).$$

With these operators, we can express the HC solution $\overline{z}$ as the solution of the operator equation

$$\overline{z} = Q_hFT^\alpha(\overline{z}).$$

5. Convergence analysis. Throughout this section, we assume that the conditions of Lemma 3.1 are satisfied. For a vector $x = [x_1, \ldots, x_N]^T \in \mathbb{R}^N$, denote $\|x\| = \max_{i=1,\ldots,N}\{|x_i|\}$, and, for a continuous function f, defined on the interval σ, denote $\|f\|_\sigma = \max_{t\in\sigma}|f(t)|$. We will use $\|f\|$ instead of $\|f\|_\sigma$ when $\sigma = [0,1]$. First, we collect some lemmas from **[4, 14, 18, 27]**.

Lemma 5.1. *Let L_{0j}, $j = 1,2,\ldots,l$, be the Lagrange functions defined by* (4.4), (4.5). *Then, there exists a positive constant c independent of N such that, for all $t_1 \in (0,1)$,*

$$\|L_{0j}\|_{[t_0,t_1]} \le c, \quad j = 1,2,\ldots,l. \tag{5.1}$$

Furthermore, for all $t_1 \in (0,1)$ and $f = f_1 + f_2$, where $f_1 \in C^m[0,1]$ and $f_2 \in V_m^\beta$, there exists a constant $C_1 > 0$ such that

$$\|f - P_{h,0}f\|_{[t_0,t_1]} \le C_1 h_0^m \|f_1^{(m)}\|_{[t_0,t_1]}. \tag{5.2}$$

Lemma 5.2. *Let Q_h be the hybrid interpolation operator defined in* (4.14) *associated with the graded partition. Suppose that f has a decomposition $f = f_1 + f_2$, where $f_1 \in C^m[0,1]$ and $f_2 \in V_m^\beta$. Then, there exists a positive constant c_1 independent of N such that*

$$\|f - Q_h f\| \le c_1 N^{-m}.$$

The following singular discrete Gronwall inequality plays an important role in our analysis.

Lemma 5.3. *Let x_i, $0 \le i \le N$, be a sequence of non-negative real numbers satisfying*

$$x_i \le \psi_i + Mh^{1-\eta}\sum_{j=0}^{i-1}\frac{x_j}{(i-j)^\eta}, \quad 0 \le i \le N,$$

where $0<\eta<1$, $M>0$ is bounded independently of h, and ψ_i, $0\le i\le N$, is a monotonic increasing sequence of non-negative real numbers. Then:

$$x_i \le \psi_i E_{1-\eta}(M\Gamma(1-\eta)(ih)^{1-\eta}), \quad 0 \le i \le N,$$

where $E_\alpha(x) = \sum_{k=0}^{+\infty} x^k/\Gamma(\alpha k+1)$ denotes the Mittag-Leffler function of order α.

We are in a position to prove the main result of the paper.

Theorem 5.4. *Assume that the conditions of Lemma* 2 *and Theorem* 2.4 *hold. Let z be the unique solution of equation* (3.11) *and $\overline{z} \in S_h^m$ the* HC *approximation to z given by* (4.1), (4.2). *Then, for sufficiently large N, there exists a positive constant C independent of N such that*

$$\|z - \overline{z}\| \le CN^{-m}. \tag{5.3}$$

Proof. First, we prove that

$$\|z - \overline{z}\|_{\sigma_0} \le CN^{-m}. \tag{5.4}$$

For $j = 1, 2, \ldots, l$, let

$$e_0(t_{0j}) = z(t_{0j}) - \overline{z}(t_{0j}), \tag{5.5}$$

and

$$\mathbf{e}_0 = [e_0(t_{0j}) : j = 1, \ldots, l]^T. \tag{5.6}$$

From (3.10) and (4.1), we have

$$\begin{aligned} e_0(t_{0j}) = f\left(t_{0j}, y_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_{0j}} (t_{0j} - s)^{\alpha-1} z(s)\, ds\right) \\ - f\left(t_{0j}, y_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_{0j}} (t_{0j} - s)^{\alpha-1} \overline{z}(s)\, ds\right). \end{aligned}$$

By the assumptions of Lemma 2 and Theorem 2.4, we know that there exists a constant L, possibly dependent upon the solution y, such that

$$|e_0(t_{0j})| \le \frac{L}{\Gamma(\alpha)} \int_0^{t_{0j}} (t_{0j} - s)^{\alpha-1} (z(s) - \overline{z}(s))\, ds.$$

Therefore,

$$|e_0(t_{0j})| \le \frac{L}{\Gamma(\alpha)} \int_0^{t_{0j}} (t_{0j} - s)^{\alpha-1} \|z - \overline{z}\|_{\sigma_0}\, ds. \tag{5.7}$$

Next, we estimate $\|z - \overline{z}\|_{\sigma_0}$ in terms of $\mathbf{e}_0$. Since the solution of equation (3.11) has the form $z = z_1 + z_2$, with $z_1 \in C^m[0,1]$ and $z_2 \in V_m^\beta$, by Lemma 5.1, we have that

$$\|z - P_{h,0} z\|_{\sigma_0} \le C_1 h_0^m \|z_1^{(m)}\|_{\sigma_0} \le C_3 N^{-m}, \tag{5.8}$$

where the fact that $h_0 \le \widetilde{c}N^{-1}$ has been used. Since $\overline{z}|_{\sigma_0} \in V_m^\beta$, $\overline{z} = P_{h,0}\overline{z}$ on σ_0. Thus,

$$\begin{aligned}\|P_{h,0}z - \overline{z}\|_{\sigma_0} &= \|P_{h,0}(z - \overline{z})\|_{\sigma_0} \\ &= \|\Sigma_{j=1}^l L_{0j}(s)(z(t_{0j}) - \overline{z}(t_{0j}))\|_{\sigma_0} \\ &\le \max_{j=1,\dots,l} \|L_{0j}\|_{\sigma_0} \Sigma_{j=1}^l |z(t_{0j}) - \overline{z}(t_{0j})| \\ &\le c\Sigma_{j=1}^l |e_0(t_{0j})|.\end{aligned}$$

Hence,

$$\begin{aligned}\|z - \overline{z}\|_{\sigma_0} &\le \|z - P_{h,0}z\|_{\sigma_0} + \|P_{h,0}z - \overline{z}\|_{\sigma_0} \\ &\le C_3N^{-m} + c\Sigma_{j=1}^l |e_0(t_{0j})|.\end{aligned} \tag{5.9}$$

Combining (5.7) and (5.9), we obtain

$$\begin{aligned}|e_0(t_{0j})| &\le \frac{L}{\Gamma(\alpha)} \int_0^{t_{0j}} (t_{0j} - s)^{\alpha-1}(C_3N^{-m} + c\Sigma_{j=1}^l |e_0(t_{0j})|)\, ds \\ &\le \frac{L}{\alpha\Gamma(\alpha)} t_{0j}^\alpha (C_3N^{-m} + c\|\mathbf{e}_0\|).\end{aligned}$$

Therefore, for sufficiently large N, we have

$$\|\mathbf{e}_0\| \le C_4N^{-m-\alpha}.$$

It follows from (5.9) that

$$\|z - \overline{z}\|_{\sigma_0} \le CN^{-m}. \tag{5.10}$$

Next, we show that

$$\|z - \overline{z}\|_{\sigma_i} \le CN^{-m}, \quad i = 1, \cdots, N' \tag{5.11}$$

where $\sigma_i = [t_i, t_{i+1}]$. Denote $e_i = (z(t) - \overline{z}(t))|_{\sigma_i}$, $i = 1, \dots, N' - 1$,

$$e_i(t_{ik}) = z(t_{ik}) - \overline{z}(t_{ik}), \tag{5.12}$$

and

$$\mathbf{e}_i = [e_i(t_{ik}) : k = 1, 2, \dots, m]^T. \tag{5.13}$$

From (3.10), (4.2) and (4.10), we have that

$$
\begin{aligned}
e_i(t_{ik}) &= f\left(t_{ik}, y_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_{ik}} (t_{ik}-s)^{\alpha-1} z(s)\, ds\right) \\
&\quad - f\left(t_{ik}, y_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_{ik}} (t_{ik}-s)^{\alpha-1} \overline{z}(s)\, ds\right) \\
&\le \frac{L}{\Gamma(\alpha)}\int_0^{t_{ik}} (t_{ik}-s)^{\alpha-1}(z(s)-\overline{z}(s))\, ds \\
&= \frac{L}{\Gamma(\alpha)}\bigg(\int_0^{t_1} (t_{ik}-s)^{\alpha-1}(z(s)-\overline{z}(s))\, ds \\
&\qquad + \sum_{j=1}^{i-1}\int_{t_j}^{t_{j+1}} (t_{ik}-s)^{\alpha-1}(z(s)-(P_{h,1}z)(s) \\
&\qquad + (P_{h,1}z)(s)-\overline{z}(s))\, ds \\
&\qquad + \int_{t_i}^{t_{ik}} (t_{ik}-s)^{\alpha-1}(z(s)-(P_{h,1}z)(s)+(P_{h,1}z)(s)-\overline{z}(s))\, ds\bigg).
\end{aligned}
$$

For the first term of the above we use (5.10) to obtain

$$
\begin{aligned}
&\int_0^{t_1} (t_{ik}-s)^{\alpha-1}(z(s)-\overline{z}(s))\, ds \\
&\qquad\qquad \le \int_0^{t_1} (t_{ik}-s)^{\alpha-1}\|z-\overline{z}\|_{\sigma_0}\, ds \le C_5 N^{-m}.
\end{aligned}
$$

Since $h_1 < h_2 < \cdots < h_{N'-1}$ and $h_i \le cN^{-1}$, $i = 1, 2, \ldots, N'-1$, we have that

$$
\begin{aligned}
\int_{t_j}^{t_{j+1}} (t_{ik}-s)^{\alpha-1} ds &\le \int_{t_j}^{t_{j+1}} (t_i - s)^{\alpha-1} ds \\
&= h_j^\alpha \int_0^1 \Big(\frac{t_i - t_j}{h_j} - \theta\Big)^{\alpha-1} d\theta \\
&\le cN^{-\alpha}(i-j)^{\alpha-1}, \qquad (5.14)
\end{aligned}
$$

and

$$
\int_{t_i}^{t_{ik}} (t_{ik}-s)^{\alpha-1} ds = h_i^\alpha \int_0^{c_k} \frac{d\theta}{(c_k-\theta)^{\alpha-1}} \le \frac{1}{\alpha} h_i^\alpha c_k \le \frac{1}{\alpha} N^{-\alpha}. \qquad (5.15)
$$

Using (5.14) and (5.15), together with Lemma 5.2, we have that

$$
\begin{aligned}
&\Big|\sum_{j=1}^{i-1}\int_{t_j}^{t_{j+1}}(t_{ik}-s)^{\alpha-1}(z(s)-(P_{h,1}z)(s)+(P_{h,1}z)(s)-\overline{z}(s))\,ds\Big| \\
&\quad\le \sum_{j=1}^{i-1}\int_{t_j}^{t_{j+1}}(t_{ik}-s)^{\alpha-1}(|z(s)-(P_{h,1}z)(s)|+|(P_{h,1}z)(s)-\overline{z}(s)|)\,ds \\
&\quad\le \sum_{j=1}^{i-1}\int_{t_j}^{t_{j+1}}(t_{ik}-s)^{\alpha-1}(C_6N^{-m}+\sum_{\gamma=1}^{m}|L_{j\gamma}(s)||z(t_{j\gamma})-\overline{z}(t_{j\gamma})|)\,ds \\
&\quad\le \sum_{j=1}^{i-1}\int_{t_j}^{t_{j+1}}(t_{ik}-s)^{\alpha-1}(C_6N^{-m}+C_7\|\mathbf{e}_j\|)\,ds \\
&\quad\le C_8N^{-m}+C_9N^{-\alpha}\sum_{j=1}^{i-1}(i-j)^{\alpha-1}\|\mathbf{e}_j\|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_{t_i}^{t_{ik}}(t_{ik}-s)^{\alpha-1}(z(s)-(P_{h,1}z)(s)+(P_{h,1}z)(s)-\overline{z}(s))\,ds\Big| \\
&\quad\le \int_{t_i}^{t_{ik}}(t_{ik}-s)^{\alpha-1}|z(s)-(P_{h,1}z)(s)|+|(P_{h,1}z)(s)-\overline{z}(s)|\,ds \\
&\quad\le \int_{t_i}^{t_{ik}}(t_{ik}-s)^{\alpha-1}(C_{10}N^{-m}+C_{11}|\mathbf{e}_i|)\,ds \\
&\quad\le C_{12}N^{-m}+C_{13}N^{-\alpha}|\mathbf{e}_i|.
\end{aligned}
$$

For N sufficiently large, $1-C_{12}N^{-\alpha}>0$. Thus,

$$\|\mathbf{e}_i\| \quad\le\quad C_{14}N^{-m}+C_{15}N^{-\alpha}\sum_{j=1}^{i-1}(i-j)^{\alpha-1}\|\mathbf{e}_j\|.$$

It follows from Lemma 5.3 that

$$\|\mathbf{e}_i\|\le C_{16}N^{-m}. \tag{5.16}$$

Next, we follow the similar procedure of estimating $\|z-\overline{z}\|_{\sigma_0}$ to estimate $\|z-\overline{z}\|_{\sigma_i}$, $i=1,2,\ldots,N'-1$.

First, note that

$$\begin{aligned}\|P_{h,1}z-\overline{z}\|_{\sigma_i} &= \left\|\sum_{\gamma=1}^{m} L_{i\gamma}(s)(z(t_{i\gamma})-\overline{z}(t_{i\gamma}))\right\|_{\sigma_i}\\ &\le \sum_{\gamma=1}^{m}\|L_{i\gamma}\|_{\sigma_i}\|e_i(t_{i\gamma})\|\\ &\le C_1\|\mathbf{e}_i\| \le C_2N^{-m}.\end{aligned}$$

Also, Lemma 5.2 indicates that

$$\|z-P_{h,1}z\|_{\sigma_i} \le C_3N^{-m}.$$

Thus,

$$\|z-\overline{z}\|_{\sigma_i} \le \|z-P_{h,1}z\|_{\sigma_i} + \|P_{h,1}z-\overline{z}\|_{\sigma_i} \le (C_2+C_3)N^{-m}.$$

The combination of (5.10) and the above completes the proof of the theorem. □

Based on the estimate for $\|z-\overline{z}\|$, it is straightforward to estimate the error between the approximation solution $\overline{y}$ defined by (4.3) and the exact solution y of fractional initial value problem (1.1)–(1.2).

Theorem 5.5. *Assume that the assumptions of Lemma* 2 *and Theorem* 2.4 *hold. Let y be the exact solution of Problem* (1.1)–(1.2) *and $\overline{y}$ the* HC *approximation of y defined by* (4.3)*. Then, there exists a constant C such that, for sufficiently large N,*

$$\|y-\overline{y}\| \le CN^{-m}. \tag{5.17}$$

Proof. Subtracting (3.10) from (4.3), we have that

$$y(t)-\overline{y}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(z(s)-\overline{z}(s))\,ds.$$

Thus,

$$\begin{aligned}\|y-\overline{y}\| &\le \|z-\overline{z}\| \sup_{0\le t\le 1}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\\ &\le C\|z-\overline{z}\|.\end{aligned}$$

□

6. Numerical experiments. In this section, we conduct two numerical experiments to demonstrate the efficiency of the HC method. In particular, we will compare the results with those of the graded collocation method (GC) based on the piecewise polynomials of degree $m-1$. The graded mesh is defined by

$$\Delta_r^N = \left\{ t_i = \left(\frac{i}{N}\right)^r, \; i = 0, 1, \ldots, N \right\},$$

where $r = m/\alpha$.

Example 6.1. In the first example, we consider a linear fractional initial value problem [**17**]:

$$D_*^{1/2} y(t) = \frac{1}{2} y(t), \quad t \in [0,1], \tag{6.1}$$

$$y(0) = 1. \tag{6.2}$$

The exact solution of (6.1) and (6.2) is

$$y(t) = E_{1/2}(0.5\sqrt{t}) = \sum_{k=0}^{\infty} 0.5^k \frac{t^{k/2}}{\Gamma(k/2)+1}.$$

Following the process described in Section 3, we define $z = z(t)$ as the solution of the following Volterra integral equation:

$$z(t) = 1 + \frac{1}{2\Gamma(1/2)} \int_0^t (t-s)^{-1/2} z(s)\, ds.$$

Then, the solution for the fractional differential equation is given by

$$y(t) = 1 + \frac{1}{2\Gamma(1/2)} \int_0^t (t-s)^{-1/2} z(s)\, ds. \tag{6.3}$$

We choose $m = 3$ for both of the methods. Accordingly, the function space defined in (3.8) is given by

$$V_3^{1/2} = \text{span}\{1, t^{1/2}, t, t^{3/2}, t^2, t^{5/2}\},$$

For the collocation parameters, we choose $c_1 = 0.1$, $c_2 = 0.2$, $c_3 = 1/3$, $c_4 = 0.5$, $c_5 = 0.8$, $c_6 = 1$ on the first subinterval for the HC method and $c_1 = 0.1$, $c_2 = 0.5$, $c_3 = 0.8$ on the rest of the subintervals for the HC method and all subintervals for the GC method.

TABLE 1. The standard GC method for Example 6.1.

N	N_{int}	$h_{\min}$	$\|e\|_\infty$	Conv. Rate
10	10	1.0000×10^{-6}	5.5232×10^{-4}	–
20	20	1.5625×10^{-8}	8.1073×10^{-5}	2.7682
40	40	2.4414×10^{-10}	1.1034×10^{-5}	2.8773
80	80	3.8147×10^{-12}	1.4432×10^{-6}	2.9346
160	160	5.9605×10^{-14}	1.8503×10^{-7}	2.9634
320	320	9.3132×10^{-16}	2.3467×10^{-8}	2.9791

TABLE 2. The fixed point GC method for Example 6.1.

N	N_{int}	$h_{\min}$	$\|e\|_\infty$	Conv. Rate
10	10	1.0000×10^{-6}	7.6629×10^{-5}	–
20	20	1.5625×10^{-8}	1.0040×10^{-5}	2.9321
40	40	2.4414×10^{-10}	1.2248×10^{-6}	3.0351
80	80	3.8147×10^{-12}	1.4618×10^{-7}	3.0667
160	160	5.9605×10^{-14}	1.7417×10^{-8}	3.0692
320	320	9.3132×10^{-16}	2.1044×10^{-9}	3.0490

The numerical results of the standard GC method, the fixed point GC method and the HC method proposed in this paper are, respectively, shown in Table 1, Table 2 and Table 3. Here, N_{int} refers to the total number of collocation intervals, $h_{\min}$ refers to the minimum length of the collocation intervals, and $\|e\|_\infty$ is the maximum norm of the error between the exact solution and the approximate solution. From these tables, we can see that, although the convergence rates are the same, the iterated GC method and the HC method are one magnitude more accurate than the standard GC method. On the other hand, although the iterated fixed point method and the iterative HC method have the same accuracy, the HC method is more efficient than the iterated GC method since it uses fewer collocation intervals. In addition, the length of the smallest interval for the HC method is much larger than that of the GC method, which reduces the potential roundoff error problem.

Example 6.2. In this example, we consider the following nonlinear problem **[29]**

$$D_*^{1/2} y(t) = -\Gamma\left(\frac{1}{2}\right) y^2(t) - \frac{8t^{3/2} + 6\pi t^2 - 3\pi - 3\pi t^4}{3\sqrt{\pi}}, \quad t \in [0,1], \tag{6.4}$$

$$y(0) = 1. \tag{6.5}$$

TABLE 3. The HC method for Example 6.1.

N	N_{int}	$h_{\min}$	$\|e\|_\infty$	Conv. Rate
10	4	1.1765×10^{-1}	7.2369×10^{-5}	–
20	8	4.2230×10^{-2}	9.6372×10^{-6}	2.9087
40	19	8.4610×10^{-3}	1.2050×10^{-6}	2.9996
80	42	2.2021×10^{-3}	1.4494×10^{-7}	3.0555
160	92	5.8000×10^{-4}	1.7341×10^{-8}	3.0632
320	198	1.6055×10^{-4}	2.0827×10^{-9}	3.0577

TABLE 4. The fixed point GC method for Example 6.2.

N	N_{int}	$h_{\min}$	$\|e\|_\infty$	Conv. Rate
10	10	1.0000×10^{-4}	1.8652×10^{-3}	–
20	20	6.2500×10^{-6}	5.4526×10^{-4}	1.7743
40	40	3.9063×10^{-7}	1.4266×10^{-4}	1.9344
80	80	2.4414×10^{-8}	3.5631×10^{-5}	2.0014
160	160	1.5259×10^{-9}	8.7705×10^{-6}	2.0224
320	320	9.5367×10^{-11}	2.1717×10^{-6}	2.0138

The exact solution is given by $y = 1 - t^2$, and the corresponding integral equation for y is given by

$$y(t) = 1 + \frac{1}{\Gamma(1/2)} \int_0^t (t-s)^{-1/2} \left(-\Gamma\left(\frac{1}{2}\right) y^2(s) - \frac{8s^{3/2} + 6\pi s^2 - 3\pi - 3\pi s^4}{3\sqrt{\pi}} \right) ds.$$

In the CG and HC methods, we choose $m = 2$. It follows that

$$V_2^{1/2} = \text{span}\{1, t^{1/2}, t, t^{3/2}\}.$$

For the collocation parameters, we choose $c_1 = 0.1$, $c_2 = 0.4$, $c_3 = 0.7$, $c_4 = 1$ on the first subinterval for the HC method and $c_1 = 0.4$ and $c_2 = 0.9$ on the rest of the subintervals for the HC method and all the subintervals for the GC method.

The numerical results are listed in Tables 4–5. From the two tables, we can draw the same conclusions as in the first example. In particular, since we only use 245 collocation intervals in the HC method, its complexity is only about 75 percent of that in the GC method.

TABLE 5. The HC method for Example 6.2.

N	N^{HC}	$h_{\min}$	$\|e\|_\infty$	Conv. Rate
10	5	1.1050×10^{-1}	1.8635×10^{-3}	–
20	11	2.9006×10^{-2}	5.4516×10^{-4}	1.7733
40	25	7.0254×10^{-3}	1.4266×10^{-4}	1.9341
80	54	2.0316×10^{-3}	3.5630×10^{-5}	2.0014
160	116	5.7500×10^{-4}	8.7705×10^{-6}	2.0224
320	245	1.7079×10^{-4}	2.1717×10^{-6}	2.0138

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