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ON THE NUMERICAL SOLUTION OF THE EXTERIOR ELASTODYNAMIC PROBLEM BY A BOUNDARY INTEGRAL EQUATION METHOD

ROMAN CHAPKO AND LEONIDAS MINDRINOS

ABSTRACT. A numerical method for the Dirichlet initial boundary value problem for the elastic equation in the exterior and unbounded region of a smooth, closed, simply connected two-dimensional domain, is proposed and investigated. This method is based on a combination of a Laguerre transformation with respect to the time variable and a boundary integral equation approach in the spatial variables. Using the Laguerre transformation in time reduces the time-dependent problem to a sequence of stationary boundary value problems, which are solved by a boundary layer approach resulting in a sequence of boundary integral equations of the first kind. The numerical discretization and solution are obtained by a trigonometrical quadrature method. Numerical results are included.

1. Introduction. The problem of numerically solving time-dependent boundary value problems has a long history. The simplest approach consists of using the finite difference method (FDM), which has the obvious limitations of simple domains and interior problems. The most used general scheme, for this type of problem, first reduces the dimensions of the problem by some semi-discretization approach and then solves the simpler problem by a numerical method. For example, using the Galerkin method with respect to the spatial variables, as a semi-discretization technique, results in a Cauchy problem for a system of ordinary differential equations. Then, single- or multi-step methods can be applied.

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On the other hand, we can apply semi-discretization with respect to the time variable and reduce the given problem to a set of stationary problems. This can be done by an integral transformation or by Rothe's method. Then, we can apply a numerical method suitable for stationary problems, for example, FDM, the finite element method, the boundary element method and others.

The main aim and result of this paper is to present an integral equation method for the time-dependent elastic equation in an unbounded two-dimensional domain. The possible variants for using an integral equation approach for time-dependent boundary value problems are discussed in [**7**]. If the given differential equation has a fundamental solution, the problem can be reduced to a time-boundary integral equation by direct or indirect methods (see [**1**] for the elastic equation). Also, in the second semi-discretization approach, the method of integral equations can be applied to obtain stationary problems.

Our goal is to extend the idea of the Fourier-Laguerre expansion of the solution and consider it as a semi-discretization approach with respect to time [**2, 3, 4, 5, 6, 8**]. Then, we need to solve a recurrent sequence of stationary problems for the Navier equations. The advantage of this semi-discretization approach, compared to the Laplace transform, is the simple representation of the numerical solution. In our case, it is a partial sum of the Fourier-Laguerre series. Since the solution domain is unbounded, the use of classical integral equation methods with a volume potential would be ineffective. Instead, we present the solution in terms of boundary potentials using explicit fundamental solutions of the sequence so obtained. This approach gives us the possibility of reducing the stationary differential problems to a sequence of boundary integral equations.

The outline of the paper is as follows. In Section 2, we describe the semi-discretization procedure in time via the Laguerre transformation. Then, the initial boundary value problem for the elastic equation is transformed to a sequence of boundary value problems for the Navier equations. In Section 3, we show the reduction of the sequence of stationary problems to integral equations merely involving boundary potentials. To do so, we create fundamental solutions for the sequence of stationary equations. In Section 4, we outline and describe how the well-established numerical methods based on trigonometrical quadratures can be adjusted and applied to numerically solve the derived sequence of boundary integral equations. In Section 5, we demonstrate the feasibility of our approach through three numerical examples.

Before closing this section, we formulate the problem to be studied. Let $D \subset \mathbb{R}^2$ be an unbounded domain such that its complement is bounded and simply connected, and assume that the boundary Γ of D is of class C^2. Consider the initial boundary value problem for the hyperbolic elastic equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta^* u \quad \text{in } D \times (0, \infty), \tag{1.1}$$

with the Lamé operator defined by $\Delta^* := c_s^2 \Delta + (c_p^2 - c_s^2)\,\mathrm{grad}\,\mathrm{div}$, supplied with the homogeneous initial conditions

$$\frac{\partial u}{\partial t}(\cdot, 0) = u(\cdot, 0) = 0 \quad \text{in } D, \tag{1.2}$$

and the boundary condition

$$u = f, \quad \text{on } \Gamma \times (0, \infty), \tag{1.3}$$

where f is a given function, satisfying the compatibility condition

$$f(x, 0) = \frac{\partial f}{\partial t}(x, 0) = 0, \quad x \in \Gamma.$$

Here, the velocities c_s and c_p have the following form

$$c_s = \sqrt{\frac{\mu}{\rho}}, \qquad c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}},$$

where ρ is the density, and λ and μ are the Lamé constants.

Since we have an unbounded solution domain, we specify that, at infinity,

$$u(x, t) \longrightarrow 0, \quad |x| \to \infty, \tag{1.4}$$

uniformly with respect to all directions $x/|x|$ and all $t \in [0, \infty)$. Other types of boundary conditions can potentially be handled as well. For uniqueness and existence results for this elastodynamic problem, see **[10,** Chapter 8] and the references therein.

2. Semi-discretization in time. For semi-discretization with respect to the time variable in problem (1.1)–(1.4), we use the Laguerre transformation.

The Laguerre polynomials have the explicit representation

$$L_n(t) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-t)^k}{k!},$$

for $n = 0, 1, \ldots$. These polynomials are orthogonal with respect to the standard L^2 inner product in $(0, \infty)$, appended with weight e^{-t}. The following recurrence relation holds:

$$(n+1)L_{n+1}(t) = (2n+1-t)L_n(t) - nL_{n-1}(t). \tag{2.1}$$

For our purposes, though, we need to note the relation

$$L'_{n+1}(t) = L'_n(t) - L_n(t), \quad n = 0, 1, \ldots, \tag{2.2}$$

which is immediate from the explicit representation of L_n.

A function being square integrable with respect to the above weight e^{-t} over the interval $(0, \infty)$ can then be expanded in a scaled Fourier-Laguerre series

$$v(t) = \kappa \sum_{n=0}^{\infty} v_n L_n(\kappa t),$$

with Fourier-Laguerre coefficients

$$v_n = \int_0^\infty e^{-\kappa t} L_n(\kappa t) v(t)\, dt, \quad n = 0, 1, \ldots.$$

These two relations are interpreted as the inverse and direct Laguerre transformations, respectively. Here, v is the given function, and the sequence $\{v_n\}$ is the image; $\kappa > 0$ is a fixed scaling parameter. In the rest of this work, we use the above scaled convention as the Laguerre transform.

Let $\{v'_n\}$ be the sequence obtained when applying the Laguerre transformation to the derivative of a sufficiently smooth function v with $v(0) = 0$. Then

$$v'_n = \kappa \sum_{m=0}^{n} v_m \tag{2.3}$$

and, for the second derivative of v with $v(0) = v'(0) = 0$, the following holds:

$$v_n'' = \kappa^2 \sum_{m=0}^{n} (n - m + 1) v_m. \tag{2.4}$$

Applying the Laguerre transformation to problem (1.1)–(1.4) with respect to the time variable, together with the relation (2.3), we obtain the following sequence of stationary boundary value problems

$$\Delta^* u_n - \kappa^2 u_n = \sum_{m=0}^{n-1} \beta_{n-m} u_m, \quad \text{in } D, \tag{2.5a}$$

$$u_n = f_n, \quad \text{on } \Gamma, \tag{2.5b}$$

$$u_n(x) \longrightarrow 0, \quad |x| \to \infty, \tag{2.5c}$$

where $n = 0, 1, \ldots$, $\beta_n = \kappa^2(n+1)$ and $\{u_n\}$ and $\{f_n\}$ are the Fourier-Laguerre sequences of coefficients of the functions u and f, respectively.

By the maximum principle and induction, we have the following uniqueness result.

Theorem 2.1. *The sequence of stationary problems* (2.5a)–(2.5c) *has at most one solution.*

The function of the form

$$u(x, t) = \kappa \sum_{n=0}^{\infty} u_n(x) L_n(\kappa t), \tag{2.6}$$

with u_n solving (2.5a)–(2.5c), is clearly a solution to the initial boundary value problem (1.1)–(1.4). On the other hand, assuming that the solution to (1.1)–(1.4) has the right smoothness properties such that it can be expanded in time in terms of the Laguerre polynomials, it follows that the coefficients will form a sequence $\{u_n\}$ and satisfy (2.5a)–(2.5c). Thus, we state the following theorem.

Theorem 2.2. *A sufficiently smooth function* (2.6) *is the solution of the time-dependent problem* (1.1)–(1.4) *if and only if its Fourier-Laguerre coefficients u_n for $n = 0, 1, \ldots$, solve the sequence of stationary problems* (2.5a)–(2.5c).

3. A boundary integral equation approach for stationary problems. First we determine a sequence of fundamental solutions for equations (2.5a).

Definition 3.1. The sequence of 2×2 matrices $\{E_n(x,y)\}$, $n = 0, 1, \ldots$, is called a *sequence of fundamental solutions* of equations (2.5a) if

$$\Delta^* E_n(x,y) - \sum_{m=0}^{n} \beta_{n-m} E_m(x,y) = \delta(x-y) I. \tag{3.1}$$

Here, I is the 2×2 identity matrix, δ denotes the Dirac function and the differentiation in (3.1) is taken with respect to x.

Let us consider the polynomials

$$v_n(\gamma, r) = \sum_{m=0}^{[n/2]} a_{n,2m}(\gamma) r^{2m}, \qquad w_n(\gamma, r) = \sum_{m=0}^{[(n-1)/2]} a_{n,2m+1}(\gamma) r^{2m+1}$$

for $n = 0, 1, \ldots, N-1$, $N \in \mathbb{N} \cup \{0\}$ $(w_0 = 0)$, where the coefficients $a_{n,m}$ satisfy the recurrence relations

$$\begin{aligned} a_{n,0}(\gamma) &= 1, && n = 0, 1, \ldots, N-1, \\ a_{n,n}(\gamma) &= -\frac{\gamma}{n}\, a_{n-1,n-1}(\gamma), && n = 1, 2, \ldots, N-1, \end{aligned}$$

and

$$a_{n,m}(\gamma) = \frac{1}{2\gamma m} \left\{ 4 \left[\frac{m+1}{2} \right]^2 a_{n,m+1}(\gamma) - \gamma^2 \sum_{k=m-1}^{n-1} (n-k+1) a_{k,m-1}(\gamma) \right\},$$

for $m = n-1, \ldots, 1$. Here, $[r]$ denotes the integer part of $r \geq 0$.

Next, we introduce the sequence of functions

$$\Phi_n(\gamma, r) = K_0(\gamma r)\, v_n(\gamma, r) + K_1(\gamma r)\, w_n(\gamma, r), \tag{3.2}$$

where K_0 and K_1 are the modified Hankel functions of orders zero and one, respectively. Throughout this paper, all functions and constants with a negative index number are set equal to zero.

Lemma 3.2. *The following formulas hold*:

$$\int_{r/a}^{\infty} \frac{e^{-\kappa t} L_n(\kappa t)}{\sqrt{t^2-(r/a)^2}}\,dt = \Phi_n\left(\frac{\kappa}{a}, r\right) \tag{3.3}$$

and

$$\int_{r/a}^{\infty} \frac{e^{-\kappa t} (\kappa t)^2 L_n(\kappa t)}{\sqrt{t^2-(r/a)^2}}\,dt = \sum_{k=-2}^{2} \chi_{k,n} \Phi_{n+k}\left(\frac{\kappa}{a}, r\right), \tag{3.4}$$

where $\chi_{-2,n} = n(n-1)$, $\chi_{-1,n} = -4n^2$, $\chi_{0,n} = 2(3n^2+3n+1)$, $\chi_{1,n} = -4(n+1)^2$ *and* $\chi_{2,n} = (n+1)(n+2)$.

Proof. Consider the fundamental solution for the wave equation

$$G(x,y;t) = \frac{\theta(t-|x-y|/a)}{\sqrt{t^2-(|x-y|/a)^2}},$$

where θ is the Heaviside function. Clearly, it satisfies the equation

$$\frac{1}{a^2}\frac{\partial^2 G(x,y;t)}{\partial t^2} - \Delta G(x,y;t) = \delta(x-y)\delta(t).$$

If we apply the Laguerre transformation to this equation with respect to time, we receive the following sequence:

$$\Delta G_n(x,y) - \frac{\kappa^2}{a^2}\sum_{m=0}^{n}(n-m+1)G_m(x,y) = \delta(x-y), \quad n = 0,1,\ldots, \tag{3.5}$$

for the Laguerre coefficients

$$G_n(x,y) = \int_0^\infty G(x,y;t) e^{-\kappa t} L_n(\kappa t)\,dt.$$

In [**6**], it was found by reducing (3.5) to ordinary differential equations and using its exact solution that $G_n(x,y) = \Phi_n(\kappa/a, |x-y|)$. Thus, formula (3.3) is proven.

The recurrence relation (2.1) gives us the following representation

$$t^2 L_n(t) = \sum_{k=-2}^{2} \chi_{k,n} L_{n+k}(t). \tag{3.6}$$

This relation, together with (3.3), results in the formula (3.4). □

Note here, that, from (3.6), it follows that $\sum_{k=-2}^{2} \chi_{k,n} = 0$. We introduce the notation $J(x) = xx^\top/|x|^2$ for $x \in \mathbb{R}^2 \setminus \{0\}$.

Theorem 3.3. *The sequence of matrices*

$$E_n(x,y) = \Phi_{1,n}(|x-y|)I + \Phi_{2,n}(|x-y|)J(x-y) \tag{3.7}$$

is a sequence of fundamental solutions of (2.5a). *Here,*

$$\begin{aligned}\Phi_{\ell,n}(r) = \frac{(-\ell)^{\ell-1}}{\kappa^2 r^2} \sum_{k=-2}^{2} \chi_{k,n} \left(\Phi_{n+k}\left(\frac{\kappa}{c_s}, r\right) - \Phi_{n+k}\left(\frac{\kappa}{c_p}, r\right)\right) \\ + \frac{(-1)^{\ell-1}}{c_p^2} \Phi_n\left(\frac{\kappa}{c_p}, r\right) + \frac{\ell-1}{c_s^2} \Phi_n\left(\frac{\kappa}{c_s}, r\right),\end{aligned}$$

for $\ell = 1, 2$.

Proof. Consider the fundamental solution of the time-dependent elastodynamic equation (1.1) (see [**1**]):

$$\begin{aligned}E(x,y;t) = \left(\frac{t^2\theta(t-r/c_s)}{r^2\sqrt{t^2-(r/c_s)^2}} - \frac{(t^2-(r/c_p)^2)\theta(t-r/c_p)}{r^2\sqrt{t^2-(r/c_p)^2}} \right) I \\ + \left(\frac{(2t^2-(r/c_p)^2)\theta(t-r/c_p)}{r^2\sqrt{t^2-(r/c_p)^2}} \right. \\ \left. - \frac{(2t^2-(r/c_s)^2)\theta(t-r/c_s)}{r^2\sqrt{t^2-(r/c_s)^2}} \right) J(x-y),\end{aligned}$$

where $r = |x-y|$. From Definition 3.1, it is clear that

$$E_n(x,y) = \int_0^\infty E(x,y;t) e^{-\kappa t} L_n(\kappa t)\, dt. \tag{3.8}$$

Thus, the statement of the theorem follows from (3.8) with the aid of formulas (3.3) and (3.4). $\square$

Note that the fundamental matrix E_0 from (3.7) coincides with the fundamental matrix for the harmonic elastodynamic equation (see [**1**]).

Now, we can analyze the singularity of the fundamental matrix. The modified Hankel functions have the following series representations:

$$\begin{aligned} K_0(z) &= -\left(\ln\frac{z}{2}+C\right)I_0(z)+S_0(z), \\ K_1(z) &= \frac{1}{z}+\left(\ln\frac{z}{2}+C\right)I_1(z)+S_1(z) \end{aligned} \tag{3.9}$$

with

$$\begin{aligned} I_0(z) &= \sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{z}{2}\right)^{2n}, \\ I_1(z) &= \sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}\left(\frac{z}{2}\right)^{2n+1}, \end{aligned}$$

and

$$\begin{aligned} S_0(z) &= \sum_{n=1}^{\infty}\frac{\psi(n)}{(n!)^2}\left(\frac{z}{2}\right)^{2n}, \\ S_1(z) &= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{\psi(n+1)+\psi(n)}{n!(n+1)!}\left(\frac{z}{2}\right)^{2n+1}. \end{aligned}$$

Here, we set $\psi(0)=0$,

$$\psi(n)=\sum_{m=1}^{n}\frac{1}{m}, \quad n=1,2,\dots,$$

and let $C=0.57721\dots$ denote Euler's constant. Thus, we can rewrite the functions Φ_n as follows:

$$\Phi_n(\gamma,r)=\phi_n(\gamma,r)\ln r+\varphi_n(\gamma,r), \quad n=0,1,\dots,$$

where

$$\phi_n(\gamma,r)=-I_0(\gamma r)v_n(\gamma,r)+I_1(\gamma r)w_n(\gamma,r)$$

and

$$\begin{aligned} \varphi_n(\gamma,r) &= \left[-\left(C+\ln\frac{\gamma}{2}\right)I_0(\gamma r)+S_0(\gamma r)\right]v_n(\gamma,r) \\ &\quad+\left[\frac{1}{\gamma r}+\left(C+\ln\frac{\gamma}{2}\right)I_1(\gamma r)+S_1(\gamma r)\right]w_n(\gamma,r). \end{aligned}$$

Clearly, we have the following asymptotic behavior with respect to r:

$$\begin{aligned}\phi_n(\gamma, r) &= \epsilon_{n,0}(\gamma) + \epsilon_{n,2}(\gamma) r^2 + O(r^4),\\ \varphi_n(\gamma, r) &= \varepsilon_{n,0}(\gamma) + \varepsilon_{n,2}(\gamma) r^2 + O(r^4),\end{aligned} \tag{3.10}$$

with

$$\epsilon_{n,0}(\gamma) = -a_{n,0}(\gamma), \qquad \epsilon_{n,2}(\gamma) = -\frac{\gamma^2}{4} a_{n,0}(\gamma) + \frac{\gamma}{2} a_{n,1}(\gamma) - a_{n,2}(\gamma)$$

and

$$\begin{aligned}\varepsilon_{n,0}(\gamma) &= -\left(C + \ln\frac{\gamma}{2}\right) a_{n,0}(\gamma) + \frac{1}{\gamma} a_{n,1}(\gamma),\\ \varepsilon_{n,2}(\gamma) &= \left(C + \ln\frac{\gamma}{2}\right)\left(-\frac{\gamma^2}{4} a_{n,0}(\gamma) + \frac{\gamma}{2} a_{n,1}(\gamma) - a_{n,2}(\gamma)\right)\\ &\quad + \frac{\gamma^2}{4} a_{n,0} - \frac{\gamma}{4} a_{n,1}(\gamma) + \frac{1}{\gamma} a_{n,3}(\gamma).\end{aligned}$$

Then, we have the following representation for the functions in (3.7):

$$\Phi_{\ell,n}(r) = \eta_{\ell,n}(r) \ln r + \xi_{\ell,n}(r), \quad \ell = 1, 2 \tag{3.11}$$

with

$$\begin{aligned}\eta_{\ell,n}(r) &= \frac{(-\ell)^{\ell-1}}{\kappa^2 r^2} \sum_{k=-2}^{2} \chi_{k,n}\left(\phi_{n+k}\left(\frac{\kappa}{c_s}, r\right) - \phi_{n+k}\left(\frac{\kappa}{c_p}, r\right)\right)\\ &\quad + \frac{(-1)^{\ell-1}}{c_p^2} \phi_n\left(\frac{\kappa}{c_p}, r\right) + \frac{\ell - 1}{c_s^2} \phi_n\left(\frac{\kappa}{c_s}, r\right),\end{aligned}$$

and

$$\begin{aligned}\xi_{\ell,n}(r) &= \frac{(-\ell)^{\ell-1}}{\kappa^2 r^2} \sum_{k=-2}^{2} \chi_{k,n}\left(\varphi_{n+k}\left(\frac{\kappa}{c_s}, r\right) - \varphi_{n+k}\left(\frac{\kappa}{c_p}, r\right)\right)\\ &\quad + \frac{(-1)^{\ell-1}}{c_p^2} \varphi_n\left(\frac{\kappa}{c_p}, r\right) + \frac{\ell - 1}{c_s^2} \varphi_n\left(\frac{\kappa}{c_s}, r\right).\end{aligned}$$

Taking into account the definition of the coefficients $\chi_{k,n}$ and $a_{n,m}(\gamma)$

and following (3.10), we obtain the asymptotic expansion

$$\eta_{\ell,n}(r) = \frac{(-\ell)^{\ell-1}}{\kappa^2} \sum_{k=-2}^{2} \chi_{k,n} \left(\epsilon_{n+k,2}\left(\frac{\kappa}{c_s}\right) - \epsilon_{n+k,2}\left(\frac{\kappa}{c_p}\right) \right) + \frac{(-1)^{\ell-1}}{c_p^2} \epsilon_{n,0}\left(\frac{\kappa}{c_p}\right) + \frac{\ell-1}{c_s^2} \epsilon_{n,0}\left(\frac{\kappa}{c_s}\right) + O(r^2)$$

and

$$\xi_{\ell,n}(r) = \frac{(-\ell)^{\ell-1}}{\kappa^2} \sum_{k=-2}^{2} \chi_{k,n} \left(\varepsilon_{n+k,2}\left(\frac{\kappa}{c_s}\right) - \varepsilon_{n+k,2}\left(\frac{\kappa}{c_p}\right) \right) + \frac{(-1)^{\ell-1}}{c_p^2} \varepsilon_{n,0}\left(\frac{\kappa}{c_p}\right) + \frac{\ell-1}{c_s^2} \varepsilon_{n,0}\left(\frac{\kappa}{c_s}\right) + O(r^2).$$

Thus, we have proved that our fundamental sequence has only the logarithmic singularity.

Next, we shall construct a solution to the sequence of problems (2.5a)–(2.5c). Let $\{U_n\}$ be a sequence of single-layer potentials

$$U_n(x) = \frac{1}{2\pi} \sum_{m=0}^{n} \int_\Gamma E_{n-m}(x,y) q_m(y)\, ds(y), \quad x \in D, \tag{3.12}$$

and $\{V_n\}$ a sequence of double-layer potentials

$$V_n(x) = \frac{1}{2\pi} \sum_{m=0}^{n} \int_\Gamma T_y E_{n-m}(x,y) q_m(y)\, ds(y), \quad x \in D, \tag{3.13}$$

$n = 0, 1, \ldots$, where $q_m \in C(\Gamma)$ are unknown densities, $\{E_n\}$ is the fundamental sequence (3.7) and T is a tracing operator

$$Tv = \lambda\, \mathrm{div} v\, \nu + 2\mu\, (\nu \cdot \mathrm{grad})\, v + \mu\, \mathrm{div}\, (Qv)\, Q\nu, \tag{3.14}$$

with the unitary matrix

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As follows from the representation of fundamental matrices (3.7) and the expansion (3.11), the classical jump and regularity properties of the logarithmic potentials (see **[9]**) can also be applied to the present situation. Hence, we have the following transformations into sequences of boundary integral equations.

Theorem 3.4. *The sequence of single-layer potentials* (3.12) *is a solution of the sequence of boundary value problems* (2.5a)–(2.5c), *provided that their densities satisfy the following sequence of boundary integral equations of the first kind*:

$$\frac{1}{2\pi}\int_\Gamma E_0(x,y)q_n(y)\,ds(y) = f_n(x) - \frac{1}{2\pi}\sum_{m=0}^{n-1}\int_\Gamma E_{n-m}(x,y)q_m(y)\,ds(y), \quad x\in\Gamma, \tag{3.15}$$

for $n = 0, 1, \ldots$.

The sequence of double-layer potentials (3.13) *is a solution of the sequence of boundary value problems* (2.5a)–(2.5c), *provided that their densities satisfy the following sequence of boundary integral equations of the second kind*:

$$\frac{1}{2}q_n(x) + \frac{1}{2\pi}\int_\Gamma T_y E_0(x,y)q_n(y)\,ds(y) = f_n(x) - \frac{1}{2}\sum_{m=0}^{n-1} q_m(x) - \frac{1}{2\pi}\sum_{m=0}^{n-1}\int_\Gamma T_y E_{n-m}(x,y)q_m(y)\,ds(y), \quad x\in\Gamma,$$

for $n = 0, 1, \ldots$.

Proof. In order to derive the above equations, we consider the boundary condition (2.5b), let $x \to \Gamma$ in equations (3.12) and (3.13), and use the continuity of the single-layer potentials and the jump relation **[10]**:

$$V_n(x) = \frac{1}{2}\sum_{m=0}^{n} q_m(x) + \frac{1}{2\pi}\sum_{m=0}^{n}\int_\Gamma T_y E_{n-m}(x,y)q_m(y)\,ds(y), \quad x\in\Gamma.$$

□

We proceed by investigating integral equations of the first kind (3.15). The case of the sequence of integral equations of the second kind contains no principal difference.

Theorem 3.5. *For any sequence f_n in $C^{1,\alpha}(\Gamma)$, the system* (3.15) *possesses a unique solution q_n in $C^{0,\alpha}(\Gamma)$.*

Proof. By standard arguments (see **[9]** for the case of the Laplace equation) it can be seen that the integral equation with logarithmic singularity

$$\frac{1}{2\pi}\int_\Gamma E_0(x,y)q_0(y)\,ds(y) = f_0(x), \quad x \in \Gamma,$$

has a unique solution $q_0 \in C^{0,\alpha}(\Gamma)$ for any f_0 in $C^{1,\alpha}(\Gamma)$. Then, the statement of the theorem follows by induction. □

4. A quadrature method for full discretization. We assume that the boundary curve Γ is given through

$$\Gamma = \{x(s) = (x_1(s), x_2(s)) : 0 \le s \le 2\pi\},$$

where $x : \mathbb{R} \to \mathbb{R}^2$ is C^1 and 2π-periodic with $|x'(s)| > 0$ for all s, such that the orientation of Γ is counter-clockwise. Then, we transform (3.15) into the parametric form:

$$\begin{aligned}\text{(4.1)}\quad \frac{1}{2\pi}\int_0^{2\pi} H_0(s,\tau)\psi_n(\tau)\,d\tau &= g_n(s)\\ &-\frac{1}{2\pi}\sum_{m=0}^{n-1}\int_0^{2\pi} H_{n-m}(s,\tau)\psi_m(\tau)\,d\tau, \quad 0 \le s \le 2\pi,\end{aligned}$$

where we have set $\psi_n(s) := |x'(s)|\, q_n(x(s))$, $g_n(s) := f_n(x(s))$, and where the kernels are given by

$$H_n(s,\tau) := E_n(x(s), x(\tau)),$$

for $s \ne \tau$ and $n = 0, 1, \ldots$.

The kernels H_n have logarithmic singularities and can be written in the form

$$H_n(s,\tau) = \ln\left(\frac{4}{e}\sin^2\frac{s-\tau}{2}\right)H_n^1(s,\tau) + H_n^2(s,\tau),$$

where

$$H_n^1(s,\tau) := \frac{1}{2}[\eta_{1,n}(|x(s)-x(\tau)|)I + \eta_{2,n}(|x(s)-x(\tau)|)J(x(s)-x(\tau))]$$

and

$$H_n^2(s,\tau) := H_n(s,\tau) - \ln\left(\frac{4}{e}\sin^2\frac{s-\tau}{2}\right) H_n^1(s,\tau)$$

with the diagonal terms

$$H_n^1(s,s) = \frac{1}{2}(\eta_{1,n}(0)I + \eta_{2,n}(0)\widetilde{J}(s,s))$$

and

$$\begin{aligned} H_n^2(s,s) = \frac{1}{2}&\ln(|x'(s)|^2 e)(\eta_{1,n}(0)I + \eta_{2,n}(0)\widetilde{J}(s,s)) \\ &+ \xi_{1,n}(0)I + \xi_{2,n}(0)\widetilde{J}(s,s). \end{aligned}$$

Here, we used the diagonal values for the matrix J,

$$\widetilde{J}(s,s) = \frac{x'(s)x'(s)^\top}{|x'(s)|^2}.$$

We chose $M \in \mathbb{N}$ and an equidistant mesh by setting $s_k := k\pi/M$, $k = 0, \ldots, 2M-1$, and use the following quadrature rules:

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\ln\left(\frac{4}{e}\sin^2\frac{s_j-\tau}{2}\right)d\tau \approx \sum_{k=0}^{2M-1} R_{|j-k|}\, f(s_k) \tag{4.2}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\,d\tau \approx \frac{1}{2M}\sum_{k=0}^{2M-1} f(s_k) \tag{4.3}$$

with the weights

$$R_j := -\frac{1}{2M}\left\{1 - 2\sum_{m=1}^{M-1}\frac{1}{m}\cos\frac{mj\pi}{M} + \frac{(-1)^j}{M}\right\}, \quad j = 0, \ldots, 2M-1.$$

These quadratures are obtained by replacing the integrand f by its trigonometric interpolation polynomial of degree M with respect to the grid points s_k, $k = 0, \ldots, 2M-1$.

We use the quadrature rules (4.2)–(4.3) to approximate the integrals in the integral equations (4.1) and collocate at the nodal points to obtain the sequence of linear systems

$$\sum_{k=0}^{2M-1} \left\{ R_{|j-k|} H_0^1(s_j, s_k) + \frac{1}{2M} H_0^2(s_j, s_k) \right\} \psi_{n,M}(s_k) = G_{n,M}(s_j),$$

$j = 0, \dots, 2M-1$, which we must solve for the nodal values $\psi_{n,M}(s_j)$. For the right hand sides, we have

$$G_{n,M}(s_j) = g_n(s_j) - \sum_{m=0}^{n-1} \sum_{k=0}^{2M-1} \left\{ R_{|j-k|} H_{n-m}^1(s_j, s_k) + \frac{1}{2M} H_{n-m}^2(s_j, s_k) \right\} \psi_{m,M}(s_k). \tag{4.4}$$

For a more detailed description of this numerical solution method, including an error and convergence analysis based on interpreting the above method as a fully discrete projection method in a Hölder space setting and in a Sobolev space setting, we refer the reader to **[9]**. In particular, this error analysis implies the exponential convergence

$$\|\psi_n - \psi_{n,M}\|_\infty \le C_n e^{-\sigma M}$$

for some positive constants C_n and σ, provided that the boundary values are also analytic. Of course, due to the accumulation of the errors, the constants C_n will increase with n.

Given the approximate solution $\psi_{n,M}$ of the integral equation (4.1), the approximate solution of the initial boundary value problem is obtained by first evaluating the parametrized form of the potential (3.12) using the trapezoidal rule, that is, by

$$\widetilde{u}_{n,M}(x) = \frac{1}{2M} \sum_{m=0}^{n} \sum_{k=0}^{2M-1} E_{n-m}(x, x(s_k)) \psi_{m,M}(s_k), \quad x \in D, \tag{4.5}$$

and then summing up

$$u_{N,M}(x,t) = \kappa \sum_{n=0}^{N-1} \widetilde{u}_{n,M}(x) L_n(\kappa t) \tag{4.6}$$

according to the series (2.6).

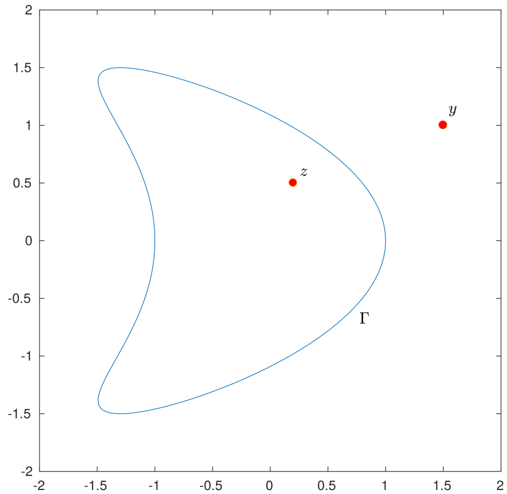

FIGURE 1. The boundary curve Γ, the source point $z \in \mathbb{R}^2 \setminus D$, and the measurement point $y \in D$.

5. Numerical results. For the numerical examples, we consider a kite-shaped boundary with parametrization

$$x(s) = (\cos(s) + 0.65\cos(2s) - 0.65, 1.5\sin(s)), \quad s \in [0, 2\pi].$$

In all of the examples, we choose the Lamé parameters to be $\lambda = 2$, $\mu = 1$, and the density $\rho = 1$.

In the first example, we set $\kappa = 1$, and we test the feasibility of the stationary problems (2.5a)–(2.5c). We choose two arbitrary points: a source point $z \in \mathbb{R}^2 \setminus D$ and a measurement point $y \in D$. We define the vector-valued boundary function

$$f_n(x) = [E_n(x, z)]_1, \quad x \in \Gamma, \tag{5.1}$$

where $[\cdot]_1$ denotes the first column of the tensor. Then, the field

$$u_n^{ex}(x) := [E_n(x, z)]_1, \quad x \in D,$$

is clearly a solution of (2.5a)–(2.5c) for the boundary function defined above. We consider the points $z = (0.2,\ 0.5)$ and $y = (1.5,\ 1)$, see Figure 1. The numerical values $\widetilde{u}_{n,M}(y)$ are presented in Table 1 and

TABLE 1. The first components of the computed and the exact solutions of (2.5a)–(2.5c), for the specific boundary function (5.1) at the measurement point $y = (1.5, 1)$.

M	$(\widetilde{u}_{0,M})_1(y)$	$(\widetilde{u}_{1,M})_1(y)$	$(\widetilde{u}_{2,M})_1(y)$
8	0.293581559232289	−0.084483725080856	−0.146079079028772
16	0.284988364785089	−0.092525310787524	−0.155666923858005
32	0.285503199323624	−0.092138738384510	−0.155404881866276
64	0.285503741272164	−0.092138337605882	−0.155404627504594
	$(u_0^{ex})_1(y)$	$(u_1^{ex})_1(y)$	$(u_2^{ex})_1(y)$
	0.285503741272020	−0.092138337605708	−0.155404627504139

Table 2, see (4.5), (and compare them with the exact solutions $u_n^{ex}(y)$ for $n = 0, 1, 2$ and varying M). The exponential convergence with respect to the spatial discretization is clearly exhibited, as we can also see in Figure 2, where we plot the L^2 norm of the difference in logarithmic scale.

TABLE 2. The second components of the computed and the exact solutions of (2.5a)–(2.5c), for the specific boundary function (5.1) at the measurement point $y = (1.5, 1)$.

M	$(\widetilde{u}_{0,M})_2(y)$	$(\widetilde{u}_{1,M})_2(y)$	$(\widetilde{u}_{2,M})_2(y)$
8	0.081036497084071	0.028667287783745	−0.011013685642670
16	0.071649152048006	0.017647071803846	−0.021814816353654
32	0.071756738147012	0.017837149908881	−0.021648258690479
64	0.071756880072043	0.017837482038337	−0.021647898034988
	$(u_0^{ex})_2(y)$	$(u_1^{ex})_2(y)$	$(u_2^{ex})_2(y)$
	0.071756880072350	0.017837482039221	−0.021647898033812

In the second example, we consider the time-dependent problem. We set in (1.3) as the boundary function:

$$f(x,t) = [E(x,z;t)]_1, \quad x \in \Gamma,\ z \in \mathbb{R}^2 \setminus D,\ t \in (0,\infty). \tag{5.2}$$

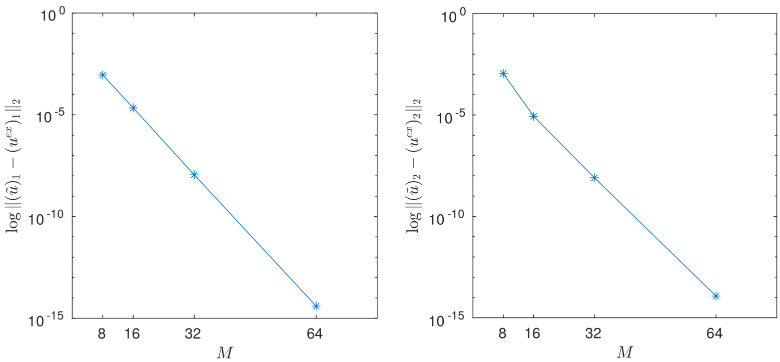

FIGURE 2. The L^2 norm of the difference between the computed and the exact solutions in logarithmic scale. In the left figure, we see the convergence for the values of Table 1, and in the right, those of Table 2.

Then, the exact solution is given by $u^{ex}(x,t) = [E(x,z;t)]_1$, $x \in D$, where its Fourier-Laguerre coefficients satisfy (2.5a)–(2.5c) for a boundary function f_n as in the first example. We consider a different source point $z = (0.4,\ 0.2)$. We compare the computed solution $u_{N,M}(y,t)$, formula (4.6), with the exact, considering the truncated form

$$u^{ex}(N;y,t) = \kappa \sum_{n=0}^{N-1} E_n(y,z) L_n(\kappa t). \tag{5.3}$$

In Table 3, we see the results of the first components, for $\kappa = 1/2$, at the position $y = (1,1)$, for different values of M and N at various time points. The values of the second components, at a different position $y = (0.5,\ -1.5)$, are presented in Table 4.

In the third example, we consider the spatial independent boundary function

$$f(x,t) = f(t)(1,1)^\top, \quad \text{for } f(t) = \frac{t^2}{4} e^{-t+2}, \tag{5.4}$$

which admits the expansion

$$f(t) = \frac{\kappa e}{4} \sum_{n=0}^{\infty} \frac{2 + \kappa n(\kappa(n-1) - 4)}{(\kappa+1)^{n+3}} L_n(\kappa t).$$

TABLE 3. Numerical values of the components of the computed $(u_{N,M})_1$ and the exact solution $(u^{ex}(N))_1$ (rows in grey) of the problem (1.1)–(1.4) for the boundary function (5.2). Here, $\kappa = 1/2$ and $y = (1,1)$.

t	M	$N = 15$	$N = 20$	$N = 25$
1	32	0.556497018896238	0.589619531491177	0.661666532791282
	64	0.556495142719243	0.589617717103517	0.661663974784660
	$(u^{ex})_1$	0.556495142721411	0.589617717107183	0.661663974772766
2	32	0.512733429371186	0.536980500925658	0.478568254585875
	64	0.512733615180447	0.536980917568962	0.478569337106983
	$(u^{ex})_1$	0.512733615179871	0.536980917566782	0.478569337119777
3	32	0.240000095287178	0.141039343209334	0.117399715612992
	64	0.240001133548879	0.141039958261864	0.117400385116931
	$(u^{ex})_1$	0.240001133547631	0.141039958259843	0.117400385110933

TABLE 4. Numerical values of the components of the computed $(u_{N,M})_2$ and the exact solution $(u^{ex}(N))_2$ (rows in grey) of the problem (1.1)–(1.4) for the boundary function (5.2). Here, $\kappa = 1/2$ and $y = (0.5, -1.5)$.

t	M	$N = 15$	$N = 20$	$N = 25$
1	32	−0.027394214935554	−0.029780601479552	−0.037360920446497
	64	−0.027394199243065	−0.029780534307372	−0.037360395287884
	$(u^{ex})_2$	−0.027394199243094	−0.029780534308750	−0.037360395289490
2	32	−0.006641666911909	−0.007677796534028	−0.001408341947825
	64	−0.006641661641063	−0.007677775150468	−0.001408727582784
	$(u^{ex})_2$	−0.006641661641210	−0.007677775149996	−0.001408727581196
3	32	0.008544988585762	0.014762278037365	0.016882729931766
	64	0.008544972022448	0.014762136068806	0.016882546539029
	$(u^{ex})_2$	0.008544972022701	0.014762136070815	0.016882546538771

The numerical solution of problem (1.1)–(1.4) is presented in Table 5 (the first component) and in Table 6 (the second component). Here, we do not know the exact solution, but we observe the convergence with respect to the discretization. We set $\kappa = 1/2$, and we compute the solution at the measurement point $y = (0.5,\ -1.5)$. Again, we see the exponential convergence with respect to M and the convergence with respect to N.

TABLE 5. The first component of the numerical solution of the problem (1.1)–(1.4) for the boundary function (5.4), for $\kappa = 1/2$, at the measurement position $y = (0.5,\ -1.5)$.

t	M	$N = 10$	$N = 15$	$N = 20$
1	16	0.062572635051987	0.047833811805535	0.046009612087451
	32	0.062555821083946	0.047821505207762	0.045995757574454
	64	0.062555817944294	0.047821498423333	0.045995752862933
2	16	0.206527137283687	0.230300799019877	0.230584463332204
	32	0.206507930918013	0.230271954401413	0.230552865835900
	64	0.206507936378995	0.230271967614367	0.230552881497197
3	16	0.281765785091472	0.290022752671760	0.293418322035422
	32	0.281753694023185	0.290010686252088	0.293412971755498
	64	0.281753701138142	0.290010693385048	0.293412971498783

TABLE 6. The second component of the numerical solution of the problem (1.1)–(1.4) for the boundary function (5.4), for $\kappa = 1/2$, at position $y = (0.5,\ -1.5)$.

t	M	$N = 10$	$N = 15$	$N = 20$
1	16	0.104385591001347	0.089274523790871	0.089029847496310
	32	0.104414411445436	0.089311568889684	0.089061508274427
	64	0.104414399045771	0.089311556699605	0.089061496285464
2	16	0.242963484605096	0.268615650361266	0.270356959200451
	32	0.242960849558260	0.268601030910481	0.270336784091780
	64	0.242963476811622	0.268601022577740	0.270336775642960
3	16	0.294596074694631	0.301687858830015	0.299978778147128
	32	0.294584218004291	0.301675713615639	0.299984785021505
	64	0.294584215781299	0.301675711488577	0.299984782636057

Note that similar convergence results can also be obtained for different types of domains with sufficiently smooth boundaries.

6. Conclusions. We presented an integral equation approach for the numerical solution of the elastodynamic problem in planar unbounded domains. Our method was based on the consecutive reduction of the dimensions of the problem. Firstly, by Laguerre transformation, we reduced the given three-dimensional time dependent problem to a recurrent sequence of stationary two-dimensional boundary value problems. We constructed a sequence of special fundamental solutions of the corresponding elliptic equations in order to introduce the boundary potential representation of the solutions. Then, the problems were transformed to a sequence of one-dimensional boundary integral equations with kernels having logarithmic singularities. Taking into account the parametric representation of the boundary, we parametrized these equations and applied the quadrature method based on trigonometrical quadrature rules. The numerical solution of the elastodynamic problem was calculated as a partial sum of a Fourier-Laguerre series. We validated the proposed method on some numerical examples and demonstrated its applicability. This approach can also be used for problems in three-dimensional domains with different boundary conditions.

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Ivan Franko National University of Lviv, Faculty of Applied Mathematics and Informatics, Lviv, Ukraine
Email address: **chapko@lnu.edu.ua**

University of Vienna, Computational Science Center, Austria
Email address: **leonidas.mindrinos@univie.ac.at**